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Existence of solutions for some quasi-variational inequalities

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Existence of Solutions for Some Quasi-Variational Inequalities.

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1. – Introduction.

We are interested in proving the existence (Theorem 1 in Sect. 2) of solutions for quasi-variational inequalities (QVI) using the same argument for continuous mappings, as for situations in which one has monotonicity and hemicontinuity assumptions. Thus, on the one hand our Theorem 1 is valid (see Example 1 in Sect. 3) in the case of QVI arisen from problems of plasma physics studied by Mossino [9], on the other hand the existence result for variational inequalities for monotone operators recently stated by Minty [8] is found again as a particular case of our mentioned theorem (see Example 2 in Sect. 3). A joining of both sort of hypotheses can be bound in the application (see Example 3 in Sect. 3) to QVI in connection with a free boundary problem of hydraulics (investigated by other methods by Baiocchi [2], [3]).

Unlike other authors ([5], [9], [11], [12]) who use different fixed point theorems for suitable selection maps of the QVI, we achieve the proof of Theorem 1 by a nonempty intersection property.

The notion of compatible topologies which we introduce just in this introductory paragraph is not of small account. The charge of this simple idea turns out specifically from the existence result established in Theorem 2 of Sect. 4.

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Finally, let us point out the application of Theorem 2 to parabolic QVI (see Corollary 2 in Sect. 4) which is of different nature than the existence results of [4].

We begin with the

DEFINITION. Consider two topological spaces (C_1, τ_1) and (C_2, τ_2); we say that the topology τ_1 is compatible on $C_1 \cap C_2$ with the topology τ_2 if the diagonal of the product space $(C_1 \cap C_2) \times (C_1 \cap C_2)$ is $\tau_1 \times \tau_2$-closed.

REMARKS. 1) Let $C_1 \subset C_2$ be Banach spaces, with duals C_1', C_2', such that the injection $C_1 \hookrightarrow C_2$ is linear and continuous and C_1 is dense in C_2. Consider τ_1 the (C_1, C_1') weak topology on C_1 and τ_2 the norm topology on C_2. τ_1 is compatible on $C_1 \cap C_2 = C_1$ with τ_2 since both topological spaces (C_1, τ_1) and (C_2, τ_2) have continuous injection in (C_2, τ_3), where τ_3 is the (C_2, C_2') weak topology. If the injection $C_1 \hookrightarrow C_2$ is even compact then the (C_1, C_1') weak topology is stronger than the trace of the norm topology of C_2 on C_1.

2) Let D be a bounded open subset with smooth boundary in the plane R^2. The weak topology τ_1 of the usual Sobolev space $H^1(D)$ is compatible on $H^1(D) \cap C^0(\bar{D})$ with the Banach topology τ_2 of the space $C^0(\bar{D})$ of continuous functions on the closure $\bar{D}$, because both $(H^1(D), \tau_1)$ and $(C^0(\bar{D}), \tau_2)$ have continuous injection in the Hilbert space $L^2(D)$.

2. – The existence theorem.

THEOREM 1. Let E_1 and E_2 be linear subspaces of a real vector space E. Suppose that E_1, E_2 are endowed with the linear separated topologies τ_1, respectively τ_2, which are compatible on $E_1 \cap E_2$. Further, let C_1 be a τ_1-closed subset of E_1 and C_2 a τ_2-closed subset of E_2. Given $h(u, w)$ and $g(u, v, w)$ functions from $C_2 \times C_1$, respectively $C_2 \times \times C_1 \times C_1$ into $R \cup \{+\infty\}$, let us assume that:

(1) For fixed $u \in C_2$, $v \in C_1$, $h(u, \cdot)$ and $g(u, v, \cdot)$ are proper convex functions on C_1.

(2) For fixed $u \in C_2$, $g(u, v, v) = 0$ and $g(u, v, w) + g(u, w, v) \leqslant 0$ for all $v, w \in C_1$.

(3) For all $u \in C_2$, $w \in C_1$, one has

$$\limsup_{t \downarrow 0^+} g(u, (1-t)v_1 + tv_2, w) = g(u, v_1, w) \quad \text{for all } v_1, v_2 \in C_1 .$$

(4) For fixed $w \in C_1$, $h(\cdot, w)$ is τ_2-upper semicontinuous (usc) on C_2.

(5) h is $\tau_2 \times \tau_1$-lower semicontinuous (lsc) on $C_2 \times C_1$.

Let us also presume that there exists a nonempty subset $K \subset C_1 \cap C_2$, compact with respect to τ_1 and τ_2-relatively compact and one of the following condition holds:

either ($C1$) $g(\cdot, v, \cdot)$ is $\tau_2 \times \tau_1$-lsc for fixed $v \in C_1$ and there exist an element $v_0 \in K$ such that $h(w, w) + g(w, v_0, w) > h(w, v_0)$ for all $w \in (C_1 \cap C_2) \setminus K$

or ($C2$) $g(\cdot, \cdot, w)$ is $\tau_2 \times \tau_1$-usc for fixed $w \in C_1$ and there exist an element $w_0 \in K$ such that $h(v, v) > h(v, w_0) + g(v, v, w_0)$ for all $v \in (C_1 \cap C_2) \setminus K$.

Under the above hypotheses the system of inequalities

$$\text{(QVI)} \qquad h(u, u) \leqslant h(u, w) + g(u, u, w) , \quad w \in C_1 \cap C_2$$

has at least one solution $u \in K$.

REMARKS. There are several significant differences between Theorem 1 and Theorems 4.1-4.2 from [9; Chapter 1]. The function $h(u, w)$ is assumed to be continuous in the first variable and clearly it is not of type of an indicator map $\delta(Q(u), w)$. Hence our theorem is not active in the case of the « classical » QVI introduced by Bensoussan-Lions [4] [5]; its aplicability will be shown by the examples from Section 3, which are QVI rather in the abstract sense of Tartar [12].

Note also that the alternate coerciveness properties ($C1$) ($C2$) have a new formulation that seems to us naturally suited. This coercivity condition together with the above mentioned continuity assumption permits us to give a proof which is not based upon Kakutani's fixed point theorem applied to a selection map; implicitly, the proof does not require the local convexity of the linear topologies.

Finally, we make use of « compatible » topologies (the argument involves their product) and in this way our theorem extends over a larger class of QVI.

PROOF OF THEOREM 1. First, it is to be established that, by compatibility of τ_1 and τ_2, K is necessarily τ_2-compact too. Moreover, one can show that τ_1 and τ_2 are even equal on K.

Next, let us define for each $w \in C_1 \cap C_2$ the subset

$$M_1(w) = \{(u, u);\ u \in C_1 \cap C_2,\ h(u, u) + g(u, w, u) \leqslant h(u, w)\}$$

if the supposition $(C1)$ is valid or alternatively

$$M_2(w) = \{(u, u);\ u \in C_1 \cap C_2,\ h(u, u) \leqslant h(u, w) + g(u, u, w)\}$$

under the supposition $(C2)$. $M_i(w)$ contains at least (w, w). By hypotheses $(C1)$, $(C2)$ one has $M_1(v_0) \subset K \times K$, respectively $M_2(w_0) \subset K \times K$.

We have in mind to show that $M_i(w)$ are $\tau_2 \times \tau_1$-closed in $C_2 \times C_1$ and that they have nonempty finite intersections.

Consider a generalized sequence $\{u_\delta, u_\delta\}_{\delta \in \Delta}$ in $M_i(w)$ with $u_\delta \xrightarrow{\tau_1} u_1$, $u_\delta \xrightarrow{\tau_2} u_2$; the compatibility of τ_2 and τ_1 yields $u_1 = u_2 = u$ and, as a consequence of hypotheses (4) (5) and $(C1)$ or $(C2)$, (u, u) must belong to $M_i(w)$.

For a finite subset $\{w_1, \ldots, w_n\} \subset C_1 \cap C_2$ we denote by T the following map defined on the $n-1$ dimensional simplex S of R^n:

$$T\Big(\sum_{i=1}^{n} a_i x_i\Big) = \Big(\sum_{i=1}^{n} a_i w_i,\ \sum_{i=1}^{n} a_i w_i\Big)\,,$$

where $a_i \geqslant 0$, $\sum_{i=1}^{n} a_i = 1$ and $\{x_1, \ldots, x_n\}$ is the canonical basis of R^n. A value $\Big(\overline{w} = \sum_{i=1}^{n} a_i w_i, \overline{w}\Big)$ of T belongs $M_1(w_1) \cup \ldots \cup M_1(w_n)$ and also to $M_2(w_1) \cup \ldots \cup M_2(w_n)$. If not *either*

$$h(\overline{w}, \overline{w}) + g(\overline{w}, w_i, \overline{w}) > h(\overline{w}, w_i) \qquad \text{for all } i = 1, \ldots, n$$

and this implies according to (2)

$$h(\overline{w}, \overline{w}) > h(\overline{w}, w_i) + g(\overline{w}, \overline{w}, w_i)\,, \qquad \forall i = 1, \ldots, n$$

or in the second situation we get straightforwardly the same inequalities. Using the convexity (1), it follows that

$$h(\overline{w}, \overline{w}) > h(\overline{w}, \overline{w}) + g(\overline{w}, \overline{w}, \overline{w})$$

and comparing to (2) we obtain a contradiction. Hence $T^{-1}(M_1(w_i))$, $i = 1, \ldots, n$, similarly $T^{-1}(M_2(w_i))$, $i = 1, \ldots, n$ must cover the simplex S and as easily seen each face of S is contained in the union of the corresponding sets from these latter. Obviously T is continuous; then, under the hypothesis $(C1)$, $T^{-1}(M_1(w_i))$ are closed and alternatively under $(C2)$, $T^{-1}(M_2(w_i))$ are closed in S. By virtue of Knaster-Kuratowski-Mazurkiewicz Lemma $\bigcap_{i=1}^{n} T^{-1}(M_1(w_i)) \neq \emptyset$, respectively $\bigcap_{i=1}^{n} T^{-1}(M_2(w_i)) \neq \emptyset$, and therefore either

$$\bigcap_{i=1}^{n} (M_1(w_i)) \neq \emptyset \quad \text{or} \quad \bigcap_{i=1}^{n} M_2(w_i) \neq \emptyset .$$

We conclude that the global intersection

$$\bigcap_{w\in C_1 \cap C_2} (M_1(w) \cap M_1(v_0)) , \qquad \text{respectively} \bigcap_{w\in C_1 \cap C_2} (M_2(w) \cap M_2(w_0))$$

is nonempty in the $\tau_2 \times \tau_1$-compact $K \times K$. In the second case, we have already proven the existence of the desired solution $u \in K$ (see Corollary 1 below).

In the first case, the assertion of the theorem will follow if we prove that any $u \in K$ with

$$h(u, u) + g(u, w, u) \leqslant h(u, w) , \qquad \forall w \in C_1 \cap C_2$$

satisfies the QVI from the statement. Otherwise, it would exist $\overline{w} \in C_1 \cap C_2$ with $h(u, u) > h(u, \overline{w}) + g(u, u, \overline{w})$. Consider then $t \in (0, 1]$ and $(1-t)u + t\overline{w} \in C_1 \cap C_2$; the lim inf as $t \downarrow 0^+$ of

$$h(u, (1-t)u + t\overline{w}) - g(u, (1-t)u + t\overline{w}, \overline{w})$$

can be computed by (3) and (5) to obtain $h(u, u) - g(u, u, \overline{w})$. Hence

$$h(u, (1-t)u + \overline{w}) - g(u, (1-t)u + t\overline{w}, \overline{w}) > h(u, \overline{w}) \qquad \text{for all } t \text{ in } (0, t_0] .$$

But it is also true that

$$h(u, (1-t)u + t\overline{w}) - g(u, (1-t)u + t\overline{w}, u) \geqslant h(u, u) .$$

We add these two inequalities multiplied by t, respectively $1-t$, and using the convexity (1) we get

$$g(u, (1-t)u + t\overline{w}, (1-t)u + t\overline{w}) < 0 .$$

This contradicts (2) and the proof is completed.

COROLLARY 1. (As one can ascertain according to the above proof) Theorem 1 remains true under the assumption $(C2)$ if we omit the monotonicity hypothesis $g(u, v, w) + g(u, w, v) \leqslant 0$ from condition (2) (we may neglect also hypothesis (3), which is implied by $(C2)$).

3. – Examples.

We shall first apply Theorem 1 to some inequalities which arose from plasma physics. Such QVI were solved by other methods in [10] [11].

EXAMPLE 1. Let $E_1 = C_1 = W^{1,p}(D)$ be the usual Sobolev space on a bounded open subset D with smooth boundary of the n-dimensional Euclidean space R^n. Consider τ_1 the weak topology on $W^{1,p}(D)$. Let $E_2 = C_2 = L^p(D)$ be the space of all (equivalence classes of) p-integrable functions on D, $p \geqslant 1$, and τ_2 its usual Banach topology. The topology τ_2 is weaker on $E_1 \cap E_2 = W^{1,p}(D)$ than τ_1. The choice of the space $E \supset E_{1,2}$ is obvious.

Let us take g identically 0 on $C_2 \times C_1 \times C_1$ and

$$h(u, w) = \frac{1}{p}\int_D\left(|w|^p + \sum_{i=1}^{n}\left|\frac{\partial w}{\partial x_i}\right|^p\right)dx - \int_D fw\,dx + \iint_{D\times D}(w(x) - u(y))_+\,dx\,dy$$

defined on $C_2 \times C_1$, with $f \in L^q(D)$, $1/p + 1/q = 1$. We shall analyse successively the conditions of Theorem 1.

To satisfy $(C1)$ or $(C2)$, we take K a bounded closed ball in

$W^{1,p}(D)$ centered at $v_0 = 0$, with a sufficiently large radius given by $\lim_{\|w\|_{W^{1,p}} \to +\infty} (h(w, w) - h(w, 0)) = +\infty$. Indeed, we have

$$h(w, w) - h(w, 0) \geq \frac{1}{p} \|w\|^p_{W^{1,p}} - k_1 \|w\|_{W^{1,p}} - \\ - \iint_{D \times D} |(w(x) - w(y))_+ - (-w(y))_+| \, dx\, dy \geq \\ \geq \frac{1}{p} \|w\|^p_{W^{1,p}} - k_1 \|w\|_{W^{1,p}} - k_2 \int_D |w(x)| \, dx ,$$

because the functional $(\cdot)_+$ is Lipschitzian. The latter shows that we can choose a convenient radius for K such that $h(w, w) > h(w, 0)$ for all $w \in W^{1,p}(D) \setminus K$. The Banach topology τ_2 of $L^p(D)$ and the weak topology τ_1 of $W^{1,p}(D)$ are equivalent on K!

It is easy to see that $h(u, \cdot)$ is strictly convex on $W^{1,p}(D)$. Hypotheses (2) and (3) are vacuous for $g \equiv 0$. Condition (4) is also verified because

$$|h(u, w) - h(u_0, w)| = \iint_{D \times D} [(w(x) - u(y))_+ - (w(x) - u_0(y))_+] \, dx\, dy \leq \\ \leq k_1 \int_D |u(y) - u_0(y)| \, dy \leq k_2 \|u - u_0\|_{L^p}$$

and it follows the continuity of $h(\cdot, w)$ on $L^p(D)$.

It only remains to check the validity of hypothesis (5); h is $\tau_2 \times \tau_1$ lsc on $L^p(D) \times W^{1,p}(D)$, because $\iint_{D \times D} (w(x) - u(y))_+ \, dx\, dy$ is continuous with respect to the $L^1(D)$ norm and $(1/p)\|w\|^p_{W^{1,p}} - \int_D fw \, dx$ is weak lsc on $W^{1,p}(D)$.

We point out that it was necessary to resort to the weak topology on $W^{1,p}(D)$, since the continuity of $h(u, \cdot)$ in the norm topology of $W^{1,p}$ does not suffice to infer the conclusion (K is only weakly compact in $W^{1,p}(D)$) and L^p-semicontinuity in the second variable is useless as $W^{1,p}(D)$ is not closed in $L^p(D)$.

We come to the conclusion that, according to Theorem 1, the

$$\text{(QVI)} \qquad h(u, u) \leq h(u, w) , \quad w \in W^{1,p}(D)$$

has a solution $u \in W^{1,p}(D)$.

EXAMPLE 2. Let $E_1 = E_2 = E$ be a linear topological space and $C_1 = C_2 = K$ a convex compact subset. Consider h identically 0 on $K \times K$ and take $g(u, v, w) = \langle w - v, f(v, u)\rangle$, where f is a mapping from $K \times K$ into a topological group Y, as in Minty's paper [8].

In view of Theorem 1, we set as $\tau_1 = \tau_2$ the topology of $E_1 = E_2$. Because $(C_1 \cap C_2) \setminus K = \emptyset$ and since $f(v, \cdot)$ is presumed to be continuous from K to the topological group Y and $\langle \cdot, \cdot \rangle$ continuous on $(K - K) \times Y$, the hypothesis $(C1)$ is exceedingly verified.

For all $y \in Y$, $\langle \cdot, y\rangle$ is linear from E to R, hence condition (1) of our theorem is also satisfied. For all $x \in E$, $\langle x, \cdot\rangle$ is a homomorphism of Y into the additive R and it is assumed that

$$\langle w - v, f(v, u)\rangle \leqslant \langle w - u, f(w, u)\rangle , \qquad \forall v, w \in K$$

and thus hypothesis (2) is verified.

Next, consider (as in [8]) Y with the weakest topology in which all $\langle x, \cdot\rangle$ are continuous and suppose that for any fixed $u \in K$, $f(\cdot, u)$ has continuous restriction to any line segment in K; it follows that condition (3) of Theorem 1 is satisfied.

Thus, we find again the existence result from a theorem for variational inequalities recently given in [8]: the inequalities

$$0 \leqslant \langle w - u, f(u, u)\rangle , \qquad w \in K ,$$

have at least a solution $u \in K$.

EXAMPLE 3. We can use Corollary 1 to give a direct proof for the existence of a solutions of a QVI formulated by Baiocchi [2] [3].

Let $D = \{(x, y); a < x < b, 0 < y < Y(x)\}$ be a subset of R^2, where $a < 0 < b$ and Y is a certain C^3 and strictly concave function on $[a, b]$. Suppose (see [3]) that $Y(a) = Y(b) = 0$ and $0 < Y(c) < Y(0)$ for a given $c \in (0, b)$. We take $E_1 = E_2 = H^1(D)$ the Sobolev space with its usual weak topology $\tau_1 = \tau_2$.

Let us set $C_1 = C_2 = \{v \in H^1(D); v(x, 0) = 0, a < x < b\}$ which is a closed subspace in E (the trace $v(\cdot, 0)$ makes sense for $v \in H^1(D)$).

The functions g and h in this example are defined on $C_1 \times C_1$ by $g(v, w) = a(v, w - v)$ with

$$a(v, w) = \int_D \left[\frac{\partial v}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial y} + Y'\left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial w}{\partial x}\right) - Y''\frac{\partial v}{\partial y} w\right] dx\, dy$$

and

$$h(u, w) = \int_{D_3} [(\gamma_3 u)^+ - w]^+ \, dx\, dy + \int_{D_1 \cup D_2} w\, dx\, dy + \int_a^b p(x)(\gamma_0 w)(x)\, dx\,,$$

where: $D_1 \cup D_2$ is an open subset of D; $D_3 = \{(x, y) \in D;\ 0 < x < c\}$; γ_3 and γ_0 are the trace operators

$$(\gamma_3 u)(x) = u(x, Y(x))\,, \qquad 0 < x < c$$

and

$$(\gamma_0 w)(x) = w(x, Y(x))\,, \qquad a < x < b\,,$$

with values in $H^{\frac{1}{2}}_{00}(0, c)$, respectively $H^{\frac{1}{2}}_{00}(a, b)$ (see [3] and [7], vol. I, Ch. 1, § 11); $p(x)$ is a continuous function on $[a, b]$.

We prove that Corollary 1 can be used to deduce the existence of a solution $u \in C_1$ for the

(QVI) $$h(u, u) \leqslant h(u, w) + g(u, w)\,, \qquad w \in C_1\,.$$

One can verify that

$$a(v, v) \geqslant \int_D \left(\left| \frac{\partial v}{\partial x} \right|^2 + \left| \frac{\partial v}{\partial y} \right|^2 \right) dx\, dy$$

for all $v \in C_1$ and $a(\cdot, \cdot)$ is continuous bilinear on C_1 taken with the norm

$$\|v\|^2_{C_1} = \int_D \left(\left| \frac{\partial v}{\partial x} \right|^2 + \left| \frac{\partial v}{\partial y} \right|^2 \right) dx\, dy;$$

this defines on C_1 the same topology as the usual norm of $H^1(D)$. Thus $g(v, w)$ is τ_1-usc in the first variable. As

$$h(v, v) - h(v, 0) + g(v, 0) = \int_{D_3} [(\gamma_3 v)^+ - v]^+ \, dx\, dy - \int_{D_3} (\gamma_3 v)^+ \, dx\, dy$$
$$+ \int_{D_1 \cup D_2} v\, dx\, dy + \int_a^b p(x)(\gamma_0 v)(x)\, dx + a(v, v) \geqslant -k\|v\|_{C_1} + \|v\|^2_{C_1}\,,$$

it is possible to choose M in $(0, +\infty)$ in such a way that the requirement $(C2)$ of Theorem 1 is satisfied for

$$K = \left\{ v \in C_1;\ \int_D \left(\left| \frac{\partial v}{\partial x} \right|^2 + \left| \frac{\partial v}{\partial y} \right|^2 \right) dx\, dy \leqslant M \right\}.$$

Concerning continuity qualities of h, let us observe that the last two integral terms are linear continuous on C_1 and as to the first of its integrals we find:

$$\left| \int_{D_3} [(\gamma_3 u_n)^+ - w_n]^+ dx\, dy - \int_{D_3} [(\gamma_3 u_0)^+ - w_0]^+ dx\, dy \right| \leqslant$$

$$\leqslant \int_{D_3} |(\gamma_3 u_n)^+ - (\gamma_2 u_0)^+|\, dx\, dy + \int_{D_3} |w_0 - w_n|\, dx\, dy\ .$$

If the weak limit $n \uparrow +\infty$ of u_n and w_n are just u_0, respectively w_0, then the right side in this inequality tends to 0, because $w_n \to w_0$ in $L_1(D_3)$ and $(\gamma_3 u_n)^+ \to (\gamma_3 u_0)^+$ in $L^1(0, c)$. Hence hypotheses (4) and (5) of Theorem 1 are positively tested.

Clearly, $g(v, v) = 0$ (as required in Corollary 1) for all $v \in C_1$. Regarding the condition (1) we need still to note the convexity of the functional $(\alpha - \cdot)^+$ on R which implies the convexity of $h(u, \cdot)$ on C_1.

It should be emphasized that a solution of the above QVI belongs to $W^{2,r}(D)$, according to [2] [3], where the existence of minimal and maximal solutions is proved constructively by an iterative method.

4. – Parabolic QVI.

Our existence result for parabolic QVI is based on the following general theorem which may be used even in stationary cases.

THEOREM 2. Consider the Banach spaces E_1, E_2 such that $E_1 \subset E_2$ algebraically and topologically and E_1 is dense in E_2. Denote by $\tau_1(\tau)$ the weak (norm) topology of E_1 and by τ_2 the norm topology on E_2. Let $C \subset E_1$ be a convex (τ_1) closed subset with compact injection in E_2 and let $Q \subset E_2 \times C$ be a $\tau_2 \times \tau$-continuous ([9], Ch. I, § 7) multifunction with convex (τ_1) closed values. Further suppose that $g(v, w)$ is a real function on $C \times C$, τ_1-usc and concave in the first variable, convex in the second one, $\tau_1 \times \tau$-usc on each $C_0 \times C$, where C_0 is a ball of E_1, with $g(v, v) = 0$, $\forall v \in C$.

If there exist a constant $k > 0$ and an element $w_0 \in Q(u)$, for all $u \in C$ with $\|u\|_{E_1} \leqslant k$, so that $g(v, w_0) < 0$ for every $v \in C$ with $\|v\|_{E_1} > k$, then the

$$\text{(QVI)} \qquad g(u, w) \geqslant 0 , \quad w \in Q(u)$$

has at least one solution $u \in Q(u)$.

PROOF. Let us define the selection map of the (QVI), namely

$$T\colon \{u \in C;\ \|u\|_{E_1} \leqslant k\} \to 2^C$$

as follows: $v \in T(u)$ if and only if v is a solution of the variational problem

$$\begin{cases} v \in Q(u) \\ g(v, w) \geqslant 0 , \quad \forall w \in Q(u) . \end{cases}$$

First, we appeal to Corollary 1 to prove that $T(u)$ are nonempty. We consider the trivial case when the two compatible topologies coincide with τ_1 on E_1 and take $C_1 = C_2 = Q(u) \neq \emptyset$. Conditions (1), (2), (4), (5) of Theorem 1 are verified if we put $g(u, v, w) = g(v, w)$ and $h \equiv 0$ on $Q(u) \times Q(u)$. The actual hypotheses assure the existence of the nonvoid τ_1-compact set

$$K = \{v \in Q(u);\ \|v\|_{E_1} \leqslant k\}$$

so that $(C2)$ is thoroughly satisfied.

Since, $Q(u)$ is τ_1-closed and $g(\cdot, w)$ is τ_1-usc and concave it follows that $T(u)$ is τ_1-closed and convex.

Next, we note that $T(u) \subset C_0$, for all $u \in C_0$, where C_0 denotes the set $C_0 = \{u \in C;\ \|u\|_{E_1} \leqslant k\}$. This set is τ_1-compact and also τ_2-relatively compact due to the compactness of the injection $C \hookrightarrow E_2$. The topologies τ_1 and τ_2 are compatible on E_1 (Example 1 in Section 1) and they are equivalent on C_0.

The assertion of the theorem follows by Kakutani's fixed point theorem [6] applied to $T\colon C_0 \to 2^{C_0}$. We must still show that T is closed in $C_0 \times C_0$; to this purpose consider the sequence $(u_n, v_n) \in T$ with the $\tau_1 \times \tau_1$ limit $(u, v) \in C_0 \times C_0$. We have $v_n \in Q(u_n)$, $g(v_n, w) \geqslant 0$, $\forall w \in Q(u_n)$ and therefore

$$g(v_n, w) + \delta(Q(u_n), w) \geqslant \delta(Q(u_n), v_n) \qquad \text{for all } w \in C ,$$

where δ is the indicator map

$$\delta(Q(u), w) = \begin{cases} 0 & \text{if } w \in Q(u) \\ +\infty & \text{if } w \notin Q(u) . \end{cases}$$

At this moment, we use the continuity property of the multivalued mapping Q together with that of g, to infer

$$g(v, w) + \delta(Q(u), w) \geqslant \delta(Q(u), v) .$$

This means that $v \in Tu$. The proof is completed.

COROLLARY 2. Let V be a real reflexive Banach space with linear and compact injection into a real Hilbert space H; V is supposed to be dense in H. Consider $\tau_1(\tau)$ the weak (norm) topology on $E_1 = L^2(0, T; V)$ and τ_2 the norm topology on $E_2 = L^2(0, T; H)$. Let $a(\cdot, \cdot)$ be a continuous coercive bilinear form on $V \times V$, $f \in L^2(0, T; H)$ and denote $(\cdot, \cdot)$ the pairing between V' and V. For an arbitrary $\alpha > 0$, denote

$$C = \left\{ v \in E_1;\ v(0) = v_0,\ \frac{dv}{dt} \in L^2(0, T;\ V'),\ \left\| \frac{dv}{dt} \right\|_{L^2(0,T;V')} \leqslant \alpha \right\} .$$

Let $Q \subset E_2 \times C$ be a $\tau_2 \times \tau$-continuous multifunction with convex values, such that there exist $w_0 \in \bigcap_{u \in B} Q(u)$ for at least one subset

$$B = \{u \in C;\ \|u\|_{L^2(0,T:V)} \leqslant k\}$$

with the radius k larger than a certain constant depending on $a(\cdot,\cdot)$, f and $\|w_0\|_{L^2(0,T;V)}$.

The QVI

$$\int_0^T \left[\left(\frac{du}{dt}, w(t) - u(t) \right) + a\big(u(t), w(t) - u(t)\big) - \big(f(t), w(t) - u(t)\big) \right] \mathrm{d}t \geqslant 0 ,$$

$$w \in Q(u)$$

has at least a solution $u \in Q(u)$.

PROOF. Apply Theorem 2 to the function

$$g(v, w) = \int_0^T \left(\frac{dv}{dt}, w - v\right) dt + \int_0^T a(v, w - v)\, dt - \int_0^T (f, w - v)\, dt$$

defined on $C \times C$.

The set C is convex and closed in $L^2(0, T; V)$: if $v_n \in C$, $v_n \to v$ in $L^2(0, T; V)$, then necessarily a subsequence of dv_n/dt converges weakly to $\bar{v}$ in $L^2(0, T; V')$ and $\bar{v} = dv/dt$; thus we get $v \in C$. By a theorem due to Aubin [1] the injection $C \hookrightarrow E_2 = L^2(0, T; H)$ is compact.

Obviously, $g(v, \cdot)$ is affine and $g(v, v) = 0$, $\forall v \in C$. Each integral term of g is concave in v on C, even $\int_0^T (dv/dt, -v)\, dt$ because of the restriction $v(0) = v_0$, $\forall v \in C$. Moreover $g(\cdot, w)$ is τ_1-usc on C, since the very same three terms are strongly continuous in v on $C \subset L^2(0, T; V)$ (here one avails himself of the fact that for the second integral dv/dt is bounded by α in the norm of $L^2(0, T; V')$, $\forall v \in C$).

From continuity and coerciveness properties it follows that

$$g(v, w_0) \leqslant \alpha(\|w_0\| + \|v\|) + M_1\|v\| \cdot \|w_0\| - M_2\|v\|^2 + \\ + \|f\|_{L^2(0,T;H)} \cdot M_3(\|w_0\| + \|v\|),$$

with $\|\cdot\|$ the norm in $L^2(0, T; V)$. Then the required inequality $g(v, w_0) < 0$ holds for all $v \in C \setminus B$, where B is as before with a sufficiently large radius $k \geqslant k_0$ dependent on α, M_1, M_2, M_3 and $\|w_0\|$.

EXAMPLE 4. We attempted a similar existence result in [13]. The hypothesis $w_0 \in \bigcap_{u \in B} Q(u)$ for a certain bounded subset $B = \{v \in C;\ \|v\|_{L^2(0,T;V)} \leqslant k\}$ seems of different nature than the usual growth assuptions on Q and is perhaps the main difference by comparison to the existence results from [4].

To reassure us concerning the requirements on Q, let us consider just the following simple case: for a bounded open set $D \subset R^n$, take

$$Q \subset L^2(0, T; L^2(D)) \times L^2(0, T; H^1(D))$$

defined by

$$Q(u) = \left\{ v;\ v(t, x) \leqslant \int_D |u(t, x)|\, dx \text{ almost everywhere on } [0, T] \times D \right\}.$$

Q is closed, i.e. if $(u_n, v_n) \in Q$ and u_n converges strongly to u in $L^2(0, T; L^2(D))$, v_n converges to v in $L^2(0, T; H^1(D))$, then $(u, v) \in Q$. Finally, $w_0 = 0 \in Q(u)$, $\forall u \in L^2(0, T; H^1(D))$ and thus the above men tioned hypothesis is trivially satisfied by Q.

Our Theorem 2 has applications as well to stationary QVI. We draw the following

COROLLARY 3. Consider the Hilbert spaces $E_1 \subset E_2$ such that the injection $E_1 \hookrightarrow E$ is linear compact and E_1 is dense in E_2. Suppose that E_1 is an ordered linear space with closed positive cone. Denote by a $(\cdot, \cdot)$ a coercive bilinear continuous form on $E_1 \times E_1$, by f an element of E_2 and by $(\cdot, \cdot)$ the inner product in E_2. Let $M\colon E_2 \to E_1$ be a mapping which is continuous from the norm topology of E_2 to the norm topology of E_1 and bounded from below, $w_0 \leqslant Mu$ on at least one ball $B = \{u \in E_1;\ \|u\|_{E_1} \leqslant k\}$ with $k \geqslant k_0$, where k_0 is a certain constant depending on $\|w_0\|_{E_1}$, $a(\cdot, \cdot)$ and f. The

$$\text{(QVI)} \qquad a(u, w - u) \geqslant (f, w - u)\,, \quad \forall w \in E_1\,,\ w \leqslant Mu$$

has at least a solution $u \leqslant Mu$.

PROOF. Apply Theorem 2 to $g(v, w) = a(v, w - v) - (f, w - v)$, $C = E_1$ and $Q(u) = \{v \in E_1;\ v \leqslant Mu\}$. Clearly, we have to do with a strict specialization of the data of that theorem. $Q \subset E_2 \times E_1$ has convex values by linearity of « $\leqslant$ ». Q is closed by continuity of M and closeness of the positive cone in E_1. From

$$g(v, w_0) \leqslant M_1 \|v\|_{E_1} \cdot \|w_0\|_{E_1} - M_2 \|v\|^2_{E_1} + \|f\|_{E_2} \cdot M_3 \big(\|w_0\|_{E_1} + \|v\|_{E_1}\big)$$

we see that $g(v, w_0) < 0$ is valid for all $v \in C$ with $\|v\|_{E_1} > k \geqslant k_0$ with

$$k_0 = \frac{1}{2M_2} \left[M_1 \|w_0\| + M_3 \|f\| + [(M_1 \|w_0\| + M_3 \|f\|)^2 + 4 M_2 M_3 \|f\| \cdot \|w_0\|]^{\frac{1}{2}} \right].$$

Note that M_1 depends on the continuity of $a(\cdot, \cdot)$, M_2 on its coercivity and M_3 on the injection $E_1 \hookrightarrow E_2$.

Remarks. Corollary 3 enlightens the connections between Theorem 2 and the existence theorems from [5] and [12]. In our Corollary 3, the space E_2 is not ordered and the positive cone is closed only in E_1, condition (11) on Q from [12] or the corresponding assumption « M decreasing » from [5] are replaced in Corollary 3 by the hypothesis that M is bounded below on a certain ball from E_1.

To illustrate let us take $M\colon E_2 = L^2(D) \to E_1 = H^1(D)$, defined by

$$(Mu)(x) = \frac{1}{\operatorname{meas} D}\int_D |u|\, dx\,, \qquad \forall x \in D\,.$$

M satisfies the required conditions: M is strongly continuous and $\forall u_1 \in E$, Mu is a constant function such that $Mu \geqslant w_0$ with $w_0 \equiv 0$ on D.

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