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RICHARD D. CARMICHAEL

Distributional boundary values in $\mathfrak{D}'_{L^p}$ (IV)

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Distributional Boundary Values in $\mathcal{D}'_{L^p}$ (IV).

RICHARD D. CARMICHAEL (*)

1. – Introduction.

In this paper we add information to [3, section IV] where we have obtained results concerning the Cauchy and Poisson integrals of distributions in $\mathcal{D}'_{L^p}$ corresponding to generalized half planes. Here we show that many of the results of [3, section IV] hold for further values of p than previously obtained and also prove additional results.

The n-dimensional notation to be used in this paper will be exactly as described in [2, section II] and in [3, section II]. We note especially the following notation. Throughout this paper $\sigma = (\sigma_1, \ldots, \sigma_n)$, n being the dimension, is an n-tuple where $\sigma_j = \pm 1$, $j = 1, \ldots, n$. For each of the 2^n n-tuples σ we put $C_\sigma = \{y \in \mathbf{R}^n: \sigma_j y_j > 0, j = 1, \ldots, n\}$. For each of these 2^n octants C_σ we correspondingly define the 2^n generalized half planes in $\mathbf{C}^n$ as $B_\sigma = \mathbf{R}^n + iC_\sigma = \{z \in \mathbf{C}^n: \sigma_j \operatorname{Im}(z_j) > 0, j = 1, \ldots, n\}$. The reader should review the definitions and properties of the function spaces $\mathcal{S}$, $\mathcal{D}_{L^p}$, $\mathcal{B} \equiv \mathcal{D}_{L^\infty}$, and $\dot{\mathcal{B}}$ and the generalized function spaces $\mathcal{S}'$ and $\mathcal{D}'_{L^p}$ contained in Schwartz [7, pp. 199-205 and pp. 233-248]. All other needed definitions, such as that of Fourier transform, are contained in [3, section II].

2. – The Cauchy and Poisson kernel functions.

For each of the $2^n \sigma$ put

$$(2.1) \qquad K_\sigma(z - t) = (2\pi i)^{-n} \prod_{j=1}^{n} \frac{\operatorname{sgn}(y_j)}{t_j - z_j}, \quad z = x + iy \in B_\sigma, \qquad t \in \mathbf{R}^n,$$

(*) Author's address: Department of Mathematics, Wake Forest University, Winston-Salem, North Carolina, 27109 U.S.A.

where

$$\operatorname{sgn}(y_j) = \begin{cases} 1, & y_j > 0, \\ & \qquad j = 1, \ldots, n. \\ -1, & y_j < 0, \end{cases}$$

$R_\sigma(z-t)$ is the Cauchy kernel corresponding to the generalized half plane B_σ. It is implicit by the analysis of Tillmann [8] that $R_\sigma(z-t) \in \mathfrak{D}_{L^q}$, $(1/p) + (1/q) = 1$, $1 < p < \infty$, as a function of $t \in \mathbf{R}^n$, for arbitrary $z \in B_\sigma$. But $\mathfrak{D}_{L^q} \subset \dot{\mathfrak{B}} \subset \mathfrak{B} \equiv \mathfrak{D}_{L^\infty}$ for every q, $1 \leqslant q < \infty$ by [7, pp. 199-200]. We thus have proved the following fact.

Lemma 2.1. *For each n-tuple σ, let $z \in B_\sigma$. As a function of $t \in \mathbf{R}^n$,*

$$(2.2) \qquad R_\sigma(z-t) \in \dot{\mathfrak{B}} \cap \mathfrak{D}_{L^q} \quad \text{for all } q, \qquad \frac{1}{p} + \frac{1}{q} = 1, \qquad 1 \leqslant p < \infty.$$

We note two false statements in [3, p. 259, lines 5-7]. As we have shown above $R_\sigma(z-t)$ is an element of $\dot{\mathfrak{B}}$ contrary to the false assertion in [3, p. 259, lines 5-6]. Further, as we shall see in section 3 of this paper, the Cauchy integral $C(U; z \in B_\sigma)$ is well defined for $U \in \mathfrak{D}'_{L^1}$ and [3, Theorem 3] does hold for $p = 1$.

Now put

$$(2.3) \qquad \begin{aligned} K_\sigma(t; z) &= (4\pi)^n \left(\prod_{j=1}^{n} (\operatorname{sgn}(y_j)) y_j \right) R_\sigma(z-t) \overline{R_\sigma(z-t)} \\ &= (\pi)^{-n} \prod_{j=1}^{n} \frac{(\operatorname{sgn}(y_j)) y_j}{(t_j - x_j)^2 + y_j^2} \end{aligned}$$

for each σ where $z = x + iy \in B_\sigma$ and $t \in \mathbf{R}^n$. $K_\sigma(t; z)$ is the Poisson kernel corresponding to B_σ. Let α be any n-tuple of nonnegative integers and let $z \in B_\sigma$ be arbitrary but fixed. By the generalized Leibnitz rule we have

$$(2.4) \qquad D_t^\alpha(K_\sigma(t; z)) = \\ = (4\pi)^n \left(\prod_{j=1}^{n} (\operatorname{sgn}(y_j)) y_j \right) \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} D_t^\beta(R_\sigma(z-t)) D_t^\gamma(\overline{R_\sigma(z-t)}),$$

where the differential operator D_t^α is defined in [2, p. 37]. From (2.2), $D_t^\beta(R_\sigma(z-t)) \in L^2 \cap L^\infty$ and similarly $D_t^\gamma(\overline{R_\sigma(z-t)}) \in L^2 \cap L^\infty$ as functions of $t \in \mathbf{R}^n$. Thus by (2.4), $D_t^\alpha(K_\sigma(t; z)) \in L^1 \cap L^\infty$. But $L^1 \cap L^\infty \subseteq L^p$, $1 \leqslant p \leqslant \infty$. We conclude that $K_\sigma(t; z) \in \mathcal{D}_{L^q}$ for all q, $1 \leqslant q \leqslant \infty$; and $K_\sigma(t; z) \in \dot{\mathcal{B}}$ also since $\mathcal{D}_{L^q} \subset \dot{\mathcal{B}}$ for every q, $1 \leqslant q < \infty$, [7, pp. 199-200]. This proves the following result.

LEMMA 2.2. *For each n-tuple σ, let $z \in B_\sigma$. As a function of $t \in \mathbf{R}^n$,*

$$K_\sigma(t; z) \in \dot{\mathcal{B}} \cap \mathcal{D}_{L^q} \text{ for all } q\,, \quad 1 \leqslant q \leqslant \infty\,. \tag{2.5}$$

3. – The Cauchy integral.

$\mathcal{D}'_{L^p}$, $1 < p \leqslant \infty$, is the dual space (space of continuous linear functionals) of $\mathcal{D}_{L^q}$, $(1/p) + (1/q) = 1$; while $\mathcal{D}'_{L^1}$ is the dual space of $\dot{\mathcal{B}}$ [7, p. 200]. Thus let $U \in \mathcal{D}'_{L^p}$ for any p, $1 \leqslant p < \infty$. For each n-tuple σ put

$$C(U; z \in B_\sigma) = \langle U_t, R_\sigma(z - t)\rangle\,, \quad z \in B_\sigma\,, \tag{3.1}$$

which is the Cauchy integral of U corresponding to B_σ. According to Lemma 2.1, $C(U; z \in B_\sigma)$ is a well defined function of $z \in B_\sigma$.

THEOREM 3.1. *Let $U \in \mathcal{D}'_{L^p}$, $1 \leqslant p < \infty$. For each σ, $C(U; z \in B_\sigma)$ is an analytic function of $z \in B_\sigma$ such that*

$$|C(U; z \in B_\sigma)| \leqslant M \prod_{j=1}^{n} \left(|y_j|^{-(1/p)} + |y_j|^{-(1/p) - m_j}\right)\,, \quad z = x + iy \in B_\sigma\,, \tag{3.2}$$

where M is a positive constant, which is independent of $z \in B_\sigma$, and each m_j, $j = 1, \ldots, n$, is a nonnegative integer.

PROOF. For $1 < p < \infty$ the desired results have been proved by Tillmann [8]. We now prove these facts for $p = 1$. By Schwartz [7, p. 201], $U \in \mathcal{D}'_{L^1}$ implies

$$U = \sum_{|\alpha| \leqslant k} D_t^\alpha(f_\alpha(t))\,, \quad f_\alpha \in L^1\,, \tag{3.3}$$

where k is some nonnegative integer and the α are n-tuples of nonnegative integers. Recall our definition of the differential operator D_t^α

given in [2, p. 37]. Using (2.1) and (3.3) we have

$$(3.4)\qquad C(U; z\in B_\sigma) = \sum_{|\alpha|\leqslant k}(-1)^{|\alpha|}\langle f_\alpha(t), D_t^\alpha(K_\sigma(z-t))\rangle =$$

$$= \sum_{|\alpha|\leqslant k}(-1)^{|\alpha|}(2\pi i)^{-n}\prod_{j=1}^{n}(\operatorname{sgn}(y_j))\left\langle f_\alpha(t), D_t^\alpha\left(\prod_{j=1}^{n}\frac{1}{t_j-z_j}\right)\right\rangle =$$

$$= \sum_{|\alpha|\leqslant k}(-1)^{|\alpha|}(2\pi i)^{-n}\prod_{j=1}^{n}(\operatorname{sgn}(y_j))\cdot$$

$$\cdot\left\langle f_\alpha(t), (2\pi i)^{-|\alpha|}\prod_{j=1}^{n}(-1)^{\alpha_j}(\alpha_j)!(t_j-z_j)^{-\alpha_j-1}\right\rangle =$$

$$= \sum_{|\alpha|\leqslant k}(-1)^{|\alpha|}(2\pi i)^{-n-|\alpha|}\cdot$$

$$\cdot\left(\prod_{j=1}^{n}(-1)^{\alpha_j}(\operatorname{sgn}(y_j))(\alpha_j)!\right)\int_{\mathbf{R}^n} f_\alpha(t)\prod_{j=1}^{n}(t_j-z_j)^{-\alpha_j-1}\,dt\,.$$

For each α in (3.4) put

$$(3.5)\qquad F_\alpha(z) = \int_{\mathbf{R}^n} f_\alpha(t)\prod_{j=1}^{n}(t_j-z_j)^{-\alpha_j-1}\,dt\,,\qquad z\in B_\sigma\,.$$

Let S be an arbitrary compact subset of B_σ and let z vary over S for the moment; there exist numbers $\gamma_j > 0$, $j = 1, \dots, n$, depending only on S such that $|y_j| \geqslant \gamma_j > 0$ for all $y = (y_1, \dots, y_n)$ for which $z = x + iy \in S$. Thus for all $z = x + iy \in S$ and all $t\in\mathbf{R}^n$ we have

$$\left|f_\alpha(t)\prod_{j=1}^{n}(t_j-z_j)^{-\alpha_j-1}\right| = |f_\alpha(t)|\prod_{j=1}^{n}\left((t_j-x_j)^2+y_j^2\right)^{-(\alpha_j+1)/2}$$

$$(3.6)\qquad \leqslant |f_\alpha(t)|\prod_{j=1}^{n}|y_j|^{-\alpha_j-1}$$

$$\leqslant |f_\alpha(t)|\prod_{j=1}^{n}(\gamma_j)^{-\alpha_j-1}\,.$$

Recalling that each $f_\alpha(t)\in L^1$, we see that the right side of (3.6) is an L^1 function of $t\in\mathbf{R}^n$ that is independent of $z = x + iy \in S$. Thus by [1, p. 295, Theorem B.4], each $F_\alpha(z)$ defined in (3.5) is analytic in B_σ; hence so is $C(U; z\in B_\sigma)$ because of (3.4). By analysis as in (3.6) we have for $z \in B_\sigma$ that

$$(3.7)\qquad \begin{aligned} |F_\alpha(z)| &\leq \int_{\mathbf{R}^n} |f_\alpha(t)| \left| \prod_{j=1}^{n} (t_j - z_j)^{-\alpha_j - 1} \right| dt \\ &\leq \prod_{j=1}^{n} |y_j|^{-\alpha_j - 1} \int_{\mathbf{R}^n} |f_\alpha(t)|\, dt\,. \end{aligned}$$

The growth (3.2) for $p = 1$ follows easily now by combining (3.4) and (3.7) where $F_\alpha(z)$ is defined in (3.5) for each α, $|\alpha| \leq k$. The proof is complete.

Of course $R_\sigma(z - t)$ does not belong to $\mathfrak{D}_{L^1}$ as a function of $t \in \mathbf{R}^n$ for z arbitrary in B_σ. Thus we can not let $p = \infty$ in Theorem 3.1 because $C(U; z \in B_\sigma)$ does not exist for $U \in \mathfrak{D}'_{L^\infty}$. Theorem 3.1 extends the corresponding information of Tillmann [8] to the case $p = 1$.

Now consider any of the 2^n n-tuples σ and the corresponding generalized half plane B_σ. Let $U \in \mathfrak{D}'_{L^p}$, $1 \leq p \leq 2$, such that $U = \hat{V}$, where $V \in \mathcal{S}'$ and $\operatorname{supp}(V) \subseteq S^0_\sigma = \{t\colon -\infty < \sigma_j t_j \leq 0, j = 1, \dots, n\}$. Let $H_\sigma(t)$ denote the characteristic function of S^0_σ and define the C^∞ function $\alpha(t)$ as in [3, p. 258] corresponding to S^0_σ. Notice that

$$(3.8)\qquad \mathcal{F}[H_\sigma(t)\alpha(t) \exp[2\pi\langle y, t\rangle]; x] = (-2\pi i)^{-n} \prod_{j=1}^{n} \frac{\operatorname{sgn}(y_j)}{z_j}\,, \qquad z = x + iy \in B_\sigma\,,$$

as in [3, p. 258, lines 19-20], where the Fourier transform in (3.8) is the L^1 transform and hence also the $\mathcal{S}'$ Fourier transform. Thus because of (3.8),

$$(3.9)\qquad \mathcal{F}[H_\sigma(t)\alpha(t) \exp[2\pi\langle y, t\rangle]; x] \in \mathfrak{D}_{L^2} \subset \mathfrak{D}'_{L^2}$$

as a function of $x \in \mathbf{R}^n$ for arbitrary $y \in C_\sigma$, and (3.8) implies

$$(3.10)\qquad H_\sigma(t)\alpha(t) \exp[2\pi\langle y, t\rangle] = \mathcal{F}^{-1}\left[(-2\pi i)^{-n} \prod_{j=1}^{n} \frac{\operatorname{sgn}(y_j)}{z_j}\right], \qquad z = x + iy \in B_\sigma\,,$$

with this inverse Fourier transform being in $\mathcal{S}'$ [7, p. 250]. Using the fact (3.9) and the proof of [7, p. 270, lines 3-17] we now have

$$(3.11)\qquad \mathcal{F}^{-1}\left[U*\left((-2\pi i)^{-n}\prod_{j=1}^{n}\frac{\operatorname{sgn}(y_j)}{z_j}\right)\right] = \\ = \mathcal{F}^{-1}[U]\,\mathcal{F}^{-1}\left[(-2\pi i)^{-n}\prod_{j=1}^{n}\frac{\operatorname{sgn}(y_j)}{z_j}\right]$$

with this equality holding in $\mathcal{S}'$ and the convolution on the left being the distributional convolution [7, Chapter 6]. But $U=\hat{V}$ in $\mathcal{S}'$ implies $V=\mathcal{F}^{-1}[U]$ in $\mathcal{S}'$. Combining this with (3.10) and (3.11) we get

$$\mathcal{F}^{-1}\left[U*\left((-2\pi i)^{-n}\prod_{j=1}^{n}\frac{\operatorname{sgn}(y_j)}{z_j}\right)\right] = H_\sigma(t)\alpha(t)\exp[2\pi\langle y, t\rangle]\,V$$

in $\mathcal{S}'$ and hence

$$(3.12)\qquad \mathcal{F}[H_\sigma(t)\alpha(t)\exp[2\pi\langle y, t\rangle]V] = U*\left((-2\pi i)^{-n}\prod_{j=1}^{n}\frac{\operatorname{sgn}(y_j)}{z_j}\right), \\ z = x+iy\in B_\sigma,$$

in $\mathcal{S}'$. Our method of obtaining (3.12) gives an alternate method of obtaining the equality [3, p. 258, (12)], and note that we have this equality now under the assumption $U\in\mathcal{D}'_{L^p}$, $1\leqslant p\leqslant 2$. (Recall that [3, Theorem 3] did not include the case $p=1$.) With the equality (3.12) now obtained under the specified assumptions for $1\leqslant p\leqslant 2$ and with Theorem 3.1 above, we now state that [3, Theorem 3] holds for $p=1$ also in which case $q=\infty$ there. The techniques to prove the stated conclusions are the same for $p=1$ as for $1<p\leqslant 2$ with the exception that we now use our proof of (3.12) above to obtain [3, p. 258, (12)]. In addition we can now state a growth condition on $C(U; z\in B_\sigma)$ because of Theorem 3.1. For completeness we now state our extension of [3, Theorem 3].

THEOREM 3.2. *Let B_σ be any of the 2^n generalized half planes in $\mathbf{C}^n$. Let $U\in\mathcal{D}'_{L^p}$, $1\leqslant p\leqslant 2$, such that $U=\hat{V}$, where $V\in\mathcal{S}'$ and* $\operatorname{supp}(V)\subseteq$ $\subseteq S^0_\sigma=\{t: -\infty<\sigma_j t_j\leqslant 0, j=1,\dots,n\}$. *Then* $V=\sum_{|\beta|\leqslant m}(-1)^{|\beta|}t^\beta h_\beta(t)$, $h_\beta(t)\in$ $\in L^q$, $(1/p)+(1/q)=1$; *$C(U; z\in B_\sigma)$ is analytic in B_σ and satisfies* (3.2);

$$(3.13)\qquad C(U; z\in B_\sigma) = \langle V, \exp[-2\pi i\langle z, t\rangle]\rangle, \qquad z\in B_\sigma,$$

as elements of $\mathcal{S}'$; *and* $C(U; z \in B_\sigma) \to U \in \mathfrak{D}'_{L^p}$ *in the strong* (*and weak*) *topology of* $\mathcal{S}'$ *as* $\operatorname{Im}(z) \to 0$.

The above convergence of $C(U; z \in B_\sigma) \to U \in \mathfrak{D}'_{L^p}$ in the strong topology of $\mathcal{S}'$ as $\operatorname{Im}(z) \to 0$ is proved as follows. After obtaining (3.13) we use the same proof as in [3, Theorem 3] to show that

$$(3.14) \qquad C(U; z \in B_\sigma) = \langle V, \exp[-2\pi i \langle z, t\rangle]\rangle \to \hat{V} = U$$

in the weak topology of $\mathcal{S}'$ as $\operatorname{Im}(z) \to 0$. But $\mathcal{S}$ is a Montel space [7, p. 235]; hence by [4, p. 510, Corollary 8.4.9] the convergence in (3.14) is in the strong topology of $\mathcal{S}'$.

As a result of Theorem 3.2, the extension of [3, Theorem 3], and its proof, the results [3, Corollary 1 and Theorem 5] hold also for $1 \leqslant p \leqslant 2$ by using the same proofs as before. Note that [3, Theorem 6] has already been obtained for $1 \leqslant p \leqslant 2$.

4. – The Poisson integral.

Let $U \in \mathfrak{D}'_{L^p}$ for any p, $1 \leqslant p \leqslant \infty$. For each σ put

$$(4.1) \qquad P(U; z \in B_\sigma) = \langle U_t, K_\sigma(t; z)\rangle, \quad z \in B_\sigma,$$

which is the Poisson integral of U. By Lemma 2.2, $P(U; z \in B_\sigma)$ is a well defined function of $z \in B_\sigma$. Note that the Poisson integral $P(U; z \in B_\sigma)$ is well defined for $U \in \mathfrak{D}'_{L^\infty}$ while $C(U; z \in B_\sigma)$ is not defined for $U \in \mathfrak{D}'_{L^\infty}$; this is because $K_\sigma(t; z) \in \mathfrak{D}_{L^1}$ while $R_\sigma(z - t)$ does not. In general $P(U; z \in B_\sigma)$ is not an analytic function which is in contrast to the Cauchy integral. However, the result [3, Theorem 7] holds for $1 \leqslant p \leqslant \infty$ by the same proof as before for the case $1 < p < \infty$; hence we have extended this result to include the case $p = 1$ and $p = \infty$ now; and $P(U; z \in B_\sigma)$ is an n-harmonic function for $U \in \mathfrak{D}'_{L^p}$, $1 \leqslant p \leqslant \infty$. We note a misprint in the proof of [3, Theorem 7]; [3, p. 262, line 19] should read

$$\prod_{j=1}^{n} \frac{y_j}{\pi |t_j - z_j|^2} = \prod_{j=1}^{n} \left(\frac{1}{2\pi i}\right)\left(\frac{1}{t_j - z_j} - \frac{1}{t_j - \bar{z}_j}\right).$$

[3, Theorems 8 and 10] related the Poisson integral with the Cauchy integral and the Fourier-Laplace transform. Because of the preceding information in this paper, these two theorems can now be seen to hold for $U \in \mathcal{D}'_{L^p}$, $1 \leq p \leq 2$, by the same proofs as given in [3, Thorems 8 and 10] since we now know that the analysis on which these proofs are based holds for $U \in \mathcal{D}'_{L^p}$, $1 \leq p \leq 2$.

We now extend [3, Theorem 9] by obtaining this result for $U \in \mathcal{D}'_{L^p}$ for any p, $1 \leq p \leq \infty$, and give a separate proof. Further, our extension is slightly more general than [3, Theorem 9]. Our result is as follows.

THEOREM 4.1. *Let $U \in \mathcal{D}'_{L^p}$, $1 \leq p \leq \infty$. For any of the 2^n n-tuples σ we have*

$$\lim_{\substack{y \to 0 \\ y \in C_\sigma}} \langle P(U; (x+iy) \in B_\sigma), \varphi(x) \rangle = \langle U, \varphi \rangle \tag{4.2}$$

for every $\varphi \in \mathcal{D}_{L^1}$.

Theorem 4.1 is more general than [3, Theorem 9] since $\mathcal{S} \subset \mathcal{D}_{L^1}$. Our present proof of Theorem 4.1 relies on the following two lemmas.

LEMMA 4.1. *Let $U \in \mathcal{D}'_{L^p}$, $1 \leq p \leq \infty$. For any of the 2^n n-tuples σ we have*

$$\begin{aligned} \langle P(U; (x+iy) \in B_\sigma), \varphi(x) \rangle &= \\ &= \langle U, \langle K_\sigma(t; x+iy), \varphi(x) \rangle \rangle , \qquad y \in C_\sigma , \end{aligned} \tag{4.3}$$

for every $\varphi \in \mathcal{D}_{L^1}$.

PROOF. Let $\varphi \in \mathcal{D}_{L^1}$. A change of variable yields

$$\int_{\mathbf{R}^n} K_\sigma(t; x+iy) \varphi(x)\, dx = \int_{\mathbf{R}^n} K_\sigma(x; y) \varphi(x+t)\, dx \tag{4.4}$$

for all $y \in C_\sigma$ and $t \in \mathbf{R}^n$ where

$$K_\sigma(x; y) = (\pi)^{-n} \prod_{j=1}^{n} \frac{(\operatorname{sgn}(y_j)) y_j}{x_j^2 + y_j^2}, \quad x \in \mathbf{R}^n , \quad y \in C_\sigma . \tag{4.5}$$

By the proof of Lemma 2.2, $K_\sigma(x; y) \in \dot{\mathcal{B}} \cap \mathcal{D}_{L^q}$ for all q, $1 \leq q \leq \infty$, as a function of $x \in \mathbf{R}^n$ for $y \in C_\sigma$ arbitrary. Thus by [7, p. 201, Théorème XXV] we have $K_\sigma(x; y) \in \mathcal{D}'_{L^q}$ for every q, $1 \leq q \leq \infty$, as a function of $x \in \mathbf{R}^n$ for $y \in C_\sigma$ arbitrary. Hence for $U \in \mathcal{D}'_{L^p}$, $1 \leqslant p \leqslant \infty$, we have that the distributional convolution

$$U * K_\sigma(x; y) \in \mathcal{D}'_{L^\infty}, \quad y \in C_\sigma, \tag{4.6}$$

by [7, p. 203, Théorème XXVI]. Thus for any $\varphi \in \mathcal{D}_{L^1}$, $\langle U * K_\sigma(x; y), \varphi\rangle$ exists because of (4.6); and

$$\langle U * K_\sigma(x; y), \varphi\rangle = \langle U, \langle K_\sigma(x; y), \varphi(x+t)\rangle\rangle, \quad y \in C_\sigma, \tag{4.7}$$

by the definition of distributional convolution ([7, Chapter 6] or [3, p. 251].) Combining (4.4) and (4.7) we obtain for $y \in C_\sigma$ that

$$\langle U, \langle K_\sigma(t; x+iy), \varphi(x)\rangle\rangle = \langle U, \langle K_\sigma(x; y), \varphi(x+t)\rangle\rangle = \\ = \langle U * K_\sigma(x; y), \varphi\rangle$$

which proves that the right side of (4.3) is well defined for any $y \in C_\sigma$. For $U \in \mathcal{D}'_{L^p}$, $1 \leqslant p \leqslant \infty$, we have by the characterization theorem of Schwartz [7, p. 201, Théorème XXV] that

$$U = \sum_{|\alpha| \leqslant m} D_t^\alpha(f_\alpha(t)), \quad f_\alpha \in L^p, \tag{4.8}$$

for some nonnegative integer m. Using (4.8), a change of order of integration, which is valid here, and the fact that differentiation can be taken under the integral sign as needed below, we obtain for any $y \in C_\sigma$ that

$$\begin{aligned}\langle U, \langle K_\sigma(t; x+iy), \varphi(x)\rangle\rangle &= \\ &= \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} \int_{\mathbf{R}^n} f_\alpha(t) \int_{\mathbf{R}^n} D_t^\alpha(K_\sigma(t; x+iy))\varphi(x)\,dx\,dt \\ &= \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} \int_{\mathbf{R}^n} \varphi(x) \int_{\mathbf{R}^n} f_\alpha(t) D_t^\alpha(K_\sigma(t; x+iy))\,dt\,dx \\ &= \left\langle \left\langle \sum_{|\alpha| \leqslant m} D_t^\alpha(f_\alpha(t)), K_\sigma(t; x+iy) \right\rangle, \varphi(x) \right\rangle \\ &= \langle P(U; (x+iy) \in B_\sigma), \varphi(x)\rangle\end{aligned}$$

which proves (4.3). The proof is complete.

LEMMA 4.2. *For any of the 2^n n-tuples σ, let $z = x + iy \in B_\sigma$. Let $\varphi \in \mathfrak{D}_{L^1}$. We have*

$$\lim_{\substack{y\to 0\\ y\in C_\sigma}} \int_{\mathbf{R}^n} K_\sigma(t; x + iy)\varphi(x)\,dx = \varphi(t) \tag{4.9}$$

in the topology of $\mathfrak{D}_{L^q}$ for all q, $1 \leqslant q \leqslant \infty$, and in the topology of $\dot{\mathfrak{B}}$.

PROOF. For $\varphi \in \mathfrak{D}_{L^1}$ and any n-tuple α of nonnegative integers, we have using (4.4) that

$$D_t^\alpha(\langle K_\sigma(t; x + iy), \varphi(x)\rangle = \int_{\mathbf{R}^n} D_t^\alpha(\varphi(x + t))\,K_\sigma(x; y)\,dx\,, \qquad y \in C_\sigma\,, \tag{4.10}$$

where $K_\sigma(x; y)$ is defined in (4.5) and the differentiation under the integral sign is valid. Now $\varphi \in \mathfrak{D}_{L^1}$ implies $\Psi_\alpha(t) = D_t^\alpha(\varphi(t)) \in \mathfrak{D}_{L^1}$. By [7, p. 200], $\mathfrak{D}_{L^1} \subseteq \mathfrak{D}_{L^q} \subset L^q$ for all q, $1 \leqslant q \leqslant \infty$, and $\mathfrak{D}_{L^1} \subset \dot{\mathfrak{B}} \subset \mathfrak{D}_{L^\infty}$. Now $K_\sigma(t; z)$ defined in (2.3) is the Poisson kernel function for the tube B_σ in $\mathbf{C}^n$ corresponding to the cone C_σ in $\mathbf{R}^n$; hence $K_\sigma(t; z)$ is an approximate identity [6, Proposition 2]. ($K_\sigma(x; y)$ is also an approximate identity.) Using (4.10) and [6, Proposition 2] we have

$$D_t^\alpha\Big(\int_{\mathbf{R}^n} K_\sigma(t; x + iy)\varphi(x)\,dx\Big) - D_t^\alpha(\varphi(t)) = \int_{\mathbf{R}^n} (\Psi_\alpha(x + t) - \Psi_\alpha(t))\,K_\sigma(x; y)\,dx \tag{4.11}$$

where $\Psi_\alpha(t) = D_t^\alpha(\varphi(t)) \in \mathfrak{D}_{L^1} \subseteq \mathfrak{D}_{L^q} \subset L^q$ for all q, $1 \leqslant q \leqslant \infty$, as noted above. Now using (4.11), [6, Proposition 2], and the same method of proof used in [5, Theorem on pp. 17-19, Theorem on p. 32] we have

$$\lim_{\substack{y\to 0\\ y\in C_\sigma}} \Big\| D_t^\alpha\Big(\int_{\mathbf{R}^n} K_\sigma(t; x + iy)\varphi(x)\,dx\Big) - D_t^\alpha(\varphi(t))\Big\|_{L^q} = \lim_{\substack{y\to 0\\ y\in C_\sigma}} \Big\| \int_{\mathbf{R}^n} (\Psi_\alpha(x + t) - \Psi_\alpha(t))\,K_\sigma(x; y)\,dx\Big\|_{L^q} = 0 \tag{4.12}$$

for any q, $1 \leqslant q < \infty$, any n-tuple α of nonnegative integers, and any $\varphi \in \mathfrak{D}_{L^1}$. (4.12) thus proves (4.9) in the topology of $\mathfrak{D}_{L^q}$ for all q,

$1 \leqslant q < \infty$. Further, $\varphi \in \mathfrak{D}_{L^1} \subset \dot{\mathfrak{B}} \subset \mathfrak{D}_{L^\infty}$ implies $\Psi_\alpha(t) = D_t^\alpha(\varphi(t)) \in \mathfrak{D}_{L^1} \subset \subset \dot{\mathfrak{B}} \subset \mathfrak{D}_{L^\infty}$ for any n-tuple α of non-negative integers; and by the definition of $\dot{\mathfrak{B}}$, $\Psi_\alpha(t) \to 0$ as $|t| \to \infty$ with $\Psi_\alpha(t)$ being continuous and bounded on $\mathbf{R}^n$. This implies that $\Psi_\alpha(t) = D_t^\alpha(\varphi(t))$ is uniformly continuous and bounded for $t \in \mathbf{R}^n$. Thus by the proof of [6, Proposition 3, (b)] we have

$$\lim_{\substack{y \to 0 \\ y \in C_\sigma}} \int_{\mathbf{R}^n} \Psi_\alpha(x + t) K_\sigma(x; y)\, dx = \Psi_\alpha(t)$$

uniformly for $t \in \mathbf{R}^n$. From this and (4.10) it follows that

$$\lim_{\substack{y \to 0 \\ y \in C_\sigma}} \left\| D_t^\alpha\left(\int_{\mathbf{R}^n} K_\sigma(t; x + iy)\varphi(x)\, dx\right) - D_t^\alpha(\varphi(t)) \right\|_{L^\infty} = 0$$

which proves (4.9) in the topology of $\dot{\mathfrak{B}}$ and in the topology of $\mathfrak{B} \equiv \mathfrak{D}_{L^\infty}$. The proof is complete.

We now give the

Proof of Theorem 4.1. For any $\varphi \in \mathfrak{D}_{L^1}$ the proof of Lemma 4.1 yields that $\langle U, \langle K_\sigma(t; x + iy), \varphi(x)\rangle\rangle$ exists for $y \in C_\sigma$. The continuity of $U \in \mathfrak{D}'_{L^p}$, $1 \leqslant p \leqslant \infty$, and Lemma 4.2 combine to prove

$$\lim_{\substack{y \to 0 \\ y \in C_\sigma}} \langle U, \langle K_\sigma(t; x + iy), \varphi(x)\rangle\rangle = \langle U, \varphi\rangle \tag{4.13}$$

and $\langle U, \varphi\rangle$ is well defined for $\varphi \in \mathfrak{D}_{L^1}$ since $\mathfrak{D}_{L^1} \subseteq \mathfrak{D}_{L^q}$ for all q, $1 \leqslant q \leqslant \infty$, and $\mathfrak{D}_{L^1} \subset \dot{\mathfrak{B}}$. The desired result (4.2) follows now by combining (4.13) and (4.3). The proof of Theorem 4.1 is complete.

If $p = \infty$ in Theorem 4.1, then (4.2) proves that $P(U; (x + iy) \in \in B_\sigma) \to U$ in exactly the weak topology of $\mathfrak{D}'_{L^\infty}$ as $y \to 0$, $y \in C_\sigma$, since (4.2) holds for each $\varphi \in \mathfrak{D}_{L^1}$ whose dual space is $\mathfrak{D}'_{L^\infty}$.

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