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Acceleration waves in thermo-viscous fluids

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Acceleration Waves in Thermo-Viscous Fluids.

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SUMMARY - Acceleration waves in heat-conducting viscous fluids are investigated through a model of fluid with hidden variables. It turns out that both longitudinal and transverse waves may occur.

1. – Introduction.

The theory of viscosity based on Navier-Stokes' law does not account for wave propagation. Differently from the analogous paradox whereby temperature waves are ruled out by Fourier's law, aside from refs. [1, 2], such a problem has been given little attention.

Really, it can be shown that at least two approaches account for wave propagation in viscous fluids. The first one employs fading memory functionals [3] whereas the second one describes the material properties via hidden variables [4, 5]. This paper aims to deliver a theory of wave propagation through heat-conducting viscous (thermo-viscous) fluids appealing to the second approach. Yet a strict application of the rule of equipresence leads to a theory ruling out the possibility of wave propagation. Accordingly, it is assumed that not all external variables, which the evolution function depends on, influence directly the response of the material. Precisely, the evolution function depends on the temperature rate, the temperature gradient, and the stretching tensor while the response function does not. In spite

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of being not common in the literature, the disregard for the rule of equipresence is not at all new since it is exhibited in the papers [6, 7] though in connection with the temperature gradient only.

The general properties of materials with hidden variables are set up in sec. 2 while an analysis of acceleration waves in temperature rate dependent thermo-viscous fluids is outlined in sec. 3. On assuming the Clausius-Duhem inequality as statement of the second law of thermodynamics, sec. 4 deals with a thermodynamic theory of a particular model of thermo-viscous fluid. Such a model appears to be the natural generalisation of the customary model of thermo-viscous fluid in that asymptotically it gives Fourier's law and Navier-Stokes' law but, meanwhile, it accounts for the existence of acceleration waves.

It is a quite remarkable result that the model proposed in this note allows the existence of transverse waves thus providing a property especially suited for testing experimentally the validity of the model itself.

2. – Materials with hidden variables.

Henceforth $\mathbf{R}$, $\mathbf{R}^+$, $\mathbf{R}^{++}$ stand for the real numbers, the positive real numbers, and the strictly positive real numbers, respectively. A dot between two vectors or tensors means inner product. The symbols $\boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{A}, \boldsymbol{\Sigma}$ denote finite-dimensional real normed vector-spaces while $\boldsymbol{L}(\boldsymbol{Y}, \boldsymbol{A})$ designates the normed vector space of all linear maps from $\boldsymbol{Y}$ into $\boldsymbol{A}$. The evolution of a body $\mathscr{B}$ may be described by supposing $\mathscr{B}$ constituted by particles labelled by the positions they occupy in a reference configuration $\mathscr{R}$. The vector $\boldsymbol{x}(\boldsymbol{X}, t)$ denotes the position of the particle $\boldsymbol{X}$ at time t. To save writing, the dependence on $\boldsymbol{X} \in \mathscr{R}$ is understood and not written. A superposed dot denotes material time differentiation. The symbols grad and div stand for the spatial gradient and divergence operators.

The present approach to materials with hidden variables hinges partially on Day's [5]. The main differences are the following ones. First, not all external variables affect the response function. Second, the hypothesis about the evolution function is used in a more restrictive form thus simplifying the next applications. So, the general properties of materials with hidden variables may be assembled as follows.

A material with hidden variables $\{\boldsymbol{y}_0, \boldsymbol{z}_0, \boldsymbol{\alpha}_0, \boldsymbol{U}, \boldsymbol{V}, \boldsymbol{\sigma}, \boldsymbol{h}\}$ on $\boldsymbol{Y} \times \boldsymbol{Z} \times \boldsymbol{A}$ consists of a ground state $(\boldsymbol{y}_0, \boldsymbol{z}_0, \boldsymbol{\alpha}_0) \in \boldsymbol{Y} \times \boldsymbol{Z} \times \boldsymbol{A}$ together with an open connected neighbourhood $\boldsymbol{U}\times\boldsymbol{V}$ of $(\boldsymbol{y}_0, \boldsymbol{z}_0)$ and the maps

$$\boldsymbol{\sigma}\in C^2(\boldsymbol{U}\times\boldsymbol{A}, \boldsymbol{\Sigma})\,, \qquad \boldsymbol{h}\in C^2(\boldsymbol{U}\times\boldsymbol{V}\times\boldsymbol{A}, \boldsymbol{A}) \tag{2.1}$$

while $\dim \boldsymbol{A}\leqslant\dim \boldsymbol{Y}+\dim \boldsymbol{Z}$. A *state* of the material is a triple $(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{\alpha})\in \boldsymbol{Y}\times\boldsymbol{Z}\times\boldsymbol{A}$, $\boldsymbol{\alpha}$ being the vector value of the hidden variables. The response of the material and the growth of the hidden variables are given by

$$\begin{cases} \boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{y}, \boldsymbol{\alpha})\,, & (\boldsymbol{y}, \boldsymbol{\alpha})\in \boldsymbol{U}\times\boldsymbol{A}\,, \\ \dot{\boldsymbol{\alpha}} = \boldsymbol{h}(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{\alpha})\,, & (\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{\alpha})\in \boldsymbol{U}\times\boldsymbol{V}\times\boldsymbol{A}\,. \end{cases} \tag{2.2}$$

To clarify the notion of equilibrium state it is convenient to introduce a map $\boldsymbol{E}\colon \boldsymbol{Y}\times\boldsymbol{Z}\to\boldsymbol{A}$ subject to the following restriction. Corresponding to each pair $(\boldsymbol{y}, \boldsymbol{z})\in\boldsymbol{U}\times\boldsymbol{V}$ there is just one hidden variable $\boldsymbol{E}(\boldsymbol{y}, \boldsymbol{z})\in\boldsymbol{A}$ such that

$$\boldsymbol{h}(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{E}(\boldsymbol{y}, \boldsymbol{z})) = \boldsymbol{0}$$

while

$$\boldsymbol{E}(\boldsymbol{y}_0, \boldsymbol{z}_0) = \boldsymbol{\alpha}_0\,.$$

The set of hidden variables

$$\boldsymbol{B} = \{\boldsymbol{E}(\boldsymbol{y}, \boldsymbol{z})\colon (\boldsymbol{y}, \boldsymbol{z})\in\boldsymbol{U}\times\boldsymbol{V}\}\,, \qquad \boldsymbol{E}\in C^2(\boldsymbol{U}\times\boldsymbol{V}, \boldsymbol{B})\,,$$

is open in $\boldsymbol{A}$, and there is a subset $\mathscr{W}\subset\boldsymbol{U}\times\boldsymbol{V}$ such that $(\boldsymbol{y}_0, \boldsymbol{z}_0)\in\mathscr{W}$ and the restriction $\hat{\boldsymbol{E}} = \boldsymbol{E}|_{\mathscr{W}}$ of $\boldsymbol{E}$ to $\mathscr{W}$ is a bijection from $\mathscr{W}$ onto $\boldsymbol{B}$ whose inverse $\hat{\boldsymbol{E}}^{-1}\in C^2(\boldsymbol{B}, \mathscr{W})$.

If $(\boldsymbol{y}, \boldsymbol{z})\in\boldsymbol{U}\times\boldsymbol{V}$, the triple $(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{E}(\boldsymbol{y}, \boldsymbol{z}))$ is an *equilibrium state.* The equilibrium response $\boldsymbol{\sigma}^*\in C^2(\boldsymbol{U}\times\boldsymbol{B}, \boldsymbol{\Sigma})$ has the form

$$\boldsymbol{\sigma}^*(\boldsymbol{y}, \boldsymbol{\alpha}) = \boldsymbol{\sigma}(\boldsymbol{y}, \boldsymbol{E}(\boldsymbol{y}, \boldsymbol{z})),\ (\boldsymbol{y}, \boldsymbol{z})\in\boldsymbol{U}\times\boldsymbol{V}\,.$$

It is assumed that there exist a map $\boldsymbol{\Lambda}\in L(\boldsymbol{A}, \boldsymbol{A})$ and a positive constat δ such that

$$|\boldsymbol{h}(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{\alpha}+\boldsymbol{\beta}) - \boldsymbol{h}(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{\alpha}) - \boldsymbol{\Lambda}\boldsymbol{\beta}| \leqslant \delta|\boldsymbol{\beta}|\,, \qquad (\boldsymbol{y}, \boldsymbol{z})\in\boldsymbol{U}\times\boldsymbol{V}\,, \quad \boldsymbol{\alpha}, \boldsymbol{\alpha}+\boldsymbol{\beta}\in\boldsymbol{A}\,. \tag{2.3}$$

So we have the uniform Lipschitz condition $\boldsymbol{h}(\boldsymbol{y}, \boldsymbol{z}, \cdot) \in \operatorname{Lip}(|\boldsymbol{\Lambda}| + \delta)$, $(\boldsymbol{y}, \boldsymbol{z}) \in \boldsymbol{U} \times \boldsymbol{V}$. Moreover the map $\boldsymbol{\Lambda} + \delta \boldsymbol{I}_A$ is supposed to be negative definite.

A *path* is a bounded and piecewise continuously differentiable map $\boldsymbol{\pi}$ from $\mathbf{R}$ into $\boldsymbol{U} \times \boldsymbol{V}$. If $\boldsymbol{\pi}$ is a path and $t \in \mathbf{R}$ then $\boldsymbol{\pi}(t)$ is termed the value of $\boldsymbol{\pi}$ at time t. A *history* is a function defined on $\mathbf{R}^+$ with values in $\boldsymbol{U} \times \boldsymbol{V}$. Given a path $\boldsymbol{\pi}$ and a time $t \in \mathbf{R}$, the history of $\boldsymbol{\pi}$ up to time t, $\boldsymbol{\pi}^t(\cdot)$, is defined by $\boldsymbol{\pi}^t(\zeta) = \boldsymbol{\pi}(t - \zeta)$, $\zeta \in \mathbf{R}^+$. A *process* is a pair $(\boldsymbol{\pi}, \boldsymbol{\alpha})$ defined on $\mathbf{R}$ and with values $\boldsymbol{\pi} \in \boldsymbol{U} \times \boldsymbol{V}$, $\boldsymbol{\alpha} \in \boldsymbol{A}$.

A path $\boldsymbol{\pi} = (\boldsymbol{y}, \boldsymbol{z})$ determines the growth of the hidden variables through the evolution equation

$$\dot{\boldsymbol{\alpha}}(t) = \boldsymbol{h}\big(\boldsymbol{\pi}(t), \boldsymbol{\alpha}(t)\big)\,, \qquad \boldsymbol{\alpha}(t_0) = \boldsymbol{\alpha}^0\,. \tag{2.4}$$

For any given continuously differentiable path $\boldsymbol{\pi}$, appealing to (2.3) a well known theorem allows us to say that the solution of (2.4) exists and is unique. Here I point out some properties of the solution.

Consider the hidden variables $\boldsymbol{\alpha}$, $\boldsymbol{\alpha} + \boldsymbol{\beta} \in \boldsymbol{A}$ corresponding to two different paths $\boldsymbol{\pi}$, $\boldsymbol{\pi} + \boldsymbol{\nu}$, that is to say

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{h}(\boldsymbol{\pi}, \boldsymbol{\alpha})\,, \qquad \boldsymbol{\alpha}(t_0) = \boldsymbol{\alpha}^0\,,$$

$$\dot{\boldsymbol{\alpha}} + \dot{\boldsymbol{\beta}} = \boldsymbol{h}(\boldsymbol{\pi} + \boldsymbol{\nu}, \boldsymbol{\alpha} + \boldsymbol{\beta})\,, \qquad (\boldsymbol{\alpha} + \boldsymbol{\beta})(t_0) = \boldsymbol{\alpha}^0 + \boldsymbol{\beta}^0\,.$$

Subtraction gives the evolution equation for the difference $\boldsymbol{\beta}$ in the form

$$\dot{\boldsymbol{\beta}} - \boldsymbol{\Lambda}\boldsymbol{\beta} = \boldsymbol{r} + \boldsymbol{\gamma} \tag{2.5}$$

where

$$\boldsymbol{r} = \boldsymbol{h}(\boldsymbol{\pi}, \boldsymbol{\alpha} + \boldsymbol{\beta}) - \boldsymbol{h}(\boldsymbol{\pi}, \boldsymbol{\alpha}) - \boldsymbol{\Lambda}\boldsymbol{\beta}\,, \quad \boldsymbol{\gamma} = \boldsymbol{h}(\boldsymbol{\pi} + \boldsymbol{\nu}, \boldsymbol{\alpha} + \boldsymbol{\beta}) - \boldsymbol{h}(\boldsymbol{\pi}, \boldsymbol{\alpha} + \boldsymbol{\beta})$$

Observe that in terms of $\boldsymbol{r}$ (2.3) takes the form $|\boldsymbol{r}| < \delta|\boldsymbol{\beta}|$. In view of (2.5) it follows that

$$\frac{d}{dt}\{\exp(-t\boldsymbol{\Lambda})\boldsymbol{\beta}(t)\} = \exp(-t\boldsymbol{\Lambda})(\boldsymbol{r} + \boldsymbol{\gamma})$$

whence

$$\boldsymbol{\beta}(t) = \exp\big((t - t_0)\boldsymbol{\Lambda}\big)\,\boldsymbol{\beta}(t_0) + \int_{t_0}^{t} \exp\big((t - s)\boldsymbol{\Lambda}\big)\big(\boldsymbol{r}(s) + \boldsymbol{\gamma}(s)\big)\,ds\,.$$

Then, letting $-m<0$ denote the real part of the eigenvalue of $\boldsymbol{\Lambda}$ which has the largest real part, we have the estimate

$$(2.6)\qquad |\boldsymbol{\beta}(t)|\leqslant|\boldsymbol{\beta}(t_0)|\exp\left(-(t-t_0)m\right)+\delta\int_{t_0}^{t}\exp\left(-(t-s)m\right)|\boldsymbol{\beta}(s)|\,ds+ +\int_{t_0}^{t}\exp\left(-(t-s)m\right)|\boldsymbol{\gamma}(s)|\,ds\,.$$

The inequality (2.6) will be essential to deriving thermodynamic restrictions in sec. 4. Yet, as it stands, (2.6) seems insufficient in that the sought function $\boldsymbol{\beta}$ occurs in either of the sides. To eliminate this insufficiency a routine procedure applies. First replace (2.6) by

$$\exp(mt)|\boldsymbol{\beta}(t)|\leqslant\exp(mt_0)|\boldsymbol{\beta}(t_0)|+\delta\int_{t_0}^{t}\exp(ms)|\boldsymbol{\beta}(s)|\,ds+ +\frac{1}{m}\max_{t_0\leqslant s\leqslant t}|\boldsymbol{\gamma}(s)|\{\exp(mt)-\exp(mt_0)\}\,.$$

Observe that

$$\frac{d}{dt}\{\exp(-\delta t)\int_{t_0}^{t}\exp(ms)|\boldsymbol{\beta}(s)|\,ds\}=-\delta\int_{t_0}^{t}\exp(-\delta t+ms)|\boldsymbol{\beta}(s)|\,ds+ +\exp\left((m-\delta)t\right)|\boldsymbol{\beta}(t)|\,.$$

Then integration and substitutions deliver

$$(2.7)\qquad |\boldsymbol{\beta}(t)|\leqslant|\boldsymbol{\beta}(t_0)|\exp\left(-(m-\delta)(t-t_0)\right)+ +\frac{1}{m-\delta}\max_{t_0\leqslant s\leqslant t}|\boldsymbol{\gamma}(s)|\{1-\exp\left(-(m-\delta)(t-t_0)\right)\}\,.$$

Notice that $m-\delta>0$ because of the negative definiteness of $\boldsymbol{\Lambda}+\delta\boldsymbol{I}_A$.

The inequality (2.7) provides a bound for the difference $\boldsymbol{\beta}$ at time t in terms of the initial value $\boldsymbol{\beta}(t_0)$ and of the difference path $\boldsymbol{\nu}$ via the quantity $\boldsymbol{\gamma}$. In the special case of equal paths, that is $\boldsymbol{\nu}\equiv\mathbf{0}$ and hence $\boldsymbol{\gamma}\equiv\mathbf{0}$, (2.7) reduces to

$$|\boldsymbol{\beta}(t)|\leqslant|\boldsymbol{\beta}(t_0)|\exp\left(-(m-\delta)(t-t_0)\right)$$

whereby the difference between the hidden variables, arising from different initial values, decreases in time at least as $\exp\big(-(m-\delta)(t-t_0)\big)$. Accordingly, letting $\boldsymbol{\pi}' = \hat{\boldsymbol{E}}^{-1}(\boldsymbol{\alpha}')$, $\boldsymbol{\alpha}' \in \boldsymbol{B}$, the evolution equation

$$\dot{\boldsymbol{\alpha}}(t) = \boldsymbol{h}\big(\boldsymbol{\pi}', \boldsymbol{\alpha}(t)\big)\,, \qquad \boldsymbol{\alpha}(t_0) \neq \boldsymbol{\alpha}'\,,$$

satisfies the condition of asymptotic stability

$$\lim_{t\to\infty} \boldsymbol{\alpha}(t) = \boldsymbol{\alpha}'\,.$$

This result lends operative meaning to the assignment of the initial condition for the hidden variables; we can get the initial value $\boldsymbol{\alpha}'$ at time t simply by holding the path $\boldsymbol{\pi}$ equal to $\hat{\boldsymbol{E}}^{-1}(\boldsymbol{\alpha}')$ up to time t.

3. – Acceleration waves.

A particle of a thermo-viscous fluid is characterised by identifying $\boldsymbol{y} \in \boldsymbol{U}$ with the pair (θ, ϱ) and $\boldsymbol{z} \in \boldsymbol{V}$ with the tiple $(\theta, \boldsymbol{g}, \boldsymbol{D})$; here θ stands for the temperature, ϱ the actual mass density, $\boldsymbol{g}$ the spatial temperature gradient, and $\boldsymbol{D}$ the stretching tensor. Indeed, once the path $\boldsymbol{\pi} = (\theta, \varrho, \dot{\theta}, \boldsymbol{g}, \boldsymbol{D})$ is given, the internal energy density e, the entropy density η, the heat flux $\boldsymbol{q}$, and the Cauchy stress tensor $\boldsymbol{T}$ are given by

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\theta, \varrho, \boldsymbol{\alpha})\,, \qquad \dot{\boldsymbol{\alpha}} = \boldsymbol{h}(\theta, \varrho, \dot{\theta}, \boldsymbol{g}, \boldsymbol{D}, \boldsymbol{\alpha})\,, \tag{3.1}$$

being $\boldsymbol{\sigma} = (e, \eta, \boldsymbol{q}, \boldsymbol{T})$.

This section deals with acceleration wave propagation in fluids described by (3.1). In this connection note that the hidden variables $\boldsymbol{\alpha}(t)$ are independent of the present value $\boldsymbol{\pi}(t)$—see, e.g., [8, 9]. Such a property allows us to assume the continuity of $\boldsymbol{\alpha}$ even though $\boldsymbol{\pi}$ suffer jump discontinuities as it happens at wave fronts. This warrants the following

DEFINITION. A wave $\mathcal{S}(t)$ is said to be an acceleration wave if:

A1) the functions $\dot{\boldsymbol{x}}, \theta, \varrho, \boldsymbol{\alpha}$ are continuous everywhere;

A2) the functions $\ddot{\boldsymbol{x}}, \boldsymbol{D}, \dot{\theta}, \boldsymbol{g}, \dot{\varrho}$, $\operatorname{grad}\varrho$, $\dot{\boldsymbol{\alpha}}$, $\operatorname{grad}\boldsymbol{\alpha}$, and the derivatives of higher order suffer jump discontinuities across $\mathcal{S}(t)$ but are continuous functions everywhere else.

By adopting standard notations — see, e.g., [10]—and denoting by ξ any of the quantities $\dot{\boldsymbol{x}}, \theta, \varrho, \boldsymbol{\alpha}$, A1) means $[\xi] = 0$. Then Maxwell's theorem gives

$$[\operatorname{grad} \xi] = \boldsymbol{n}[\boldsymbol{n} \cdot \operatorname{grad} \xi] \tag{3.2}$$

while the kinematic condition of compatibility may be written as

$$[\dot{\xi}] + U\boldsymbol{n} \cdot [\operatorname{grad} \xi] = 0\,, \tag{3.3}$$

$\boldsymbol{n}$ being the unit normal to $\mathscr{s}(t)$ and U the local speed of propagation. The definition of acceleration wave and the assumption (3.1) imply that $[\boldsymbol{T}] = \boldsymbol{0}$, $[\boldsymbol{q}] = \boldsymbol{0}$. Accordingly, (3.3) yields

$$[\operatorname{div} \boldsymbol{T}] = -\frac{1}{U}[\dot{\boldsymbol{T}}]\boldsymbol{n}\,, \qquad [\operatorname{div} \boldsymbol{q}] = -\frac{1}{U}[\dot{\boldsymbol{q}}] \cdot \boldsymbol{n}\,. \tag{3.4}$$

The propagation of waves is governed by the balance equations. If, as usual, the body force and the energy supply are supposed continuous across the wave front, the balance equations provide the jump relations

$$\left\{\begin{array}{l} [\dot{\varrho}] + \varrho[\operatorname{div} \boldsymbol{v}] = 0\,, \\ \varrho[\dot{\boldsymbol{v}}] - [\operatorname{div} \boldsymbol{T}] = \boldsymbol{0}\,, \\ \varrho[\dot{\varepsilon}] - \boldsymbol{T} \cdot [\boldsymbol{D}] + [\operatorname{div} \boldsymbol{q}] = 0\,, \end{array}\right. \tag{3.5}$$

being $\boldsymbol{v} \equiv \dot{\boldsymbol{x}}$. Let Θ, $\boldsymbol{a}$ stand for the thermal amplitude $[\dot{\theta}]$ and the acceleration amplitude $[\dot{\boldsymbol{v}}]$, respectively. In view of (3.3) we get the identities

$$[\boldsymbol{g}] = -\frac{1}{U}\Theta\boldsymbol{n}\,, \qquad [\boldsymbol{D}] = -\frac{1}{2U}(\boldsymbol{a} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{a})\,, \tag{3.6}$$

whence $[\operatorname{div} \boldsymbol{v}] = -U^{-1}\boldsymbol{a} \cdot \boldsymbol{n}$. Hence it is an immediate consequence of (3.1) that

$$[\dot{\boldsymbol{\sigma}}] = \boldsymbol{\sigma}_\theta \Theta + \frac{\varrho}{U}\boldsymbol{\sigma}_\varrho \boldsymbol{a} \cdot \boldsymbol{n} + \boldsymbol{\sigma}_{\boldsymbol{\alpha}}[\boldsymbol{h}] \tag{3.7}$$

with the subscripts denoting partial differentiations. Since $\theta, \varrho, \boldsymbol{\alpha}$ are continuous across the wave front, the jump $[\boldsymbol{h}]$ arises only from the contributions of $\dot{\theta}, \boldsymbol{g}$, and $\boldsymbol{D}$. Moreover these contributions can be handled easily if $\boldsymbol{h}$ is assumed linear with respect to $\dot{\theta}, \boldsymbol{g}, \boldsymbol{D}$ (1). If such is the case, we have

$$(3.8) \qquad [\boldsymbol{h}] = \boldsymbol{h}_{\dot{\theta}}\Theta - \frac{1}{U}\boldsymbol{h}_{\boldsymbol{g}}\boldsymbol{n}\Theta - \frac{1}{2U}(\boldsymbol{a}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{a})\,.$$

Then, appealing to (3.4)-(3.8), a straightforward calculation delivers

$$(3.9) \qquad \begin{cases} \varrho U^2\boldsymbol{a} + \boldsymbol{n}\mathscr{T}(\boldsymbol{a}) + (\boldsymbol{f}_1 U + \boldsymbol{f}_2)\Theta = \boldsymbol{0}\,, \\ \chi_1(\boldsymbol{a})\,U + \chi_2(\boldsymbol{a}) + (\varphi_1 U^2 + \varphi_2 U + \varphi_3)\Theta = 0\,, \end{cases}$$

where

$$(3.10) \qquad \begin{cases} \mathscr{T}(\boldsymbol{a}) = \varrho \boldsymbol{T}_\varrho \boldsymbol{n}\cdot\boldsymbol{a} - \frac{1}{2}\boldsymbol{T}_\alpha \boldsymbol{h}_D(\boldsymbol{a}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{a})\,, \\ \boldsymbol{f}_1 \;\;= \boldsymbol{n}(\boldsymbol{T}_\theta + \boldsymbol{T}_\alpha \boldsymbol{h}_{\dot{\theta}})\,, \quad \boldsymbol{f}_2 = -\boldsymbol{n}\boldsymbol{T}_\alpha \boldsymbol{h}_{\boldsymbol{g}}\boldsymbol{n}\,, \\ \chi_1(\boldsymbol{a}) = (\boldsymbol{T}\boldsymbol{n})\cdot\boldsymbol{a} + \varrho^2 e_\varrho \boldsymbol{n}\cdot\boldsymbol{a} - \frac{1}{2}\varrho e_\alpha \boldsymbol{h}_D(\boldsymbol{a}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{a})\,, \\ \chi_2(\boldsymbol{a}) = \frac{1}{2}\boldsymbol{n}\cdot\{\boldsymbol{q}_\alpha \boldsymbol{h}_D(\boldsymbol{a}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{a})\} - \varrho\boldsymbol{n}\cdot\boldsymbol{q}_\varrho \boldsymbol{n}\cdot\boldsymbol{a}\,, \\ \varphi_1 \;\;= \varrho(e_\theta + e_\alpha \boldsymbol{h}_{\dot{\theta}})\,, \quad \varphi_3 = \boldsymbol{n}\cdot(\boldsymbol{q}_\alpha \boldsymbol{h}_{\boldsymbol{g}}\boldsymbol{n})\,, \\ \varphi_2 \;\;= -\varrho e_\alpha\cdot(\boldsymbol{h}_{\boldsymbol{g}}\boldsymbol{n}) + \boldsymbol{n}\cdot\boldsymbol{q}_\theta + \boldsymbol{n}\cdot(\boldsymbol{q}_\alpha \boldsymbol{h}_{\dot{\theta}})\,. \end{cases}$$

A more explicit form of the relations (3.10) is examined in the next section through an example of constitutive equations closely related to Fourier's and Navier-Stokes' laws. Here it is worth emphasising the general property that the present account of the dependence on the temperature rate $\dot{\theta}$, the temperature gradient $\boldsymbol{g}$, and the stretching tensor $\boldsymbol{D}$ are not in contrast with the existence of acceleration waves. Indeed, the relations (3.9), (3.10) show how such dependences give rise to new additive terms besides the standard ones.

(1) The linearity with respect to $\dot{\theta}$, $\boldsymbol{D}$ is consistent with Lubliner's proposal [11] whereby the evolution function must depend linearly on the time derivatives of the external variables ($\dot{\varrho} = -\varrho \operatorname{tr}\boldsymbol{D}$). If $\boldsymbol{h}$ is a non-linear function of $\dot{\theta}, \boldsymbol{g}, \boldsymbol{D}$, the result (3.9) holds in connection with infinitesimal waves.

4. – An example.

The preceding examination of acceleration waves shows that the dependence of $\boldsymbol{h}$ on θ, ϱ is not at all essential. On the other hand, the general assumption (2.3) means that $\boldsymbol{h}$ must be a nearly linear function of the hidden variables. Bearing this in mind, to adhere as close as possible to Fourier's law of heat conduction and Navier-Stokes' law of viscosity it is convenient to suppose that the hidden variables $\boldsymbol{\alpha}$, at each particle $\boldsymbol{X}$, consist of a vector $\boldsymbol{\alpha}_2 \in \mathscr{V}$ and a symmetric tensor $\boldsymbol{\alpha}_2 \in \mathrm{Sym}(\mathscr{V}, \mathscr{V})$ governed by the evolution equations

$$(4.1) \qquad \left|\begin{aligned} &\dot{\boldsymbol{\alpha}}_1 = \frac{1}{\tau_1}(\boldsymbol{g} - \boldsymbol{\alpha}_1), \qquad \boldsymbol{\alpha}_1(t_0) = \boldsymbol{\alpha}_1^0 \in \boldsymbol{C}^1(\mathscr{R}), \\ &\dot{\boldsymbol{\alpha}}_2 = \frac{1}{\tau_2}(\boldsymbol{D} - \boldsymbol{\alpha}_2), \qquad \boldsymbol{\alpha}_2(t_0) = \boldsymbol{\alpha}_2^0 \in \boldsymbol{C}^1(\mathscr{R}). \end{aligned}\right.$$

The negative definiteness of $\boldsymbol{\Lambda}$ makes $\tau_1, \tau_2 \in \mathbf{R}^{++}$. The two parameters τ_1, τ_2 can be thought of as relaxation times. So eqs. (4.1) allow heat conduction and viscosity to be affected by different relaxation times.

The equilibrium map is $\boldsymbol{\alpha} \equiv (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = (\boldsymbol{g}, \boldsymbol{D})$. The obvious solutions of (4.1) are

$$\boldsymbol{\alpha}_1(t) = \boldsymbol{g}(t; \tau_1) + \boldsymbol{\alpha}_1^0 \exp\left(-(t-t_0)/\tau_1\right), \qquad t - t_0 \in \mathbf{R}^+,$$

$$\boldsymbol{\alpha}_2(t) = \boldsymbol{D}(t; \tau_2) + \boldsymbol{\alpha}_2^0 \exp\left(-(t-t_0)/\tau_2\right), \qquad t - t_0 \in \mathbf{R}^+,$$

the symbol $\xi(t; \tau)$ being defined by

$$\xi(t; \tau) = \frac{1}{\tau}\int_{t_0}^{t} \exp\left(-(t-\zeta)/\tau\right)\xi(\zeta)\,d\zeta .$$

The response function $\boldsymbol{\sigma}(\theta, \varrho, \boldsymbol{\alpha})$ must be compatible with the second law of thermodynamics. Unfortunately this assertion has not a unique mathematical counterpart since the current literature exhibits several statements of the second law. Here it is considered the compatibility with the second law in the form of the Clausius-Duhem inequality which appears to be the most restrictive statement. Accordingly, on introducing the free energy $\psi = \varepsilon - \theta\eta$, the inequality

$$-\varrho(\dot{\psi} + \eta\dot{\theta}) + \boldsymbol{T}\cdot\boldsymbol{D} - \frac{1}{\theta}\boldsymbol{q}\cdot\boldsymbol{g} \geqslant 0 \tag{4.2}$$

is assumed to hold for every C^1 path $\boldsymbol{\pi} = (\theta, \varrho, \boldsymbol{g}, \boldsymbol{D})$ on $\mathbb{R}$. Substitution of (4.1) allows (4.2) to be written as

$$-\varrho(\psi_\theta + \eta)\dot{\theta} + \left(\varrho^2\psi_\varrho\boldsymbol{I} + \boldsymbol{T} + \frac{\varrho}{\tau_2}\psi_{\boldsymbol{\alpha}_2}\right)\cdot\boldsymbol{D} - \left(\frac{1}{\theta}\boldsymbol{q} + \frac{\varrho}{\tau_1}\psi_{\boldsymbol{\alpha}_1}\right)\cdot\boldsymbol{g} + \\ + \varrho\left(\frac{1}{\tau_1}\psi_{\boldsymbol{\alpha}_1}\cdot\boldsymbol{\alpha}_1 + \frac{1}{\tau_2}\psi_{\boldsymbol{\alpha}_2}\cdot\boldsymbol{\alpha}_2\right) \geqslant 0\,. \tag{4.3}$$

Given a path $\boldsymbol{\pi} \in \boldsymbol{C}^1(\mathbb{R})$ and the present time t, consider a path $\boldsymbol{\pi} + \boldsymbol{\nu} \in \boldsymbol{C}^1(\mathbb{R})$ such that $\boldsymbol{\nu}(t') \equiv \boldsymbol{0}$, $t' \in [t_0, t-\varepsilon]$, and, meanwhile, $\dot{\theta}(t)$, $\boldsymbol{g}(t)$, and $\boldsymbol{D}(t)$ are arbitrary. First, the estimate (2.7) tells us that the change of the hidden variables, $\boldsymbol{\beta}$, due to the change from $\boldsymbol{\pi}$ to $\boldsymbol{\pi} + \boldsymbol{\nu}$ vanishes identically outside $[t-\varepsilon, t]$. Second, (2.6) enables us to say that $|\boldsymbol{\beta}(t)|$ may be as little as we please provided ε is small enough. In conclusion, upon the choice of a small enough ε we obtain that, via the change $\boldsymbol{\pi} \to \boldsymbol{\pi} + \boldsymbol{\nu}$, the quantities $\dot{\theta}, \boldsymbol{D}$, and $\boldsymbol{g}$ change arbitrarily while the quantities between brackets change as little as we please. So, (4.3) holds if and only if

$$\eta = -\psi_\theta\,, \qquad \boldsymbol{T} = -\varrho^2\psi_\varrho\boldsymbol{I} + \frac{\varrho}{\tau_2}\psi_{\boldsymbol{\alpha}_2}\,, \qquad \boldsymbol{q} = -\frac{\varrho\theta}{\tau_1}\psi_{\boldsymbol{\alpha}_1}\,, \tag{4.4}$$

$$\frac{1}{\tau_1}\psi_{\boldsymbol{\alpha}_1}\cdot\boldsymbol{\alpha}_1 + \frac{1}{\tau_2}\psi_{\boldsymbol{\alpha}_2}\cdot\boldsymbol{\alpha}_2 \geqslant 0\,. \tag{4.5}$$

Then a function $\psi(\theta, \varrho, \boldsymbol{\alpha})$ satisfying (4.5) makes the response functions (4.4) identically compatible with the second law of thermodynamics.

To specialise the example under consideration look now at a free energy function ψ dependent on θ, ϱ and on the quadratic invariants $\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2\cdot\boldsymbol{\alpha}_2$, and $(\operatorname{tr}\boldsymbol{\alpha}_2)^2$ in the form

$$\psi(\theta, \varrho, \boldsymbol{\alpha}) = \Psi(\theta, \varrho) + \frac{1}{\varrho}\left\{\frac{\varkappa\tau_1}{2\theta}\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_1 + \mu\tau_2\boldsymbol{\alpha}_2\cdot\boldsymbol{\alpha}_2 + \frac{1}{2}\lambda\tau_2(\operatorname{tr}\boldsymbol{\alpha}_2)^2\right\} \tag{4.6}$$

where $\varkappa, \mu, \lambda$ are non-vanishing constants. It is a simple matter to show that the function (4.6) satisfies (4.5) if and only if

$$\mu > 0\,, \quad 3\lambda + 2\mu \geqslant 0\,, \quad \varkappa > 0\,. \tag{4.7}$$

Substitution of (4.6) into (4.4) yields

$$\eta = -\Psi_\theta + \frac{\varkappa\tau_1}{2\varrho\theta^2}\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_1\,, \tag{4.8}$$

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mu\boldsymbol{\alpha}_2 + \lambda(\operatorname{tr}\boldsymbol{\alpha}_2)\boldsymbol{I}\,, \qquad \boldsymbol{q} = -\varkappa\boldsymbol{\alpha}_1\,, \tag{4.9}$$

where $p = \varrho^2\psi_\varrho$. As to the meaning of (4.7) note that, if $\boldsymbol{g}$ and $\boldsymbol{D}$ are constant in time, on account of the asymptotic stability and of the equilibrium map we have

$$\lim_{t\to\infty}\left(\boldsymbol{\alpha}_1(t), \boldsymbol{\alpha}_2(t)\right) = (\boldsymbol{g}, \boldsymbol{D})\,.$$

This allows us to say that when $\boldsymbol{g}$ and $\boldsymbol{D}$ are constant in time eqs. (4.9) asymptotically become the Navier-Stokes and Fourier constitutive equations. So, (4.7) may be regarded as the Stokes-Duhem and Fourier inequalities.

The behaviour of the fluid as to the propagation of acceleration waves is described by (3.9). Indeed, account of the response functions (4.6)-(4.9) allows us to write (3.9) in the form

$$\begin{aligned}(4.10)\quad &\left(\varrho U^2 - \frac{\mu}{\tau_2}\right)\boldsymbol{a} - \left\{\left(\varrho p_\varrho + \frac{\mu+\lambda}{\tau_2} + 2\mu\mathscr{N} + \lambda\operatorname{tr}\boldsymbol{\alpha}_2\right)\boldsymbol{a}\cdot\boldsymbol{n} + \right.\\ &\qquad\left. + 2\mu\mathscr{S}\boldsymbol{a}\cdot\boldsymbol{t}\right\}\boldsymbol{n} - \left(Up_\theta + \frac{\varkappa}{\theta}\boldsymbol{\alpha}_1\cdot\boldsymbol{n}\right)\Theta\boldsymbol{n} = \boldsymbol{0}\,,\\ &U(\varrho^2\varepsilon_\varrho - p)\boldsymbol{a}\cdot\boldsymbol{n} + \left(\varrho\varepsilon_\theta U^2 - \frac{2\varkappa}{\theta}\boldsymbol{\alpha}_1\cdot\boldsymbol{n}U - \frac{\varkappa}{\tau_1}\right)\Theta = 0\,,\end{aligned}$$

where $\mathscr{N} = \boldsymbol{n}\cdot\boldsymbol{\alpha}_2\boldsymbol{n}$ and $\mathscr{S} = |\boldsymbol{n}\wedge\boldsymbol{\alpha}_2\boldsymbol{n}|$ are the components of $\boldsymbol{\alpha}_2\boldsymbol{n}$, that is

$$\boldsymbol{\alpha}_2\boldsymbol{n} = \mathscr{N}\boldsymbol{n} + \mathscr{S}\boldsymbol{t}\,, \qquad \boldsymbol{t}\cdot\boldsymbol{n} = 0\,.$$

The explicit expressions of the quantities $p, p_\theta, p_\varrho, \varepsilon_\theta, \varepsilon_\varrho$ are

$$
(4.11)\quad \begin{cases}
p = \varrho^2\Psi_\varrho - \left\{\dfrac{\varkappa\tau_1}{2\theta}\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_1 + \mu\tau_2\boldsymbol{\alpha}_2\cdot\boldsymbol{\alpha}_2 + \dfrac{1}{2}\lambda\tau_2(\operatorname{tr}\boldsymbol{\alpha}_2)^2\right\}, \\[2ex]
p_\theta = \varrho^2\Psi_{\varrho\theta} + \dfrac{\varkappa\tau_1}{2\theta^2}\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_1, \qquad p_\varrho = (\varrho^2\Psi_\varrho)_\varrho, \\[2ex]
\varepsilon_\theta = -\theta\Psi_{\theta\theta} - \dfrac{\varkappa\tau_1}{\varrho\theta^2}\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_1, \\[2ex]
\varepsilon_\varrho = (\Psi - \theta\Psi_\theta)_\varrho - \dfrac{1}{\varrho^2}\left\{\dfrac{\varkappa\tau_1}{\theta}\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_1 + \mu\tau_2\boldsymbol{\alpha}_2\cdot\boldsymbol{\alpha}_2 + \dfrac{1}{2}\lambda\tau_2(\operatorname{tr}\boldsymbol{\alpha}_2)^2\right\}.
\end{cases}
$$

The propagation condition associated with the homogeneous system (4.10) in the unknown amplitudes $\boldsymbol{a}$, Θ follows straightaway.

Setting aside a comprehensive investigation of (4.10), it seems interesting to point out some particular solutions. First, if $\mathscr{I} = 0$ the system (4.10) bears evidence of the existence of purely mechanical transverse waves, that is to say $\boldsymbol{a}\cdot\boldsymbol{n} = 0$, $\Theta = 0$. Such waves propagate through the fluid with the local speed of propagation

$$U_T = \left(\frac{\mu}{\varrho\tau_2}\right)^{\frac{1}{2}}.$$

So these transverse waves closely resemble the customary transverse waves in elastic materials—possibly with microstructure [12]—once μ/τ_2 is viewed as Lamé's coefficient.

Second, longitudinal waves too may exist. For, letting $\boldsymbol{a} = (\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n}$. $\Theta \neq 0$, the system (4.10) is associated with the propagation condition

$$(4.12)\qquad c_4U^4 + c_3U^3 + c_2U^2 + c_1U + c_0 = 0$$

where

$$
(4.13)\quad \begin{cases}
c_4 = \varrho^2\varepsilon_\theta, \\[1ex]
c_3 = -\dfrac{2\varkappa\varrho}{\theta}\boldsymbol{\alpha}_1\cdot\boldsymbol{n}, \\[2ex]
c_2 = -\dfrac{\varkappa\varrho}{\tau_1} - \varrho\varepsilon_\theta\Omega + p_\theta(\varrho^2\varepsilon_\varrho - p), \\[2ex]
c_1 = \dfrac{\varkappa}{\theta}(\varrho^2\varepsilon_\varrho - p + 2\Omega)\boldsymbol{\alpha}_1\cdot\boldsymbol{n}, \\[2ex]
c_0 = \dfrac{\varkappa}{\tau_1}\Omega, \qquad \Omega = \varrho p_\varrho + \dfrac{2\mu+\lambda}{\tau_2} + 2\mu\mathscr{N} + \lambda\operatorname{tr}\boldsymbol{\alpha}_2.
\end{cases}
$$

As to the waves accounted for by (4.12), examine first the possibility of symmetric waves—symmetric roots of (4.12). These waves occur if and only if the coefficients c_1, c_3 vanish. According to (4.13) this happens if $\boldsymbol{\alpha}_1\cdot\boldsymbol{n} = 0$. On the other hand, since the possibility of symmetric waves has already been examined by Suliciu [13], a comparison is in order. Using the present notations Suliciu's requirements (3.11), (3.12) read

$$\boldsymbol{n}\cdot p_{\boldsymbol{\alpha}_1} = 0\,, \qquad \boldsymbol{n}\cdot(\psi_{\boldsymbol{\alpha}_1} + \theta\eta_{\boldsymbol{\alpha}_1}) = 0\,, \tag{4.14}$$

$$\boldsymbol{q}_\theta = \boldsymbol{0}\,, \qquad \boldsymbol{q}_\varrho = \boldsymbol{0}\,. \tag{4.15}$$

Here the relations (4.15) are trivially true while both relations (4.14) hold if $\boldsymbol{\alpha}_1\cdot\boldsymbol{n} = 0$. So, Suliciu's theory and ours ultimately lead to the same condition for the existence of symmetric waves. This is due also to the fact that the introduction of hidden variables accounting for viscosity does not give rise to additive terms in c_1 and c_3.

A simple special case of (4.12) is provided by non-heat conducting viscous fluids. Indeed, letting $\varkappa = 0$ we have $c_0 = 0$, $c_1 = 0$, $c_3 = 0$ and (4.12) yields

$$U^2 = \frac{1}{\varrho}\left(\frac{2\mu + \lambda}{\tau_2} + 2\mu\mathcal{N} + \lambda\,\mathrm{tr}\,\boldsymbol{\alpha}_2\right) + p_\varrho - \frac{p_\theta(\varrho^2 e_\varrho - p)}{\varrho^2 e_\theta}\,. \tag{4.16}$$

displaying the additive contributions arising from viscosity (2). Accordingly longitudinal waves exist ($U^2 > 0$) provided $\tau_2|\mathcal{N}| < 1$ and $\tau_2|\mathrm{tr}\,\boldsymbol{\alpha}_2| < 1$.

A final comment about (4.12) concerns the limiting case $\tau_1 \to 0$, $\tau_2 \to 0$; for the sake of simplicity suppose $\tau_2 = \mathcal{O}(\tau)$, $\tau \equiv \tau_1$: The physical meaning of this limit may be realised by observing that, in respect of the response functions, the presence of hidden variables induces memory effects. In particular, the example at hand accounts for memory effects associated with heat conduction and viscosity. In this sense the limit $\tau \to 0$ corresponds to an extremely short memory and hence we expect such a limit to establish a connection with Fourier's and Navier-Stokes' theories. To clarify this point observe that

$$\boldsymbol{\alpha}_1 = \boldsymbol{g} + \mathcal{O}(\tau)\,, \qquad \boldsymbol{\alpha}_2 = \boldsymbol{D} + \mathcal{O}(\tau)\,. \tag{4.17}$$

(2) If $\dot{\varrho} = 0$ it is evident from (4.11) that p_θ, e_θ, $\varrho^2 e_\varrho - p$ depend only on θ and ϱ.

Hence, if $\tau \to 0$, c_1 and c_2 increase as τ^{-1} while c_0 as τ^{-2}. Then, because of (4.12), the speed U goes to infinity as $\tau^{-\frac{1}{2}}$. This provides the sought connection in that Fourier's and Navier-Stokes' laws, forbidding acceleration wave propagation, may be obtained from the constitutive equations (4.9) via the limit $\tau \to 0$.

5. – Remarks.

On the basis of different viewpoints, two objections to the analysis of sec. 4 may be in order. First, the evolution equations (4.1) are not objective—or material indifferent [14]. Second the analysis of acceleration waves relies on the Clausius-Duhem inequality as statement of the second law of thermodynamics. An answer to these objections may be given as follows.

Objective evolution equations. The material time derivative of vectors and tensors is not objective. In respect of (4.1), if $\boldsymbol{\alpha}_1$ ($\boldsymbol{\alpha}_2$) is an objective vector (tensor) then objective evolution equations can be obtained by replacing $\dot{\boldsymbol{\alpha}}_1, \dot{\boldsymbol{\alpha}}_2$ by the co-rotational time rates $\overset{\circ}{\boldsymbol{\alpha}}_1, \overset{\circ}{\boldsymbol{\alpha}}_2$, respectively, defined by

$$\overset{\circ}{\boldsymbol{\alpha}}_1 = \dot{\boldsymbol{\alpha}}_1 - \boldsymbol{W}\boldsymbol{\alpha}_1 ,$$
$$\overset{\circ}{\boldsymbol{\alpha}}_2 = \dot{\boldsymbol{\alpha}}_2 - \boldsymbol{W}\boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_2 \boldsymbol{W} ,$$

$\boldsymbol{W}$ being the skew-symmetric part of the velocity gradient. This makes eqs. (4.1) to be replaced by

$$\dot{\boldsymbol{\alpha}}_1 = \frac{1}{\tau_1}(\boldsymbol{g} - \boldsymbol{\alpha}_1) + \boldsymbol{W}\boldsymbol{\alpha}_1 ,$$
$$\dot{\boldsymbol{\alpha}}_2 = \frac{1}{\tau_2}(\boldsymbol{D} - \boldsymbol{\alpha}_2) + \boldsymbol{W}\boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_2 \boldsymbol{W} .$$

So, in writing the counterpart of the relations (4.10) we have new contributions because

$$[\dot{\boldsymbol{\alpha}}_1] = \frac{1}{\tau_1}[\boldsymbol{g}] + [\boldsymbol{W}]\boldsymbol{\alpha}_1$$
$$[\dot{\boldsymbol{\alpha}}_2] = \frac{1}{\tau_2}[\boldsymbol{D}] + [\boldsymbol{W}]\boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_2[\boldsymbol{W}]$$

being $[\boldsymbol{W}] = -(2U)^{-1}(\boldsymbol{a}\otimes\boldsymbol{n} - \boldsymbol{n}\otimes\boldsymbol{a})$. Although this gives rise to more cumbersome formulae, the new contributions do not exhibit qualitatively new terms. Then, setting aside an exhaustive description of waves according to objectivity, here it is pointed out that the new terms vanish if $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ vanish at the wave. This happens when $\boldsymbol{g} = \boldsymbol{0}$, $\boldsymbol{D} = \boldsymbol{0}$ until the arrival of the wave.

Thermodynamic restrictions. Doubtless the results about wave propagation hinge on the thermodynamic restrictions placed by the statement assumed for the second law of thermodynamics. An analysis of the consequences of the various statements of the second law is obviously beyond the scope. Yet it is significant to show that account for transverse acceleration waves is not typical of the previous example only. To this end, independently of any thermodynamic consideration, generalise Fourier's and Navier-Stokes' laws by means of the constitutive equations (4.9) and assume that ψ, η, ϵ, and p depend only on θ and ϱ. In such a case we find that the relations

$$\left(-\varrho U^2 + \frac{\mu}{\tau_2}\right)\boldsymbol{a} + \left(\varrho p_\varrho + \frac{\mu+\lambda}{\tau_2}\right)(\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n} + U p_\theta \Theta \boldsymbol{n} = 0\,,$$

$$U\{(\varrho^2 \epsilon_\varrho - p + 2\mu\mathscr{N} + \lambda\,\mathrm{tr}\,\boldsymbol{\alpha}_2)\boldsymbol{a}\cdot\boldsymbol{n} + 2\mu\mathscr{I}\boldsymbol{a}\cdot\boldsymbol{t}\} + \left(\varrho\epsilon_\theta U^2 - \frac{\varkappa}{\tau_1}\right)\Theta = 0\,,$$

must hold at the wave. Again $\mathscr{I} = 0$ allows the existence of purely mechanical transverse waves with the local speed of propagation U_T.

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