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A link between $C^\infty$ and analytic solvability for P.D.E. with constant coefficients

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A Link Between $C^\infty$ and Analytic Solvability for P.D.E. with Constant Coefficients.

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0. Let $\Omega$ be an open set of $\mathbb{R}^n$ and $P(= P(D))$ a linear partial differential operator with constant coefficients; Hörmander and Malgrange proved that:

$$PC^\infty(\Omega) = C^\infty(\Omega),$$

if and only if $\Omega$ is $P$-convex in the sense of the following definition:

(2) $\Omega$ is $P$-convex if to every compact set $K_0 \subset \Omega$ there exists another compact set $K \subset \Omega$ s.t. $g \in C^\infty(\Omega)$ and $\text{supp } P(-D)g \subset K_0$ implies $\text{supp } g \subset K$.

Of course (2) is not necessary to get $PC^\infty(\Omega') \supset r^\Omega_{\Omega'} C^\infty(\Omega)$ (where $r^\Omega_{\Omega'}$ denotes the restriction map from $\Omega$ to $\Omega'$) for every relatively compact open set $\Omega'$ of $\Omega$, because every differential operator with constant coefficients is semiglobally solvable in view of the existence of the fundamental solution. Denoting by $A(\Omega)$ the space of the real analytic functions on $\Omega$, we prove here that (2) is also necessary in order to solve analytically the equations $Pu = f$, $\forall f \in A(\Omega)$, over compact subsets of $\Omega$; namely:

**Theorem 1.** Let $\Omega$ be an open subset of $\mathbb{R}^n$. If $PA(\Omega') \supset r^\Omega_{\Omega'} A(\Omega)$ for every relatively compact open subset $\Omega'$ of $\Omega$, then $PC^\infty(\Omega) = C^\infty(\Omega)$.

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Since in our work [6] we proved that (1) is sufficient to have $PA(\Omega) = A(\Omega)$ when $\Omega \subset \mathbb{R}^2$, we can state:

**Theorem 2.** Let $\Omega$ be an open set of $\mathbb{R}^2$; $PA(\Omega) = A(\Omega)$ if and only if $PC^\omega(\Omega) = C^\omega(\Omega)$.

Note that if $n > 2$ the result isn’t generally true. Indeed in [4] Hörmander proved that $PA(\mathbb{R}^n) \neq A(\mathbb{R}^n)$ unless every irreducible germ of the real characteristic asymptotic variety $\{x \in \mathbb{R}^n \sim 0: P_m(x) = 0\}$ (where $P_m$ is the principal part of $P$) is of dimension $n - 1$. For the heat equation in $\mathbb{R}^3$ the real characteristics form a line from which follows the nonsurjectivity of the heat operator thought as endomorphism of $A(\mathbb{R}^3)$; this explains a conjecture by E. De Giorgi and L. Cattabriga [1] which L. Piccinini first proved.

1.

We need some preliminary information. We call $(\mathcal{F})$-space every Hausdorff T.V.S. which is the union of an increasing sequence $\{A_i\}_i$ of $(\mathcal{F})$-spaces, the imbedding of $E_i$ into $E_{i+1}$ being continuous, endowed with the inductive limit topology of $E_i$. We call strictly bornological space every Hausdorff space which is the inductive limit of a family of Banach spaces. It is easy to see that every Hausdorff quasi-complete space is strictly bornological if (and only if) it is bornological. That said, if $E$ is a strictly bornological space and $F$ a $(\mathcal{F})$ space, every linear map of $E$ into $F$ is continuous if and only if it has closed graph (see [2] pg. 271).

Let $x$ be the variable in $\mathbb{R}^n$ and $(x, t)$ that in consider $\mathbb{R}^n$ as a subset of $\mathbb{R}^{n+1}$ where $\mathbb{R}^{n+1}$ is the Alexandroff compactification of $\mathbb{R}^{n+1}$. Let $\Omega$ be a subset of $\mathbb{R}^n$ not necessary open, set $A(\Omega) = \lim A(B)$ (in the algebraic sense) where $B$ varies in the family of the open sets of $\mathbb{R}^n$ containing $\Omega$ which are connected with $\Omega$; $\forall f \in A(\Omega)$ there is one and only one harmonic symmetric (with respect to $\mathbb{R}^n$) function $\tilde{f}(x, t)$ in an open symmetric neighbourhood of $\Omega$ in $\mathbb{R}^{n+1}$ s.t. $\tilde{f}(x, 0) = f(x) \forall x$ in an open neighbourhood of $\Omega$ in $\mathbb{R}^n$. So we algebraically and topologically identify the space $A(\Omega)$ with $\lim A_i(B)$ when $B$ varies in the family of symmetric neighbourhoods of $\Omega$ in $\mathbb{R}^{n+1}$ and $A_i(B)$ denotes the $(\mathcal{F})$-space of the harmonic symmetric functions on $B$ (that are infinitesimal at $\infty$ when $\infty \in B$). One can prove that $A(\Omega) = \lim A(K)$ with $K$ varying in the family of
compact subsets of $\Omega$; with such a topology, $A(\Omega)$ is a Hausdorff complete barreled bornological (and so strictly bornological) space and, if $\Omega$ is a compact set, a $(\mathcal{L}, \mathcal{F})$-space. Denoting by $A'(\Omega)$ the dual of $A(\Omega)$ there is an algebraic and topological isomorphism:

$$\Psi: A'(\Omega) \rightarrow A_s(\mathbb{R}^{n+1} \sim \Omega)$$

defined as follows: if $(x, t) \in \mathbb{R}^{n+1} \sim \Omega$ and $T \in A'(\Omega)$; $\Psi T(x, t) = \langle T, E(x - \xi, t) \rangle$ where $E$ is the fundamental solution of $A$ in $\mathbb{R}^{n+1}$ infinitesimal at $\infty$; precisely $E(x, t) = \alpha/|\langle x, t \rangle|^{n-1}$ (here we suppose $n \geq 2$) with $\alpha$ suitable constant. One obtains $\Psi T$ on a neighbourhood of $\mathbb{R}^{n+1} \sim \Omega$ by means of an analytic continuation. Such an identification enables us to say that every analytic functional has compact support and that the polynomials are dense in $A(\Omega) \forall \Omega \subset \mathbb{R}^n$ (1).

2.

PROOF OF THEOREM 1. Given a generic function $g \in C^\infty(\Omega)$ we associate to $g$ the linear functional on $A(\Omega)$ defined by:

$$\langle T, f \rangle = \int g f dx \quad \forall f \in A(\Omega).$$

$T_g$ is continuous on $A(\Omega)$ for the seminorm:

$$f \mapsto \sup_{x \in \text{supp} f} |f(x)| \quad f \in A(\Omega)$$

is continuous on $A(\Omega)$. First we prove that $\text{supp} \ g = \text{supp} \ T_g$, where $\text{supp} \ T_g$ is the smallest compact set of $\mathbb{R}^n$ s.t. $T_g \in A'(\text{supp} \ T_g)$ or equivalently the smallest compact set of $\mathbb{R}^{n+1}$ on the complement of which $\Psi T_g$ has a harmonic continuation ($\Psi$ is the representing isomorphism of $A'(\Omega)$). Indeed observe that:

$$\Psi T(x, t) = \int \frac{\alpha g(\xi)}{|\langle \xi - x, t \rangle|^{n-1}} d\xi = g \otimes \delta_x * E (x, t) \quad \forall (x, t) \in \mathbb{R}^{n+1} \sim \Omega.$$

Since $g \otimes \delta_x * E$ is continuous in $\mathbb{R}^{n+1}$ because it is the newtonian

(1) For more information see [5].
potential of the masses with density $g$, it follows that $\mathcal{P}T_\nu(x, t) = g \otimes \delta_t \ast E(x, t) \forall (x, t) \in \mathbb{R}^{n+1} \sim \text{supp } T_\nu$. Finally $\text{supp } T_\nu$ is the support, in $\mathbb{R}^{n+1}$, of the distribution $\Lambda(g \otimes \delta_t \ast E) = g \otimes \delta_t$ or equivalently it is the support, in $\mathbb{R}^n$, of $g$. We want to prove now that the distances from $\mathbb{R}^n \sim \Omega$ to $\text{supp } T_\nu$ and to $\text{supp } t^\nu \mathcal{P}T_\nu(t)$, which obviously coincides with $\text{supp } T_{\nu(\mathcal{P}^{-1}T_\nu)}$, are equal. Let $\Omega_n$ be the open set of all $x \in \Omega$ s.t. $|x| < n$ and the distance from $x$ to $\mathbb{R}^n \sim \Omega$ is larger than $1/n$ and note that there is a $n_0$ s.t., $\forall n > n_0$, $T_\nu \in A'(\Omega_n)$. Fix a $n$ among them and set $d(\text{supp } t^\nu \mathcal{P}T_\nu, \mathbb{R}^n \sim \Omega) = d$; consider $\forall y \in \mathbb{R}^n$ s.t. $|y| < \inf \{d - 1/n, n\}$ the functional $\tau_y t^\nu \mathcal{P}T_\nu$ where $\tau_y$ is the translation operator by means of $y$. Obviously $\tau_y t^\nu \mathcal{P}T_\nu$ has its support in $\Omega$ and moreover belongs to $t^\nu PA'(\Omega_2n)$ (weak closure). In fact, for every fixed $f \in A(\Omega_2n)$ s.t. $P(f) = 0$, the map:

$$y \mapsto \langle \tau_y t^\nu \mathcal{P}T_\nu, f \rangle \quad \forall |y| < \inf \{d - 1/n, n\}$$

is analytic and, since it vanishes with all its derivatives at $y = 0$, it is identically zero. So $\forall |y| < \inf \{d - 1/n, n\}$ $t^\nu \mathcal{P}T_\nu \in t^\nu PA'(\tau_y \Omega_2n)$; and, since by hypothesis $PA(\tau_y \Omega_2n) \supset r_{\tau_y \Omega_2n}^{-1} A(\tau_y \Omega)$, it follows that there is some $T_\nu \in A'(\tau_y \Omega)$ s.t. $t^\nu \mathcal{P}T_\nu = t^\nu T_\nu$. In fact consider the (commutative) diagram:

$$A(\tau_y \Omega_2n) \xrightarrow{r} A(\tau_y \Omega)$$

The space $A(\tau_y \Omega_2n)$ is of type $(\mathcal{L} \mathcal{F})$ because $\Omega_2n$ is a compact set, while $A(\tau_y \Omega)$ is a strictly bornological space; so we can use the closed graph theorem as we saw in paragraph 1; thus we conclude that $PA(\tau_y \Omega_2n) \supset r_{\tau_y \Omega_2n}^{-1} A(\tau_y \Omega)$ implies $t^\nu PA'(\tau_y \Omega_2n) \supset \tau_y PA'(\tau_y \Omega)$ (2).

Since, $\forall y$, $T_\nu = T_\nu$ (indeed the map $P : A(\mathbb{R}^n) \to A(\mathbb{R}^n)$ has dense range because the polynomials are dense in $A(\mathbb{R}^n)$) it follows that $T_\nu \in \bigcap_{|y| < \inf \{d - 1/n, n\}} A'(\tau_y \Omega)$.

(2) $t^\nu$ is the transpose of $P : A(\Omega) \to A(\Omega)$.

(3) See Theorem 2 of [7] and repeat step by step the demonstration of the analogous implication. Note that for spaces like $A(\Omega)$ and $A(\Omega_2n)$ we couldn’t obtain the same result since, $\Omega_2n$ being open, $A(\Omega_2n)$ isn’t an inductive limit of a sequence of $(\mathcal{F})$-spaces.
Thus $d(\text{supp } T_g, \mathbb{R}^n \sim \Omega) > \inf \{d - 1/n, n\}$ and, with $n$ tending to $\infty$
$d(\text{supp } T_g, \mathbb{R}^n \sim \Omega) > d$.

Summarizing we proved that $\forall g \in C^\infty_c(\Omega) \ d(\text{supp } g, \mathbb{R}^n \sim \Omega) = d(\text{supp } P(-D)g, \mathbb{R}^n \sim \Omega)$ which obviously implies that $\Omega$ is $P$-convex.

q.e.d.

3. – Remark.

It is very easy to prove theorem 1 when $\Omega$ is a subset of $\mathbb{R}^2$ and $P$ is homogeneous; to see this we’ll use an idea suggested by prof. Bratti. If $PC^\infty(\Omega) \neq C^\infty(\Omega)$ we know there exists a characteristic line of $P$ that intersects $\Omega$ in more than one interval; by change of the affine coordinate system we can suppose that such a line is the $x_1$-axis (and so $P(Dx_1, Dx_2) = Dx_1 R(Dx_1, Dx_2)$) and that $\Omega$ contains an open subset

$$\Omega^0 = \Omega^1 \cup \Omega^2 \cup \Omega^3 \ \text{s.t.}$$

$$\Omega^1 = \{(x_1, x_2) : -\epsilon_1 < x_1 < \epsilon_2, -c < x_2 < 0\};$$

$$\Omega^2 = \{(x_1, x_2) : -\epsilon_1 < x_1 < -a_1 < 0, -c < x_2 < d, d > 0\};$$

$$\Omega^2 = \{(x_1, x_2) : 0 < a_2 < x_1 < \epsilon_2, -c < x_2 < d\};$$

and the point $(0, 0) \notin \Omega$.

From the hypothesis $PA(\Omega') \supseteq r^2_{\Omega^0} A(\Omega), \forall \Omega'$ relatively compact open subset of $\Omega$, it follows that $D_{x_1} C^\infty(\Omega^0) \supseteq r^2_{\Omega^0} A(\Omega)$. In fact given $f \in A(\Omega)$ and given, $\forall n, \ u_n \in A(\Omega^0_n)$ (4) s.t. $Pu_n = f$ in $\Omega^0_n$ then $R(u_{n+1} - u_n)$ is analytic in $\Omega^0_n$ and since it verifies there

$$D_{x_1} R(u_{n+1} - u_n) = 0,$$

it has an analytic extension on the convex hull of $\Omega^0_n$. Since the $C^\infty$ solutions in $\mathbb{R}^2$ of $D_{x_1} u = 0$ are dense in the space of the $C^\infty$ solutions in convex regions of the same equation, we can use the well known device of the telescopic series to find a function $u \in C^\infty(\Omega^0)$ which resolves $D_{x_1} u = f$. But such a solution $u$ can’t exist when the datum $f$

(4) $\Omega^0_n$ is the subset of $\Omega^0$ defined in the proof of theorem 1.
is $1/(x_1^2 + x_2^2)$; in fact if it existed we would have, in $\Omega^1$:

$$u(x_1, x_2) = 1/x_2 \arctg (x_1/x_2) + u(0, x_2).$$

This gives $\lim_{x_2 \to 0^+} u(0, x_2) = +\infty = -\infty$.

\section*{BIBLIOGRAPHY}


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