RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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A link between C^{∞} and analytic solvability for P.D.E. with constant coefficients

Rendiconti del Seminario Matematico della Università di Padova, tome 63 (1980), p. 145-150

<http://www.numdam.org/item?id=RSMUP_1980__63__145_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 63 (1980)

A Link Between C^∞ and Analytic Solvability for P.D.E. with Constant Coefficients.

GIUSEPPE ZAMPIERI (*)

0. Let Ω be an open set of $\mathbf{R}^n$ and $P(=P(D))$ a linear partial differential operator with constant coefficients; Hörmander and Malgrange proved that:

$$PC^\infty(\Omega) = C^\infty(\Omega)\,, \tag{1}$$

if and only if Ω is P-convex in the sense of the following definition:

(2) Ω is P-convex if to every compact set $K_0 \subset \Omega$ there exists another compact set $K \subset \Omega$ s.t. $g \in C_c^\infty(\Omega)$ and $\operatorname{supp} P(-D)g \subset K_0$ implies $\operatorname{supp} g \subset K$.

Of course (2) is not necessary to get $PC^\infty(\Omega') \supset r^\Omega_{\Omega'} C^\infty(\Omega)$ (where $r^\Omega_{\Omega'}$ denotes the restriction map from Ω to Ω') for every relatively compact open set Ω' of Ω, because every differential operator with constant coefficients is semiglobally solvable in view of the existence of the fundamental solution. Denoting by $A(\Omega)$ the space of the real analytic functions on Ω, we prove here that (2) is also necessary in order to solve analytically the equations $Pu = f$, $\forall f \in A(\Omega)$, over compact subsets of Ω; namely:

THEOREM 1. *Let Ω be an open subset of $\mathbf{R}^n$. If $PA(\Omega') \supset r^\Omega_{\Omega'} A(\Omega)$ for every relatively compact open subset Ω' of Ω, then $PC^\infty(\Omega) = C^\infty(\Omega)$.*

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L'autore usufruisce di borsa di studio C.N.R. per l'Analisi Funzionale.

Since in our work [6] we proved that (1) is sufficient to have $PA(\Omega) = A(\Omega)$ when $\Omega \subset \mathbf{R}^2$, we can state:

THEOREM 2. *Let Ω be an open set of $\mathbf{R}^2$; $PA(\Omega) = A(\Omega)$ if and only if $PC^\infty(\Omega) = C^\infty(\Omega)$.*

Note that if $n > 2$ the result isn't generally true. Indeed in [4] Hörmander proved that $PA(\mathbf{R}^n) \neq A(\mathbf{R}^n)$ unless every irreducible germ of the real characteristic asymptotic variety $\{x \in \mathbf{R}^n \sim 0 : P_m(x) = 0\}$ (where P_m is the principal part of P) is of dimension $n-1$. For the heat equation in $\mathbf{R}^3$ the real characteristics form a line from which follows the nonsurjectivity of the heat operator thought as endomorphism of $A(\mathbf{R}^3)$; this explaines a conjecture by E. De Giorgi and L. Cattabriga [1] which L. Piccinini first proved.

1.

We need some preliminary information. We call $(\mathfrak{L}\mathcal{F})$-space every Hausdorff T.V.S. which is the union of an increasing sequence $\{A_i\}_i$ of $(\mathcal{F})$-spaces, the imbedding of E_i into E_{i+1} being continuous, endowed with the inductive limit topology of E_i. We call strictly bornological space every Hausdorff space which is the inductive limit of a family of Banach spaces. It is easy to see that every Hausdorff quasi-complete space is strictly bornological if (and only if) it is bornological. That said, if E is a strictly bornological space and F a $(\mathfrak{L}\mathcal{F})$ space, every linear map of E into F is continuous if and only if it has closed graph (see [2] pg. 271).

Let x be the variable in $\mathbf{R}^n$ and (x, t) that in $\mathbf{R}^{n+1}$; consider $\mathbf{R}^n$ as a subset of $\tilde{\mathbf{R}}^{n+1}$ where $\tilde{\mathbf{R}}^{n+1}$ is the Alexandroff compactification of $\mathbf{R}^{n+1}$. Let Ω be a subset of $\mathbf{R}^n$ not necessary open, set $A(\Omega) = \varinjlim A(B)$ (in the algebraic sense) where B varies in the family of the open sets of $\mathbf{R}^n$ containing Ω which are connected with Ω; $\forall f \in A(\Omega)$ there is one and only one harmonic symmetric (with respect to $\mathbf{R}^n$) function $\tilde{f}(x, t)$ in an open symmetric neighbourhood of Ω in $\tilde{R}^{n+1}$ s.t. $\tilde{f}(x, 0) = f(x)$ $\forall x$ in an open neighbourhood of Ω in $\mathbf{R}^n$. So we algebraically and topologically identify the space $A(\Omega)$ with $\varinjlim \mathcal{A}_s(B)$ when B varies in the family of symmetric neighbourhoods of Ω in $\tilde{R}^{n+1}$ and $\mathcal{A}_s(B)$ denotes the $(\mathcal{F})$-space of the harmonic symmetric functions on B (that are infinitesimal at ∞ when $\infty \in B$). One can prove that $A(\Omega) = \varprojlim A(K)$ with K varying in the family of compact subsets of Ω; with such a topology, $A(\Omega)$ is a Hausdorff complete barreled bornological (and so strictly bornological) space and, if Ω is a compact set, a $(\mathfrak{LF})$-space. Denoting by $A'(\Omega)$ the dual of $A(\Omega)$ there is an algebraic and topological isomorphism:

$$\Psi\colon A'(\Omega) \to \mathcal{A}_s(\tilde{R}^{n+1} \sim \Omega)$$

defined as follows: if $(x, t) \in \overbrace{\mathbf{R}^{n+1} \sim \Omega}^{\circ}$ and $T \in A'(\Omega)$, $\Psi T(x, t) = \langle T_\xi, E(x-\xi, t)\rangle$ where E is the fundamental solution of Δ in $\mathbf{R}^{n+1}$ infinitesimal at ∞; precisely $E(x, t) = \alpha/|(x, t)|^{n-1}$ (here we suppose $n \geq 2$) with α suitable constant. One obtains ΨT on a neighbourhood of $\tilde{\mathbf{R}}^{n+1} \sim \Omega$ by means of an analytic continuation. Such an identification enables us to say that every analytic functional has compact support and that the polinomials are dense in $A(\Omega)$ $\forall \Omega \subset \mathbf{R}^n$ (1).

2.

Prof of theorem 1. Given a generic function $g \in C_c^\infty(\Omega)$ we associate to g the linear functional on $A(\Omega)$ defined by:

$$\langle T_g, f\rangle = \int g f\, dx \qquad \forall f \in A(\Omega)\,.$$

T_g is continuous on $A(\Omega)$ for the seminorm:

$$f \to \sup_{x \in \operatorname{supp} g} |f(x)| \qquad f \in A(\Omega)$$

is continuous on $A(\Omega)$. First we prove that $\operatorname{supp} g = \operatorname{supp} T_g$, where $\operatorname{supp} T_g$ is the smallest compact set of $\mathbf{R}^n$ s.t. $T_g \in A'(\operatorname{supp} T_g)$ or equivalently the smallest compact set of $\mathbf{R}^{n+1}$ on the complement of which ΨT_g has a harmonic continuation (Ψ is the representing isomorphism of $A'(\Omega)$). Indeed observe that:

$$\Psi T_g(x, t) = \int \frac{\alpha g(\xi)}{|(\xi - x, t)|^{n-1}}\, d\xi = g \otimes \delta_t * E\ (x, t) \qquad \forall (x, t) \in \mathbf{R}^{n+1} \sim \bar{\Omega}\,.$$

Since $g \otimes \delta_t * E$ is continuous in $\mathbf{R}^{n+1}$ because it is the newtonian

(1) For more information see [5].

potential of the masses with density g, it follows that $\Psi T_g(x,t) = g\otimes\delta_t * E\ (x,t)\ \forall(x,t)\in\mathbf{R}^{n+1}\sim\operatorname{supp} T_g$. Finally $\operatorname{supp} T_g$ is the support, in $\mathbf{R}^{n+1}$, of the distribution $\Delta(g\otimes\delta_t * E) = g\otimes\delta_t$ or equivalently it is the support, in $\mathbf{R}^n$, of g. We want to prove now that the distances from $\mathbf{R}^n\sim\Omega$ to $\operatorname{supp} T_g$ and to $\operatorname{supp}{}^tPT_g$ (2), which obviously coincides with $\operatorname{supp} T_{P(-D)g}$, are equal. Let Ω_n be the open set of all $x\in\Omega$ s.t. $|x|<n$ and the distance from x to $\mathbf{R}^n\sim\Omega$ is larger than $1/n$ and note that there is a n_0 s.t., $\forall n\geqslant n_0$, $T_g\in A'(\Omega_n)$. Fix a n among them and set $d(\operatorname{supp}{}^tPT_g, \mathbf{R}^n\sim\Omega) = d$; consider $\forall y\in\mathbf{R}^n$ s.t. $|y|<\inf\{d-1/n, n\}$ the functional $\tau_y{}^tPT_g$ where τ_y is the translation operator by means of y. Obviously $\tau_y{}^tPT_g$ has its support in Ω and moreover belongs to ${}^tPA'(\Omega_{2n})^-$ (weak closure). In fact, for every fixed $f\in A(\Omega_{2n})$ s.t. $Pf=0$, the map:

$$y\mapsto\langle\tau_y{}^tPT_g, f\rangle\qquad\forall|y|<\inf\{d-1/n, n\}$$

is analytic and, since it vanishes with all its derivatives at $y=0$, it is identically zero. So $\forall|y|<\inf\{d-1/n, n\}\,{}^tPT_g\in{}^tPA'(\tau_y\Omega_{2n})^-$; and, since by hypothesis $PA(\tau_y\overline{\Omega_{2n}})\supset r^{\tau_y\Omega}_{\tau_y\overline{\Omega_{2n}}}A(\tau_y\Omega)$, it follows that there is some $T^y\in A'(\tau_y\Omega)$ s.t. ${}^tPT_g = {}^tPT^y$. In fact consider the (commutative) diagram:

$$\begin{array}{ccc} A(\tau_y\overline{\Omega_{2n}}) & \xrightarrow{P} & A(\tau_y\overline{\Omega_{2n}}) \\ {\scriptstyle r^{\tau_y\Omega}_{\tau_y\overline{\Omega_{2n}}}}\uparrow & & \uparrow{\scriptstyle r^{\tau_y\Omega}_{\tau_y\overline{\Omega_{2n}}}} \\ A(\tau_y\Omega) & \xrightarrow{P} & A(\tau_y\Omega) \end{array}$$

The space $A(\tau_y\overline{\Omega_{2n}})$ is of type $(\mathfrak{L}\mathcal{F})$ because $\overline{\Omega_{2n}}$ is a compact set, while $A(\tau_y\Omega)$ is a strictly bornological space; so we can use the closed graph theorem as we saw in paragraph 1; thus we conclude that $PA(\tau_y\overline{\Omega_{2n}})\supset r^{\tau_y\Omega}_{\tau_y\overline{\Omega_{2n}}}A(\tau_y\Omega)$ implies ${}^tPA'(\tau_y\overline{\Omega_{2n}})^-\subset{}^tPA'(\tau_y\Omega)$ (3).

Since, $\forall y$, $T_g = T^y$ (indeed the map $P\colon A(\mathbf{R}^n)\to A(\mathbf{R}^n)$ has dense range because the polynomials are dense in $A(\mathbf{R}^n)$) it follows that $T_g\in\bigcap_{|y|<\inf\{d-1/n,n\}} A'(\tau_y\Omega)$.

(2) ${}^tP$ is the transpose of $P\colon A(\Omega)\to A(\Omega)$.

(3) See Theorem 2 of [7] and repeat step by step the demonstration of the analogous implication. Note that for spaces like $A(\Omega)$ and $A(\Omega_{2n})$ we couldn't obtain the same result since, Ω_{2n} being open, $A(\Omega_{2n})$ isn't an inductive limit of a sequence of $(\mathcal{F})$-spaces.

Thus $d(\operatorname{supp} T_g, \mathbf{R}^n \sim \Omega) \geqslant \inf\{d - 1/n, n\}$ and, with n tending to ∞ $d(\operatorname{supp} T_g, \mathbf{R}^n \sim \Omega) \geqslant d$.

Summarizing we proved that $\forall g \in C_c^\infty(\Omega)$ $d(\operatorname{supp} g, \mathbf{R}^n \sim \Omega) = d(\operatorname{supp} P(-D)g, \mathbf{R}^n \sim \Omega)$ which obviously implies that Ω is P-convex.

q.e.d.

3. – Remark.

It is very easy to prove theorem 1 when Ω is a subset of $\mathbf{R}^2$ and P is homogeneous; to see this we'll use an idea suggested by prof. Bratti. If $PC^\infty(\Omega) \neq C^\infty(\Omega)$ we know there exists a characteristic line of P that intersects Ω in more than one interval; by change of the affine coordinate system we can suppose that such a line is the x_1-axis (and so $P(D_{x_1}, D_{x_2}) = D_{x_1} R(D_{x_1}, D_{x_2})$) and that Ω contains an open subset

$$\Omega^0 = \Omega^1 \cup \Omega^2 \cup \Omega^3 \text{ s.t.:}$$

$$\Omega^1 = \{(x_1, x_2): -\varepsilon_1 < x_1 < \varepsilon_2,\ -c < x_2 < 0\};$$

$$\Omega^2 = \{(x_1, x_2): -\varepsilon_1 < x_1 < -a_1 < 0,\ -c < x_2 < d,\ d > 0\};$$

$$\Omega^3 = \{(x_1, x_2): 0 < a_2 < x_1 < \varepsilon_2,\ -c < x_2 < d\};$$

and the point $(0, 0) \notin \Omega$.

From the hypothesis $PA(\Omega') \supset r^{\Omega}_{\Omega'} A(\Omega)$, $\forall \Omega'$ relatively compact open subset of Ω, it follows that $D_{x_1} C^\infty(\Omega^0) \supset r^{\Omega}_{\Omega^0} A(\Omega)$. In fact given $f \in A(\Omega)$ and given, $\forall n$, $u_n \in A(\Omega_n^0)$ (4) s.t. $Pu_n = f$ in Ω_n^0 then $R(u_{n+1} - u_n)$ is analytic in Ω_n^0 and since it verifies there

$$D_{x_1} R(u_{n+1} - u_n) = 0\ ,$$

it has an analytic extension on the convex hull of Ω_n^0. Since the C^∞ solutions in $\mathbf{R}^2$ of $D_{x_1} u = 0$ are dense in the space of the C^∞ solutions in convex regions of the same equation, we can use the well known device of the telescopic series to find a function $u \in C^\infty(\Omega^0)$ which resolves $D_{x_1} u = f$. But such a solution u can't exist when the datum f

(4) Ω_n^0 is the subset of Ω^0 defined in the proof of theorem 1.

is $1/(x_1^2 + x_2^2)$; in fact if it existed we would have, in Ω^1:

$$u(x_1, x_2) = 1/x_2 \operatorname{arctg}(x_1/x_2) + u(0, x_2) .$$

This gives $\lim\limits_{x_2 \to 0^-} u(0, x_2) = +\infty = -\infty$.

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Manoscritto pervenuto in redazione il 4 settembre 1979.