Rendiconti
del
Seminario Matematico
della
Università di Padova

S. Zaidman

Some remarks concerning regularity of solutions for abstract differential equations

Rendiconti del Seminario Matematico della Università di Padova, tome 62 (1980), p. 47-64

<http://www.numdam.org/item?id=RSMUP_1980__62__47_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 62 (1980)

Some Remarks Concerning Regularity of Solutions for Abstract Differential Equations.

S. ZAIDMAN (*)

Introduction.

Theorems on regularity (or differentiability, or smoothness) of (weak) solutions of partial differential equations were studied intensively (see for example Hörmander's book [7 - Ch. IV]). In Lions' monograph [8] various regularity theorems are proved for (weak) solutions of operational-differential equations: $u'(t) = A(t)u(t) + f(t)$, and more recently, in Treves [10, Sect. 42] are again explained regularity results for weak solutions of abstract evolution equations. The propositions that we present in this Note are most closely related to some of the facts exposed in Chapter IV of the fundamental memoir [1] of Agmon and Nirenberg which is concerned with differentiability (or analyticity) of solutions of $Lu = (1/i)(du/dt) - Au = f$ (in a Banach space), assuming f to be differentiable (or analytic).

They prove necessary conditions which are obtained by employing the closed graph theorem in a suitable space and sufficient conditions which are not too far removed from the necessary ones.

They consider the classes C^n, C^∞ and the class of analytic functions, and note that various other classes of functions could be treated by the same procedure and that weak solutions of equations $(1/i)(du/dt) - Au = f$ (in some sense) have also similar regularity properties.

In our paper [12] we have considered the second order differential

(*) Indirizzo dell'A.: Department of Mathematics, Case postale 6128, Succursale « A », Montreal, P.Q., H3C3J7.

This research is supported by a grant of the Natural Sciences and Engineering Research Council Canada.

equation $u''(t) - Au(t) = f(t)$, $-\infty < t < \infty$ and have proved regularity theorems for L^2-weak solutions (in Hilbert spaces) of this equation. Equations with time dependent operator: $(1/i)(du/dt) - A(t)u = f(t)$ where considered by A. Friedman in [5] (see also our Lecture Notes [13] and his book [4]).

Regularity of distributions solutions of the above indicated equations in Hilbert spaces was demonstrated by V. Barbu in [2], [3] and more recently, in our Note [14] we have given for the first order equation $u'(t) - Au(t) = f(t)$ a sufficient condition for differentiability of weak solutions which is closely related to the statements in [1].

In the present work we concentrate ourselves mainly on the study of the necessary conditions for smoothness of weak solutions in both Hilbert and Banach spaces. After proving some preliminary remarks concerning weak solutions we present in Theorems 1 and 2 a necessary condition for regularity in an interval (a, b) up to the boundary of this interval whereas in Theorem 3 we explain a similar condition for interior regularity (i.e., regularity in any strict subinterval $(\alpha, \beta) \subset \subset (a, b)$). Whereas the above Theorems 1, 2, 3 concern L^2-weak solutions in Hilbert spaces, in our Theorem 4 we give also a result in Banach spaces, where *continuous* weak solutions are instead considered.

The paper ends by indicating how one can extend our paper [4] in order to obtain sufficient conditions for interior L^2-regularity in finite intervals (the above indicated Note [14] is dedicated to L^2_{loc}-solutions on the whole real line).

The methods are essentially those used by Agmon and Nirenberg, with convenient modifications corresponding to the class of weak solutions that are here investigated.

While preparing the manuscript we had the opportunity to have some interesting discussions with professor J. Goldstein.

1. Let be H a Hilbert space and A a linear closed operator: $\mathfrak{D}(A) \subset H \to H$ where the domain $\mathfrak{D}(A)$ is dense in H.

Denote as usual by A^* the Hilbertian adjoint to A, so that $(Ah, k)_H = (h, A^*k)_H$ holds for any $h \in \mathfrak{D}(A)$ and $k \in \mathfrak{D}(A^*)$ (the domain $\mathfrak{D}(A^*)$ is also dense in H—see [11]).

Given an interval $I = (a, b) \subset \mathbf{R}$ (the real line) let us define the class $K_{A^*}(a, b)$ of vector-valued test-functions:

$$K_{A^*}(a, b) = \{\varphi(t) \in C^1(a, b; H),\ \operatorname{supp}\varphi \text{ is compact in } (a, b),\ \varphi(t) \in \mathfrak{D}(A^*)\ \forall t \in I,\ (A^*\varphi)(t) \in C^0(a, b; H)\} \tag{1.1}$$

Consider now the operator (abstract differential) $L = (d/dt) - A$ which is classically defined on functions $u(t) \in C^1(a, b; H)$ such that $u(t) \in \mathfrak{D}(A)\ \forall t \in (a, b)$ and $(Au)(t) \in C(a, b; H)$; next we define a natural weak extension ωL as follows: the domain $\mathfrak{D}(\omega L)$ consists of those functions $u(t) \in L^2(a, b; H)$ such that there exists $f(t) \in L^2(a, b; H)$ with the property that the integral identity

$$\int_a^b (\varphi'(t) + (A^*\varphi)(t), u(t))_H \, dt = -\int_a^b (\varphi(t), f(t))_H \, dt, \qquad \forall \varphi(t) \in K_{A^*}(a, b) \tag{1.2}$$

is verified.

We say that: $(\omega L)u \ni f$. A few observations concerning this definition are necessary:

PROPOSITION 1. *To a given (single) element $u(\cdot) \in \mathfrak{D}(\omega L)$ it corresponds a single element $f(\cdot) \in L^2(a, b; H)$ such that $(\omega L)u \ni f$.*

Otherwise we could have, say $f_1, f_2 \in L^2(a, b; H)$ such that $(\omega L)u \ni f_1$, $(\omega L)u \ni f_2$: this means, using 1.2) the equality

$$\int_a^b (\varphi(t), f_1(t) - f_2(t))_H \, dt = 0 \qquad \forall \varphi \in K_{A^*}(a, b) \tag{1.3}$$

Let us take $\varphi(t) = \nu(t)h$, where $h \in \mathfrak{D}(A^*)$ and $\nu(t) \in C_0^1(a, b)$. We obtain

$$\left(\int_a^b \nu(t)(f_1(t) - f_2(t))\, dt, h\right) = 0 \tag{1.4}$$

Using density of $\mathfrak{D}(A^*)$ in H we deduce equality

$$\int_a^b \nu(t)(f_1(t) - f_2(t))\, dt = \theta \qquad \forall \nu(t) \in C_0^1(a, b) \tag{1.5}$$

If we take now an arbitrary $t_0 \in (a, b)$ and positive τ small enough, and if we consider the function

$$\nu_{t_0}(t) = \begin{cases} 1/\tau, & t_0 \leqslant t \leqslant t_0 + \tau \\ 0, & t \notin [t_0, t_0 + \tau], \end{cases}$$

we can find a sequence $\nu_p(t) \in C_0^1(a, b)$, such that $\nu_p(t) \to \nu_{t_0}(t)$ in $L^2(a, b)$ sense. Then

$$\int_a^b \nu_p(t)\big(f_1(t) - f_2(t)\big)\, dt \to \int_a^b \nu_{t_0}(t)\big(f_1(t) - f_2(t)\big)\, dt$$

as easily seen by Schwartz inequality

$$\left\|\int_a^b [\nu_p(t) - \nu_{t_0}(t)]\,[f_1(t) - f_2(t)]\, dt\right\| \leqslant \int_a^b |\nu_p(t) - \nu_{t_0}(t)|\cdot$$

$$\cdot \|f_1(t) - f_2(t)\|\, dt \leqslant \Big(\int_a^b |\nu_p - \nu_{t_0}|^2 dt\Big)^{\frac{1}{2}} \Big(\int_a^b \|f_1(t) - f_2(t)\|^2 dt\Big)^{\frac{1}{2}} \to 0 \qquad \text{as } p \to \infty$$

Hence,

$$(1.6)\qquad \int_a^b \nu_{t_0}(t)\big(f_1(t) - f_2(t)\big)\, dt = \frac{1}{\tau}\int_{t_0}^{t_0+\tau} [f_1(t) - f_2(t)]\, dt = \theta$$

$$\forall t_0 \in (a, b)\,, \qquad \tau > 0$$

Now, by a well-known result (see [6] p. 88) we obtain as $\tau \to 0$, that $f_1(t) - f_2(t) = \theta$ almost everywhere.

Proposition 2. *The domain $\mathfrak{D}(\omega L)$ is a linear set in $L^2(a, b; H)$ and ωL; $\mathfrak{D}(\omega L) \to L^2(a, b; H)$ is a linear operator on it.*

In fact, if $u_1, u_2 \in \mathfrak{D}(\omega L)$, $\exists f_1, f_2 \in L^2(a, b; H)$, such that

$$(1.7)\qquad \int_a^b \big(\varphi'(t) + (A^*\varphi)(t), u_i(t)\big)_H\, dt = -\int_a^b \big(\varphi(t), f_i(t)\big)_H\, dt\,, \qquad i = 1, 2$$

is true, $\forall \varphi \in K_{A^*}(a, b)$.
Summing, we get

$$(1.8)\qquad \int_a^b \big(\varphi'(t) + (A^*\varphi)(t), u_1(t) + u_2(t)\big)_H\, dt = -\int_a^b \big(\varphi(t), f_1(t) + f_2(t)\big)_H\, dt$$

$$\forall \varphi \in K_{A^*}(a, b)$$

where $f_1 + f_2 \in L^2(a, b; H)$.

Hence, $u_1 + u_2 \in \mathfrak{D}(\omega L)$ and

$$(\omega L)(u_1 + u_2) = f_1 + f_2 = (\omega L) f_1 + (\omega L) f_2 .$$

Similarly $(\omega L)(\lambda u) = \lambda(\omega L) u$, $\forall \lambda \in \mathbf{C}$, $u \in \mathfrak{D}(\omega L)$.

PROPOSITION 3. *(ωL) is a closed operator, $\mathfrak{D}(\omega L) \to H$.*

Let us consider a sequence $(u_n)_1^\infty \subset \mathfrak{D}(\omega L)$, such that $u_n \to u$ in $L^2(a, b; H)$ and $(\omega L) u_n = f_n \to f \in L^2(a, b; H)$ in the same sense. Hence, we can write equalities

$$\int_a^b \big(\varphi'(t) + (A^*\varphi)(t), u_n(t)\big)_H dt = -\int_a^b \big(\varphi(t), f_n(t)\big)_H dt ,$$

$$\forall \varphi \in K_{A^*}(a, b) , \quad n = 1, 2, \ldots$$

as $n \to \infty$ we get readily

$$\int_a^b \big(\varphi'(t) + (A^*\varphi)(t), u(t)\big)_H dt = -\int_a^b \big(\varphi(t), f(t)\big)_H dt \qquad \forall \varphi \in K_{A^*}(a, b)$$

This means that $u(t) \in \mathfrak{D}(\omega L)$ and $(\omega L) u = f$.

Finally, we shall prove the

PROPOSITION 4. *The domain $\mathfrak{D}(\omega L)$ is dense in $L^2(a, b; H)$.*

This is a consequence of the more general fact:

For any dense set $\mathcal{M} \subset H$, the set of finite sums $\left\{\sum_1^n h_i \psi_i(t)\right\}_{n=1,2,\ldots}$ where $h_i \in \mathcal{M}$ and $\psi_i(t) \in C_0^\infty(a, b)$ is dense in $L^2(a, b; H)$.

A straightforward proof can be given in the following way: given $f \in L^2(a, b; H)$ and $\varepsilon > 0$, $\exists$

i) a continuous function $f_\varepsilon(t) \in C[a, b; H]$ s.t. $\|f - f_\varepsilon\|_{L^2(a,b;H)} < \varepsilon$.

ii) using the vector-form of Bernstein's approximation theorem (see [9], Ch. IV, § 5, T.2) $\exists$ a polynomial $P_\varepsilon(t) = \sum_1^{N(\varepsilon)} a_k t^k$, $a_k \in H$, s.t.,

$$\sup_{[a,b]} \|P_\varepsilon(t) - f_\varepsilon(t)\|_H < \varepsilon \text{ which} \Rightarrow \|P_\varepsilon - f_\varepsilon\|_{L^2(a,b;H)} \leqslant \varepsilon(b - a)^{\frac{1}{2}} .$$

iii) $\forall a_k \; \exists h_k \in \mathcal{M}$, s.t. $\|a_k - h_k\|_H \leqslant \varepsilon\left[\sum_1^N \|t^k\|_{L^2(a,b)}\right]^{-1}$, $k = 1, 2, \ldots, N(\varepsilon)$

Then

$$\|P_\varepsilon - \sum_1^N h_k t^k\|_{L^2(a,b;H)} = \|\sum_1^N (h_k - a_k) t^k\|_{L^2(a,b;H)} \leqslant \sum_1^N \|(h_k - a_k) t^k\|_{L^2(a,b;H)} \leqslant$$

$$\leqslant \frac{\varepsilon \sum_1^N \|t^k\|_{L^2(a,b)}}{\sum_1^N \|t^k\|_{L^2(a,b)}} = \varepsilon$$

iv) $\forall t^k \; \exists \psi_k(t) \in C_0^\infty(a, b)$ s.t. $\|t^k - \psi_k\|_{L^2(a,b)} < \dfrac{\varepsilon}{\sum_1^{N(\varepsilon)} \|h_k\|_H}$.

Then

$$\|\sum_1^N h_k t^k - \sum_1^N \psi_k h_k\|_{L^2(a,b;H)} = \|\sum_1^N h_k (t^k - \psi_k)\|_{L^2} \leqslant$$

$$\sum_1^N \|h_k (t^h - \psi_k)\|_{L^2(a,b;H)} = \sum_1^N \|h_k\| \, \|t^h - \psi_k\|_{L^2(a,b)} \leqslant \frac{\varepsilon}{\sum \|h_k\|} \sum \|h_k\| = \varepsilon .$$

Consequently

$$\|f - \sum_1^{N(\varepsilon)} h_k \psi_k\| \leqslant \|f - f_\varepsilon\| + \|f_\varepsilon - P_\varepsilon\| + \|P_\varepsilon - \sum_1^N h_k t^k\| +$$

$$+ \|\sum_1^N h_k t^k - \sum_1^N h_k \psi_k\| \leqslant \varepsilon + \varepsilon (b-a)^{\frac{1}{2}} + \varepsilon + \varepsilon .$$

2. We shall consider the graph of the operator ωL, $G_{\omega L}$, as a subset of the cartesian product $L^2(a, b; H) \times L^2(a, b; H)$. It consists of all pairs $\{u, (\omega L) u\}$ where u belongs to $\mathfrak{D}(\omega L)$; it becomes a Hilbert space with the usual scalar product and norm, due to closedness of the operator (ωL).

Let us consider now the important particular case when $L = d/dt$ (so that $A = \theta$, the null operator).

Now $\omega L = \omega(d/dt)$ has as domain the set of functions $u(t) \in L^2(a, b; H)$ such that $\exists v(t) \in L^2(a, b; H)$ with property that

$$(2.1) \quad \int_a^b (\varphi'(t), u(t))\, dt = -\int_a^b (\varphi(t), v(t))\, dt \text{ is verified, } \forall \varphi \in C_0^1(a, b; H)$$

(we remark that for $A = \theta$, $K_{A^*}(a, b)$ reduces precisely to $C_0^1(a, b; H)$).
Also, for $u \in \mathfrak{D}(\omega(d/dt))$, $\omega(d/dt)u = v$. Now, as above, the graph $\{u, \omega(d/dt)u\}_{u \in \mathfrak{D}(\omega(d/dt))}$ becomes a Hilbert space with usual norm:

$$\left\| \left\{ u, \omega \frac{d}{dt} u \right\} \right\|^2 = \|u\|^2_{L^2(a,b;H)} + \left\| \omega \frac{du}{dt} \right\|^2_{L^2(a,b;H)} . \tag{2.2}$$

We denote this space by $\mathcal{H}^1(a, b; H)$. We assume the following *regularity hypothesis*:

(R.H.) *if* $u(t) \in L^2(a, b; H)$ *belongs to* $\mathfrak{D}(\omega L)$, (*so that* $(\omega L)u = f \in L^2(a, b; H)$) *then* $u(t) \in \mathfrak{D}(\omega(d/dt))$, *so that* $\omega(d/dt)u \in L^2(a, b; H)$.

We shall prove below the following

THEOREM 1. *Let us suppose* R.H. *true. Then, there exists a positive constant* $K > 0$ *such that if* λ *is a complex number and* $|\lambda| \geqslant K$, *then the inequality*

$$\|(i\lambda - A)\phi\|_H \geqslant \frac{1}{K} \|\phi\|_H , \qquad \forall \phi \in \mathfrak{D}(A) \tag{2.3}$$

is verified.

PROOF. The hypothesis R.H. means the inclusion: Graph $(\omega L) \subset \mathcal{H}^1(a, b; H)$. Consider the inclusion map i; each element in Graph (ωL) is transformed in the same element considered in $\mathcal{H}^1(a, b; H)$. We remark that i is a closed map. Assume $(U_n)_1^\infty \subset$ Graph (ωL), $U_n \to u$ in $G(\omega L)$ and $iU_n \to v$ in $\mathcal{H}^1(a, b; H)$.

Then [1], $U_n \to u$ in $L^2(a, b; H)$ and $iU_n = U_n \to v$ in $L^2(a, b; H)$ so that $u = v = iu$. Using the closed graph theorem [11] we obtain that i is a continuous map. Hence, there exists a positive constant K such that

$$\|iu\|_{\mathcal{H}^1(a,b;H)} \leqslant K \|u\|_{\mathrm{Graph}(\omega L)} \tag{2.4}$$

[1] Because both topologies, in $G\omega L$ and in $\mathcal{H}^1(a, b; H)$ are stronger than the topology of $L^2(a, b; H)$.

which means

$$(2.5)\qquad \left(\|u\|^2_{L^2(a,b;H)} + \left\|\omega\frac{d}{dt}u\right\|^2_{L^2(a,b;H)}\right)^{\frac12} \leqslant K\left(\|u\|^2_{L^2(a,b;H)} + \right.$$
$$\left. + \|(\omega L)u\|^2_{L^2(a,b;H)}\right)^{\frac12} \qquad \forall u \in \mathfrak{D}(\omega L)\,.$$

Using the standard procedure in [1] we shall apply the « a priori estimate » 2.5) to all functions $u(t) = \phi \exp[i\lambda t]$, where $\phi \in \mathfrak{D}(A)$ and λ is a complex number. Any such function is a classical (strong) solution of the equation

$$u'(t) - Au(t) = (i\lambda \exp[i\lambda t])\phi - \exp[i\lambda t]A\phi = \exp[i\lambda t](i\lambda - A)\phi = f(t)$$

and consequently is in $\mathfrak{D}(\omega L)$, $(\omega L)u = Lu = (d/dt - A)u = f$ as easily seen. Also $\omega(d/dt)u = (d/dt)u = i\lambda\exp[i\lambda t]\phi$.

We see now that

$$\|u(t)\|^2_{L^2(a,b;H)} = \int_a^b \|u(t)\|^2_H\,dt;$$

also, for $\lambda = \mu + i\nu$,

$$\|u(t)\|_H = \|\phi\|_H \exp[-\nu t]$$

hence

$$\|u(t)\|^2_{L^2(a,b;H)} = \|\phi\|^2_H \int_a^b \exp[-2\nu t]\,dt$$

Also

$$\|u'(t)\|^2_{L^2(a,b;H)} = \int_a^b \|u'(t)\|^2_H\,dt = \int_a^b \|i\lambda\exp[i(\mu + i\nu)t]\phi\|^2_H\,dt =$$
$$= |\lambda|^2\|\phi\|^2_H \int_a^b \exp[-2\nu t]\,dt$$

Next $(\omega L)u = (i\lambda - A)\exp[i\lambda t]\phi$, so that

$$\|(\omega L)u(t)\|^2_H = \|(i\lambda - A)\phi\|^2_H \exp[-2\nu t]$$

and

$$\int_a^b \|(\omega L)u(t)\|^2\,dt = \left(\int_a^b \exp[-2\nu t]\,dt\right)\|(i\lambda - A)\phi\|_H^2$$

After these computations (2.5) becomes

$$(2.6)\qquad \{\|\phi\|_H^2\int_a^b \exp[-2\nu t]\,dt + |\lambda|^2\|\phi\|_H^2\int_a^b \exp[-2\nu t]\,dt\} \leqslant K^2\left(\|\phi\|_H^2\int_a^b \exp[-2\nu t]\,dt + \|(i\lambda - A)\phi\|_H^2\int_a^b \exp[-2\nu t]\,dt\right)$$

This obviously simplifies to

$$(2.7)\qquad \|\phi\|_H^2(1 + |\lambda|^2) \leqslant K^2(\|\phi\|_H^2 + \|(i\lambda - A)\phi\|_H^2)$$

and also

$$(2.8)\qquad \frac{1 + |\lambda|^2}{K^2} \leqslant \left(1 + \left\|\frac{(i\lambda - A)\phi}{\|\phi\|_H}\right\|_H^2\right)$$

and

$$(2.9)\qquad \frac{\|(i\lambda - A)\phi\|_H^2}{\|\phi\|_H^2} \geqslant \frac{1 + |\lambda|^2}{K^2} - 1 = \frac{1 + |\lambda|^2 - K^2}{K^2}\,.$$

Assume $|\lambda| \geqslant K$: then $\dfrac{1 + |\lambda|^2 - K^2}{K^2} \geqslant \dfrac{1}{K^2}$ and (2.9) becomes

$$(2.10)\qquad \frac{\|(i\lambda - A)\phi\|_H}{\|\phi\|_H} \geqslant \frac{1}{K}, \qquad \forall\phi \in \mathfrak{D}(A)$$

which proves our Theorem 1.

A variant to Theorem 1, similar to Theorem 11.1, page 145 in [4] is given in

THEOREM 2. *Let us suppose* R.H. *true. Then, there exists a positive constant* $N > 0$ *such that for* $\lambda \in \mathbf{C}$ *and* $|\lambda| \geqslant N$, *the inequality*

$$(2.11)\qquad \|(i\lambda - A)\phi\|_H \geqslant C|\lambda|\,\|\phi\|_H\,, \qquad \forall\phi \in \mathfrak{D}(A)$$

is verified, $C > 0$ *being a positive constant.*

The proof is similar to that of Theorem 1. We obtain again

$$\frac{\|(i\lambda - A)\phi\|_H^2}{\|\phi\|_H^2} \geqslant \frac{1+|\lambda|^2}{K^2} - 1 .$$

Now, $\dfrac{1+|\lambda|^2}{K^2} - 1 \geqslant \dfrac{|\lambda|^2}{2K^2}$ iff $\dfrac{1}{2K^2}|\lambda|^2 \geqslant 1 - \dfrac{1}{K^2}$ iff $|\lambda|^2 \geqslant 2K^2 - 2$ iff $|\lambda| \geqslant \sqrt{2(K^2-1)}$ (when $K > 1$ as we can always assume). Hence, for $|\lambda| \geqslant \sqrt{2(K^2-1)}$ we have

$$\frac{\|(i\lambda - A)\phi\|_H}{\|\phi\|_H} \geqslant \frac{|\lambda|}{K\sqrt{2}} \quad \text{or} \quad \|(i\lambda - A)\phi\|_H \geqslant \frac{1}{K\sqrt{2}}|\lambda|\|\phi\|_H \qquad \forall \phi \in \mathfrak{D}(A) .$$

This proves Theorem 2.

3. The results expressed in Theorems 1 and 2 give necessary conditions for regularity in an interval (a, b) up to the boundary of this interval; this has permitted us to derive the estimates (2.11) and (2.3) for all *complex* numbers λ lying outside some disk in the complex plane.

If we restrict our requests and look only for interior regularity results (of the form; $u \in L^2(a, b; H)$ and $(\omega L)u \in L^2(a, b, H) \Rightarrow \omega(d/dt)u \in L^2(\alpha, \beta; H)$ for any $a < \alpha < \beta < b$) then we obtain estimates which are valid for only *real* values of λ outside a certain interval. We state the precise result in form of

THEOREM 3. *Let us assume A to be a closed linear operator in the Hilbert space H with dense domain $\mathfrak{D}(A)$ and let be $L = d/dt - A$ and ωL the strong and weak abstract differential operators associated to it. Let $a < \alpha < \beta < b$. Assume that if $u \in L^2(a, b; H)$ belongs to $\mathfrak{D}(\omega L)$ — so that $(\omega L)u = f \in L^2(a, b; H)$, — then the weak derivative $\omega(d/dt)u$ exists and belongs to $L^2(\alpha, \beta; H)$. It follows that*

$$\|(i\lambda - A)\phi\| \geqslant C|\lambda|\|\phi\| \quad \text{if } \lambda \text{ real}, \ |\lambda| \geqslant N , \ \phi \in \mathfrak{D}(A) \tag{3.1}$$

where C, N are some positive constants.

The proof is similar to the one exposed in Theorems 1 and 2. In the same way, we obtain the *a priori* estimate

$$\left(\|u\|_{L^2(\alpha,\beta;H)}^2 + \left\|\omega\frac{du}{dt}\right\|_{L^2(\alpha,\beta;H)}^2\right)^{\frac12} \leqslant K\left(\|u\|_{L^2(a,b;H)}^2 + \|(\omega L)u\|_{L^2(a,b;H)}^2\right)^{\frac12} \qquad \forall u \in \mathfrak{D}(\omega L) . \tag{3.2}$$

Again, we shall apply (3.2) to functions $u(t) = \exp[i\lambda t]\phi$, where $\phi \in \mathfrak{D}(A)$ and λ is any real number. We obtain the inequality

$$\{\|\phi\|^2(1+|\lambda|^2)(\beta-\alpha)\} \leqslant K^2\{\|\phi\|^2(b-a) + \|(i\lambda - A)\phi\|^2(b-a)\} \tag{3.3}$$

hence

$$\|\phi\|^2\left(1+|\lambda|^2 - K^2\frac{b-a}{\beta-\alpha}\right) \leqslant K^2\frac{b-a}{\beta-\alpha}\|(i\lambda-A)\phi\|^2 \tag{3.4}$$

and also

$$\frac{\|(i\lambda-A)\phi\|^2}{\|\phi\|^2} \geqslant \left(\frac{1+|\lambda|^2}{K^2((b-a)/(\beta-\alpha))} - 1\right), \tag{3.5}$$

for any real λ and $\phi \in \mathfrak{D}(A)$.

Remark now that

$$\lim_{|\lambda|\to\infty} \frac{1}{|\lambda|^2}\left(\frac{1+|\lambda|^2}{K^2((b-a)/(\beta-\alpha))} - 1\right) = \frac{1}{K^2((b-a)/(\beta-\alpha))} > 0\,.$$

Hence,

$$\frac{1}{|\lambda|^2}\left(\frac{1+|\lambda|^2}{K^2((b-a)/(\beta-\alpha))} - 1\right) > \frac{1}{2K^2((b-a)/(\beta-\alpha))} \quad \text{for } |\lambda| \geqslant \Lambda$$

so that

$$\frac{\|(i\lambda-A)\phi\|^2}{\|\phi\|^2} > \frac{|\lambda|^2}{2K^2((b-a)/(\beta-\alpha))} \quad \text{for } |\lambda| \geqslant \Lambda$$

and

$$\|(i\lambda-A)\phi\| > \frac{|\lambda|}{\sqrt{2}K\sqrt{(b-a)/(\beta-\alpha)}}\|\phi\| \tag{3.6}$$

for $|\lambda| \geqslant \Lambda$, λ real and $\phi \in \mathfrak{D}(A)$, q.e.d.

We shall consider now necessary conditions for regularity in more general Banach spaces (see A. Friedman, Th. 11.1, page 145 of [4]).

Following definitions of our paper [15] let us consider a Banach space $\mathfrak{X}$, $\mathfrak{X}^*$ its dual space and a linear closed operator A with dense domain $\mathfrak{D}(A) \subset \mathfrak{X} \to \mathfrak{X}$; let be A^* the dual operator with usual definition; given a (finite) interval $(a, b) \subset \mathbf{R}$, define the class $K_{A^*}(a, b)$ of test-functions consisting of continuously differentiable functions $\phi(t)$, $a \leqslant t \leqslant b \to \mathfrak{X}^*$ having compact support in the open interval (a, b); $\phi(t) \in \mathfrak{D}(A^*)$, $\forall t$ and $A^*\phi$ is $\mathfrak{X}^*$-continuous in (a, b).

Associated to the (classical) abstract differential operator $L = (d/dt) - A$ defined on $u(t) \in C^1\{(a, b); \mathfrak{X}\} \cap C^0\{[a, b]; \mathfrak{X}\}$ such that $u(t) \in \mathfrak{D}(A)$ $\forall t \in (a, b)$ and $(Au)(t) \in C^0\{(a, b); \mathfrak{X}\}$ we consider the weak extension ωL which is defined in the following way:

The domain $\mathfrak{D}(\omega L)$ consists of those $u(t) \in C([a, b]; \mathfrak{X})$ such that there exists $f(t) \in C([a, b]; \mathfrak{X})$ with property that

$$(3.7) \quad \int_a^b \langle \phi'(t) + A^*\phi(t), u(t)\rangle\, dt = -\int_a^b \langle \phi(t), f(t)\rangle\, dt \quad \forall \phi \in K_{A^*}(a, b)$$

is satisfied. Then, by definition, $(\omega L)u \ni f$. We have

PROPOSITION 5. *The map $u \to (\omega L)u$ is single- valued.*

In fact, otherwise we would have, say, $f_1, f_2 \in C([a, b]; \mathfrak{X})$, such that $(\omega L)u \ni f_1$ and $(\omega L)u \ni f_2$. This means, using (3.7), the equality

$$(3.8) \qquad \int_a^b \langle \phi(t), f_1(t)\rangle\, dt = \int_a^b \langle \phi(t), f_2(t)\rangle\, dt \qquad \forall \phi \in K_{A^*}(a, b)\, .$$

Take here $\phi(t) = \nu(t)\chi^*$ where $\nu(t) \in C_0^1(a, b)$ and $\chi^* \in \mathfrak{D}(A^*)$. We obtain equality

$$(3.9) \quad \int_a^b \langle \chi^*, f_1(t) - f_2(t)\rangle \nu(t)\, dt = 0 \qquad \forall \nu(t) \in C_0^1(a, b)\, ,\ \forall \chi^* \in \mathfrak{D}(A^*)$$

Now, remark that $\langle \chi^*, f_1(t) - f_2(t)\rangle \in C[a, b]$; as usual (3.9) will imply that $\langle \chi^*, f_1(t) - f_2(t)\rangle = 0$ $\forall t \in [a, b]$ and $\forall \chi^* \in \mathfrak{D}(A^*)$.

But, as well known, the domain $\mathfrak{D}(A^*)$ is a total set in $\mathfrak{X}^*$. Consequently, we obtain that $f_1(t) = f_2(t)$ on $[a, b]$.

It is easy to see (cf. Prop. 2) the following

PROPOSITION 6. *The domain $\mathfrak{D}(\omega L)$ is a linear set in $C([a, b]; \mathfrak{X})$ and the operator (ωL), $\mathfrak{D}(\omega L) \to C([a, b]; \mathfrak{X})$ is a linear map.*

Finally, we have

PROPOSITION 7. *The operator* (ωL) *is a closed operator.*

Let $(u_n)_1^\infty \subset \mathfrak{D}(\omega L)$, $u_n \to u$ in $C([a, b]; \mathfrak{X})$ and $(\omega L)u_n = f_n \to f$ in $C([a, b]; \mathfrak{X})$. Then we have

$$\int_a^b \langle \phi'(t) + (A^*\phi)(t), u_n(t)\rangle\, dt = -\int_a^b \langle \phi(t), f_n(t)\rangle\, dt\,,$$
$$\forall \phi \in K_{A^*}(a, b)\,,\ n = 1, 2, \ldots$$

as $n \to \infty$ we get readily

$$(3.10)\qquad \int_a^b \langle \phi'(t) + (A^*\phi)(t), u(t)\rangle\, dt = -\int_a^b \langle \phi(t), f(t)\rangle\, dt \qquad \forall \phi \in K_{A^*}(a, b)$$

which means that $u \in \mathfrak{D}(\omega L)$ and $(\omega L)u = f$.

We shall now prove a counterpart to the previous Theorem 3, in the following

THEOREM 4. *Let be* A *a linear closed operator with dense domain* $\mathfrak{D}(A)$ *in the Banach space* $\mathfrak{X}$, *and* $(\omega L) = \omega(d/dt - A)$ *the above defined weak extension. Let* $a < \alpha < \beta < b$. *Assume that if* $u \in C([a, b]; \mathfrak{X})$ *and* $u \in \mathfrak{D}(\omega L)$ — *so that* $(\omega L)u = f \in C([a, b]; \mathfrak{X})$, — *then* $u \in C^1([\alpha, \beta]; \mathfrak{X})$ *It follows*

$$(3.11)\qquad \|(i\lambda - A)\chi\|_{\mathfrak{X}} \geqslant C|\lambda|\,\|\chi\|_{\mathfrak{X}}\,, \qquad \text{if } \lambda \text{ real, } |\lambda| \geqslant N\,,\ \chi \in \mathfrak{D}(A)$$

where C, N *are some positive constants.*

PROOF. Consider the graph $G_{\omega L}$ as a subset of $C([a, b]; \mathfrak{X}) \times C([a, b]; \mathfrak{X})$ consisting of pairs $\{u, (\omega L)u\}_{u \in \mathfrak{D}_{\omega L}}$; *it is a Banach space as easily seen (with usual graph topology).*

Consider now the mapping W from $G_{\omega L}$ into $C^1([\alpha, \beta]; \mathfrak{X})$ defined by: $W\{u, (\omega L)u\} = u$. By the assumptions of the theorem, this map is well-defined on the whole space $G_{\omega L}$. Furthermore, it is a closed map. Let in fact be: $\{u_n, (\omega L)u_n\} \subset G_{\omega L}$, converges to $\{u_0, v_0\} \in G_{\omega L}$ and $W\{u_n, (\omega L)u_n\} = u_n$ converges to u_1 in $C^1([\alpha, \beta]; \mathfrak{X})$.

Actually, $\{u_0, v_0\} \in G_{\omega L}$ so that $u_0 \in \mathfrak{D}(\omega L)$ and $(\omega L)u_0 = v_0$. Hence $u_n \to u_0$ in $C([a, b]; \mathfrak{X})$, $(\omega L)u_n \to (\omega L)u_0$ in $C([a, b]; \mathfrak{X})$ and also $u_n \to u_1$ in $C^1([\alpha, \beta]; \mathfrak{X})$. This implies $u_1 = u_0$ in $[\alpha, \beta]$. Hence $W\{u_n, (\omega L)u_n\} \to W\{u_0, (\omega L)u_0\} = u_0$ in $C^1([\alpha, \beta]; \mathfrak{X})$.

Using the closed graph theorem we conclude that W is a continuous mapp, that is

$$\|u\|_{C^1([\alpha,\beta];\mathfrak{X})} \leq K\left(\|u\|_{C([a,b];\mathfrak{X})} + \|(\omega L)u\|_{C([a,b];\mathfrak{X})}\right) \qquad \forall u \in \mathfrak{D}(\omega L) \tag{3.12}$$

K being a positive constant.

Let us take now $u(t) = \chi \exp[i\lambda t]$, where $\chi \in \mathfrak{D}(A)$ and λ is a real number. Obviously.

$$(\omega L)u = \left(\frac{d}{dt} - A\right)u = i\lambda \exp[i\lambda t]\chi - \exp[i\lambda t]A\chi = \exp[i\lambda t](i\lambda - A)\chi .$$

In the space $C^1([\alpha, \beta]; \mathfrak{X})$ we use the norm

$$\|u\|_{C^1([\alpha,\beta];\mathfrak{X})} = \max_{j=0,1} \max_{\alpha \leq t \leq \beta} \|u^{(j)}(t)\| .$$

Consequently we obtain

$$\|\chi \exp[i\lambda t]\|_{C^1([\alpha,\beta];\mathfrak{X})} = \max\left\{\max_{\alpha \leq t \leq \beta} \|\chi \exp[i\lambda t]\|, \max_{\alpha \leq t \leq \beta} \|i\lambda \exp[i\lambda t]\chi\|\right\} = $$
$$= \max\{\|\chi\|, |\lambda|\|\chi\|\}$$

which $= |\lambda|\|\chi\|$ if $|\lambda| \geq 1$ which we shall assume.

On the other hand

$$\|u\|_{C([a,b];\mathfrak{X})} = \|\chi\|; \quad \|(\omega L)u\|_{C([a,b];\mathfrak{X})} = \sup_{a \leq t \leq b} \|\exp[i\lambda t](i\lambda - A)\chi\| =$$
$$= \|(i\lambda - A)\chi\| ,$$

Hence, it follows that, for $|\lambda| \geq 1$,

$$|\lambda|\|\chi\| < K(\|\chi\| + \|(i\lambda - A)\chi\|), \qquad \forall \chi \in \mathfrak{D}(A) .$$

This transforms into

$$\|(i\lambda - A)\chi\| \geq \frac{(|\lambda| - K)\|\chi\|}{K}, \quad |\lambda| \geq 1, \qquad \chi \in \mathfrak{D}(A) . \tag{3.13}$$

Remark now that $\dfrac{|\lambda| - K}{K|\lambda|} > \dfrac{1}{2K}$ iff $\dfrac{|\lambda| - K}{|\lambda|} > \dfrac{1}{2}$ iff $|\lambda| - K > \dfrac{|\lambda|}{2}$ iff

$\frac{|\lambda|}{2} > K$ iff $|\lambda| > 2K$. Hence, for $|\lambda| > 2K$, $\frac{|\lambda| - K}{K} > \frac{|\lambda|}{2K}$ and $\|(i\lambda - A)\chi\| \geqslant \frac{1}{2K} |\lambda| \|\chi\|$, $\forall \chi \in \mathfrak{D}(A)$, $|\lambda| > 2K$, $|\lambda| \geqslant 1$.

This proves Theorem 4.

4. In this final section we shall see how some simple modifications of the argument used in our previous paper [14] permits us to prove interior regularity on finite intervals for weak solutions in Hilbert spaces.

Consider as in [14] a Hilbert space H and a linear closed operator A, $\mathfrak{D}(A) \subset H \to H$, where $\mathfrak{D}(A)$ is dense in H. Assume that the inverse operator $(i\lambda - A)^{-1}$ exists and belongs to $\mathfrak{L}(H; H)$ for any real λ with $|\lambda| > N > 0$, and that the estimate

$$\|(i\lambda - A)^{-1}\| \leqslant \frac{C}{|\lambda|}, \quad |\lambda| > N \text{ is verified .} \tag{4.1}$$

Consider now a finite interval in $\mathbf{R}$, $-\infty < a < b < \infty$ and define as above the class of test-functions $K_{A^*}(a, b)$. Next, consider a pair of functions: $u(t) \in L^2(a, b; H)$, $f(t) \in L^2(a, b; H)$ connected through the integral identity

$$\int_a^b \big(u(t), \varphi'(t) + (A^*\varphi)(t)\big)_H dt = -\int_a^b \big(f(t), \varphi(t)\big)_H dt, \quad \forall \varphi \in K_{A^*}(a, b). \tag{4.2}$$

Then we shall prove the following result of interior regularity:

THEOREM 5. *If* (4.1) *and* (4.2) *are verified, then $u(t)$ equals almost-everywhere in (a, b) a strongly continuous function; $u(t)$ belongs to $\mathfrak{D}(A)$ almost-everywhere in (a, b); the strong derivative $u'(t)$ exists almost-everywhere in (a, b) and belongs to $L^2(\alpha, \beta; H)$ for any $a < \alpha < \beta < b$; $Au(t)$ also belongs to $L^2(\alpha, \beta; H)$ for any $a < \alpha < \beta < b$. The equality $u' = Au + f$ holds almost-everywhere on (a, b).*

PROOF. Let us consider a real-valued function $\zeta(t) \in C_0^\infty(\mathbf{R})$, which equals one for $a + \delta \leqslant t \leqslant b - \delta$, equals 0 for $-\infty < t \leqslant a + \delta/2$, $b - \delta/2 \leqslant t < \infty$, and $0 \leqslant \zeta \leqslant 1$ for any $t \in \mathbf{R}$.

Define then a function $v(t)$, $\mathbf{R} \to H$ by the formula: $v(t) = \zeta(t) u(t)$ $a \leqslant t \leqslant b$, $v(t) = \theta$, for $t \notin [a, b]$. Then $v = u$ on $[a + \delta, b - \delta]$ and v has compact support on $\mathbf{R}$. Now we can state

Lemma. *Let be* $h(t) = \zeta'(t)u(t) + \zeta(t)f(t)$ *for* $a \leqslant t \leqslant b$, $h(t) = \theta$ *outside* (a, b). *Then, the integral identity*

$$\int_{\mathbf{R}} \big(v(t), \varphi'(t) + (A^*\varphi)(t)\big)\, dt = -\int_{\mathbf{R}} \big(h(t), \varphi(t)\big)\, dt \tag{4.3}$$

is verified for $\varphi \in K_{A^*}(\mathbf{R}) = K_{A^*}$.

(K_{A^*} is defined in a similar way to $K_{A^*}(a, b)$—see [14]).

Proof. Let us take any $\varphi \in K_{A^*}$. We have

$$\int_{\mathbf{R}} \big(v(t), \varphi'(t) + (A^*\varphi)(t)\big)\, dt = \int_a^b \big(\zeta(t)u(t), \varphi'(t) + (A^*\varphi)(t)\big)\, dt = \tag{4.4}$$

$$\int_a^b \big(u(t), \zeta(t)\varphi'(t) + \zeta(t)(A^*\varphi)(t)\big)\, dt = \int_a^b \Big(u(t), (\zeta\varphi)'(t)$$

$$+ A^*\big(\zeta(t)\varphi(t)\big)\Big)\, dt - \int_a^b \big(u(t), \zeta'(t)\varphi(t)\big)\, dt$$

Now, it is easy to see that $\varphi \in K_{A^*}(\mathbf{R}) \Rightarrow \zeta\varphi \in K_{A^*}(a, b)$. Using (4.2) we shall obtain

$$\int_a^b \big(u(t), (\zeta\varphi)'(t) + A^*(\zeta\varphi)(t)\big)\, dt = -\int_a^b \big(f(t), \zeta(t)\varphi(t)\big)\, dt \tag{4.5}$$

and hence

$$\int_{\mathbf{R}} \big(v(t), \varphi'(t) + (A^*\varphi)(t)\big)\, dt = -\int_a^b \Big[\big(f(t), \zeta(t)\varphi(t)\big) \tag{4.6}$$

$$+ \big(u(t), \zeta'(t)\varphi(t)\big)\Big]\, dt = -\int_a^b \big(\zeta(t)f(t) + \zeta'(t)u(t), \varphi(t)\big)\, dt =$$

$$= -\int_{\mathbf{R}} \big(h(t), \varphi(t)\big)\, dt\,,$$

which proves Lemma.

From now on we can deduce, exactly as in [14], that the Fourier transform $\hat{v}(\lambda)$ belongs to $L^1(\mathbf{R}; H)$, so that $v(t)$ equals almost everywhere on $\mathbf{R}$ a strongly continuous function. Hence $u(t)$, which equals $v(t)$ on $[a+\delta, b-\delta]$ will be continuous on this interval, outside a null-set $\mathcal{E}_\delta$. If we take a sequence $\delta_n = 1/n$, we find a sequence of null-sets $\mathcal{E}_n \subset [a+1/n, b-1/n]$, such that $u(t)$ is continuous for $t \in [a+1/n, b-1/n]/\mathcal{E}_n$. Then, obviously $u(t)$ is continuous on $(a, b)/\bigcup_1^\infty \mathcal{E}_n$, that is almost-everywhere on $[a, b]$. The remaining of the proof follows the lines in [14] with a few minor modifications.

REFERENCES

[1] S. AGMON - L. NIRENBERG, *Properties of solutions of ordinary differential equations in Banach spaces*, Comm. Pure Appl. Math., **16**, no. 2 (May 1963), pp. 121-239.

[2] V. BARBU, *On regularity of weak solutions of abstract differential equations in Hilbert spaces*, Acc. Naz. dei Lincei, Rendiconti Classe Scienze, Serie VIII, **45**, fasc. 3-4 (1968).

[3] V. BARBU, *On the regularity of the weak solutions of abstract differential equations*, Osaka J. Math., **6** (1969), pp. 49-56.

[4] A. FRIEDMAN, *Partial Differential Equations*, Holt, Rinehart and Winston Inc., 1969.

[5] A. FRIEDMAN, *Differentiability of solutions of ordinary differential equations in Hilbert space*, Pacific J. Math., **16** (1966), pp. 267-271.

[6] E. HILLE - R. S. PHILLIPS, *Functional Analysis and Semi-Groups*, A.M.S. Colloquium Publ., 1957.

[7] L. HÖRMANDER, *Linear Partial Differential Operators* (Third revised printing), Springer-Verlag, New York Inc., 1969.

[8] J. L. LIONS, *Equations differentielles operationnelles et problémes aux limites*, Springer-Verlag, Berlin and New York, 1961.

[9] I. P. NATANSON, *Theory of Functions of a Real Variable* (in Russian), Moscow, 1950.

[10] F. TREVES, *Basic Linear Partial Differential Equations*, Academic Press, New York-San Francisco-London, 1975.

[11] K. YOSIDA, *Functional Analysis*, Springer-Verlag, 1965.

[12] S. ZAIDMAN, *Soluzioni quasi-periodiche per alcune equazioni differenziali in spazi hilbertiani*, Ricerche Mat., **13** (1964), pp. 118-134.

[13] S. ZAIDMAN, *Equations differentielles abstraites*, Les Presses de l'Université de Montréal, Montréal, Québec, 1966.
[14] S. ZAIDMAN, *Remarks on regularity theorems for weak solutions of abstract differential equations*, Boll. U.M.I. (4), **6** (1972), pp. 55-60.
[15] S. ZAIDMAN, *Remarks on weak solutions of differential equations in Banach spaces*, Boll. U.M.I. (4), **9** (1974), pp. 638-643.

Manoscritto pervenuto in redazione il 22 marzo 1979.