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Rendiconti del Seminario Matematico della Università di Padova, tome 62 (1980), p. 251-259

<http://www.numdam.org/item?id=RSMUP_1980__62__251_0>
Abelian Groups in which Every $\Gamma$-Isotype Subgroup is a Pure Subgroup, Resp. an Isotype Subgroup.

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All groups considered in this paper are abelian. Concerning the terminology and notation, we refer to [3]. In addition, if $G$ is a group then $G_t$ and $G_p$ are the torsion part of $G$ and the $p$-component of $G_t$ respectively. Let $G$ be a group and $p$ a prime. Following Rangaswamy [10] we say that a subgroup $H$ of $G$ is $p$-absorbing, resp. absorbing in $G$ if $(G/H)_p = 0$, resp. $(G/H)_t = 0$. A subgroup $H$ of $G$ is said to be isotype in $G$ if $p^a H = H \cap p^a G$ for all primes $p$ and all ordinals $\alpha$. Recall that if $H$ is $p$-absorbing in $G$ then $p^a H = H \cap p^a G$ for every ordinal $\alpha$ (see lemma 103.1 [3]).

Let $N$ be the set of all positive integers, $p_1, p_2, \ldots$ be the sequence of all primes in the natural order and $\mathcal{K}$ the class of all sequences $(\alpha_1, \alpha_2, \ldots)$, where each $\alpha_i$ is either an ordinal or the symbol $\infty$ which is considered to be larger than any ordinal. Let $G$ be a group and $\Gamma = (\alpha_1, \alpha_2, \ldots) \in \mathcal{K}$. A subgroup $H$ of $G$ is said to be $\Gamma$-isotype in $G$ if $p^\beta H = H \cap p^\beta G$ for every $i \in N$ and for every ordinal $\beta < \alpha_i$. If $\Gamma = (0, 0, \ldots)$, $\Gamma = (1, 1, \ldots)$, $\Gamma = (\omega, \omega, \ldots)$, $\Gamma = (\infty, \infty, \ldots)$ then $\Gamma$-isotype subgroups of $G$ are precisely subgroups, neat subgroups, pure subgroups, isotype subgroups respectively. Note that if $\Gamma = (\alpha_1, \alpha_2, \ldots)$, $\Gamma' = (\alpha'_1, \alpha'_2, \ldots) \in \mathcal{K}$ and $\Gamma \leq \Gamma'$ (i.e. $\alpha_i < \alpha'_i$ for each $i \in N$) then every $\Gamma'$-isotype subgroup of $G$ is $\Gamma$-isotype in $G$. Let $G$ be a $p$-group, $\gamma$ be an ordinal or the symbol $\infty$. A subgroup $H$ of $G$ is said to be $\gamma$-isotype in $G$ if $p^\beta H = H \cap p^\beta G$ for every ordinal $\beta < \gamma$.

A direct sum of cyclic groups of the same order $p^\varepsilon$ is denoted by $B_\varepsilon$.

The purpose of this paper is to describe the classes of all groups in which every $\Gamma$-isotype subgroup is a neat, a pure, an isotype subgroup, a direct summand, an absolute direct summand, an absorbing subgroup respectively. Here are so generalized the results of this type from [1], [2], [4], [6]-[9], [11], [13] (see [1]-introduction).

**Lemma 1.** Let $G$ be a torsion group and $\Gamma = (\alpha_1, \alpha_2, \ldots) \in \mathfrak{X}$. A subgroup $H$ of $G$ is $\Gamma$-isotype in $G$ iff $H_{\alpha_i}$ is $\alpha_i$-isotype in $G_{\alpha_i}$ for every $i \in \mathbb{N}$.

**Proof.** Obvious.

**Lemma 2.** Let $G$ be a $p$-group, $g \in G$ an element of order $p^i$, $k \in \mathbb{N}$, $k < i$. The subgroup $\langle g \rangle$ is $k$-isotype in $G$ iff $\langle g \rangle$ is $k$-isotype in $G$. Moreover, the subgroup $\langle g \rangle$ is pure (isotype) in $G$ iff $\langle g \rangle$ is pure (isotype) in $G$.

**Proof.** Easy.

**Lemma 3.** Let $H$ be a subgroup of a group $G$ and $p$ a prime. If $G$ is divisible and $pH = H \cap G$ then $p^xH = H \cap G$ for every ordinal $x$.

**Proof.** Obviously, $H_p$ is neat in $G_p$ and hence $H_p$ is divisible. Write $H = H_p \oplus H'$ and $G = G_p \oplus G'$, where $H' \subset G'$. Since $H'$ is $p$-absorbing in $G'$, the result follows.

**Lemma 4.** Let $G$ be a $p$-group and $k \in \mathbb{N}$. If every $k$-isotype subgroup of $G$ is a pure subgroup of $G$ then either $G = D \oplus B$, where $D$ is divisible and $p^{k-1}B = 0$, or $p^{k-1}G = B_e \oplus B_{e+1}$ for some $e \in \mathbb{N}$.

**Proof.** Let $G = D \oplus B$, where $D$ is nonzero divisible and $B$ is reduced. Suppose $B = \langle a \rangle \oplus B'$, where $o(a) = p^j$ and $j > k$; let $d \in D$ be an element of order $p^{j+1}$. By lemma 2, $\langle a + d \rangle$ is $k$-isotype in $G$ but is not pure in $G$—a contradiction. Hence $p^{k-1}B = 0$.

Let $G$ be reduced. Suppose $G = \langle a \rangle \oplus \langle b \rangle \oplus G'$, where $o(a) = p^j$, $o(b) = p^m$ and $m - 2 > j > k$. By lemma 2, the subgroup $\langle a + pb \rangle$ is $k$-isotype in $G$ but is not pure in $G$—a contradiction. If $B$ is a basic subgroup of $G$ then obviously $G = B = B_1 \oplus \ldots \oplus B_{k-1} \oplus B_m \oplus B_{m+1}$, where $m > k$, and hence $p^{k-1}G = B_e \oplus B_{e+1}$ ($e = m - k + 1$).

**Lemma 5.** Let $G$ be a group, $p$ a prime and $\alpha < \beta$ ordinals. If $p^\beta G_{\alpha}$ is not essential in $p^\alpha G_{\alpha}$ and either $p^{\beta+1}G_{\alpha}$ is nonzero or $p^\beta G$ is not torsion then there is a subgroup $H$ of $G$ with following properties: $H$ is $q$-absorbing in $G$ for every prime $q \neq p$, $p^\gamma H = H \cap p^\gamma G$ for every ordinal $\gamma < \alpha + 1$ and $p^{\beta+1}H \neq H \cap p^{\beta+1}G$.
Proof (see lemma 3 [1]). There is a nonzero element \( n \in p^aG_p[p] \) such that \( \langle n \rangle \cap p^aG_p = 0 \). Let \( g \in p^aG \) such that either \( 0 = pg \in G_p \) or \( o(g) = \infty \). Write \( X = \langle p^aG[p], pg, n + g \rangle \). It is easy to see that \( \langle n \rangle \cap X = 0 \). Let \( H \) be an \( \langle n \rangle \)-high subgroup of \( G \) containing \( X \). By \([5]\), \( p^\gamma H = H \cap p^\gamma G \) for every ordinal \( \gamma < \alpha + 1 \). Since \( p^aG[p] \subseteq H \), \( p^{\alpha+1}H \neq H \cap p^{\alpha+1}G \). By lemma 6 [1], \( H \) is \( q \)-absorbing in \( G \) for every prime \( q \neq p \).

Theorem 1. Let \( G \) be a group and \( \Gamma = (\alpha_1, \alpha_2, \ldots) \in \mathcal{C} \). The following are equivalent:

(i) Every \( \Gamma \)-isotype subgroup of \( G \) is a pure subgroup of \( G \).

(ii) For every \( i \in \mathbb{N} \), if \( \alpha_i < \omega \) then either \( G_{\alpha_i} = D \oplus B \), where \( D \) is divisible and \( p_i^{\omega}B = 0 \), or \( G \) is torsion and \( G_{\alpha_i} \) is elementary or \( G \) is torsion and \( p_i^{\omega}G_{\alpha_i} = B_e \oplus B_{e+1} \) for some \( e \in \mathbb{N} \).

Proof. Assume (i). For each \( i \in \mathbb{N} \), every \( \alpha_i \)-isotype subgroup of \( G_{\alpha_i} \) is \( \Gamma \)-isotype in \( G \) and hence pure in \( G_{\alpha_i} \). By lemma 4 and by \([4]\), \( G_{\alpha_i} \) is as claimed. Suppose \( G \) is not torsion and \( \alpha_i = 0 \) for some \( i \in \mathbb{N} \). If \( g \in G \) is an element of infinite order then \( g \notin \langle p, g, G_{\alpha_i} \rangle \) and a subgroup \( H \) maximal with respect to the properties \( g \notin H \), \( \langle p, g, G_{\alpha_i} \rangle \subseteq H \) is \( \Gamma \)-isotype in \( G \) by lemma 6 [1]. Since \( H \) is pure in \( G \), there is an element \( h \in H \) such that \( p_i g = p_i h \), hence \( g - h \in G_{\alpha_i} \subseteq H \) — a contradiction. Finally, if \( G \) is not torsion, \( \alpha_i < \omega \) for some \( i \in \mathbb{N} \) and \( p_i^{\omega}G_{\alpha_i} = B_e \oplus B_{e+1} \neq 0 \) then \( p_i^{\omega}G_{\alpha_i} \) is not essential in \( p_i^{\omega}G_{\alpha_i} \). \( p_i^{\omega}G \) is not torsion and lemma 5 implies a contradiction.

Assume (ii). Let \( H \) be an \( \Gamma \)-isotype subgroup of \( G \) and \( i \in \mathbb{N} \). Write \( \beta = \alpha_i \) and \( p = p_i \). If \( \beta > \omega \) then \( H \) is \( p \)-pure in \( G \). If \( \beta = 0 \) then by assumption \( G \) is torsion and \( G_p \) is elementary; write \( G = G_p \oplus G' \) and \( H = H_p \oplus H' \). For every \( k \in \mathbb{N} \), \( p^kH = H' = H \cap G' = H \cap p^kG \), i.e. \( H \) is \( p \)-pure in \( G \). Let \( 0 < \beta < \omega \). Suppose that \( p^{\beta-1}G_p = B_e \oplus B_{e+1} \) and \( G \) is torsion. By lemma 1,

\[
p(p^{\beta-1}H_p) = H_p \cap p(p^{\beta-1}G_p) = p^{\beta-1}H_p \cap p(p^{\beta-1}G_p),
\]

i.e. \( p^{\beta-1}H_p \) is neat in \( p^{\beta-1}G_p \). By \([9]\), \( p^{\beta-1}H_p \) is pure in \( p^{\beta-1}G_p \) and hence \( H_p \) is pure in \( G_p \). Consequently, \( H \) is \( p \)-pure in \( G \). Suppose that \( G_p = D \oplus B \), where \( D \) is divisible and \( p^{\beta-1}B = 0 \). Now,

\[
p(p^{\beta-1}H) = H \cap p(p^{\beta-1}G) = p^{\beta-1}H \cap p(p^{\beta-1}G).
\]
Since $p^{\phi-1}G_p$ is divisible, $p^{\phi-1}H$ is $p$-pure in $p^{\phi-1}G$ by lemma 3. Therefore $H$ is $p$-pure in $G$.

**Theorem 2.** Let $G$ be a group and $\Gamma = (\alpha_1, \alpha_2, \ldots) \in \mathcal{K}$. Every $\Gamma$-isotype subgroup of $G$ is a direct summand of $G$ iff the following conditions hold:

(i) $G = T \oplus D \oplus N$, where $T$ is torsion reduced, $D$ is divisible and $N$ is a direct sum of a finite number mutually isomorphic torsion-free rank one groups;

(ii) if $\alpha_i < \omega$ then either $p^{\alpha_i-1}T_{p_i} = 0$ or $G$ is torsion and $G_{p_i}$ is elementary or $G$ is torsion and $p^{\alpha_i-1}G_{p_i} = B_e \oplus B_{e+1}$ for some $e \in \mathbb{N}$;

(iii) if $\omega < \alpha_i$ then $T_{p_i}$ is bounded.

**Proof.** If every $\Gamma$-isotype subgroup of $G$ is a direct summand of $G$ then every isotype subgroup of $G$ is a direct summand of $G$ and every $\Gamma'$-isotype subgroup of $G$ is pure in $G$. Now, theorem 2 [1] and theorem 1 imply (ii). Conversely, by theorem 1, every $\Gamma$-isotype subgroup of $G$ is pure in $G$ and by [2], every pure subgroup of $G$ is a direct summand of $G$.

For the similar result see [12].

**Theorem 3.** Let $G$ be a group and $\Gamma = (\alpha_1, \alpha_2, \ldots) \in \mathcal{K}$. Every $\Gamma$-isotype subgroup of $G$ is an absolute direct summand of $G$ if $G$ satisfies one of the following two conditions:

(i) $G$ is torsion and for every $i \in \mathbb{N}$,

- if $\alpha_i = 0$ then $G_{p_i}$ is elementary,
- if $0 < \alpha_i$ then either $G_{p_i}$ is divisible or $G_{p_i} = B_e$ for some $e \in \mathbb{N}$.

(ii) $\alpha_i \neq 0$ for every $i \in \mathbb{N}$ and either $G$ is divisible or $G = G_i \oplus R$, where $G_i$ is divisible and $R$ is of rank one.

**Proof.** Every $\Gamma$-isotype subgroup of $G$ is an absolute direct summand of $G$ iff every $\Gamma$-isotype subgroup of $G$ is a direct summand of $G$ and every direct summand of $G$ is an absolute direct summand of $G$. Now, theorem 2 and [11] imply the desired result.

**Theorem 4.** Let $G$ be a group and $\Gamma = (\alpha_1, \alpha_2, \ldots) \in \mathcal{K}$. The following are equivalent:

(i) Every $\Gamma$-isotype subgroup of $G$ is a neat subgroup of $G$.

(ii) If $\alpha_i = 0$ for some $i \in \mathbb{N}$ then $G_{p_i}$ is elementary and $G$ is torsion.
PROOF. Assume (i). Every $\alpha_i$-isotype subgroup of $G_{\pi_i}$ is obviously $\Gamma$-isotype in $G$ and hence neat in $G_{\pi_i}$. By [11], if $\alpha_i = 0$ then $G_{\pi_i}$ is elementary. Suppose that $G$ is not torsion and $\alpha_i = 0$ for some $i \in \mathbb{N}$. If $g \in G$ is an element of infinite order then a subgroup $H$ of $G$ maximal with respect to the properties $g \notin H$, $\langle p \cdot g, G_{\pi_i} \rangle \subset H$ is $\Gamma$-isotype in $G$ by lemma 6 [1] but obviously it is not a neat subgroup of $G$.

Assume (ii). If $\alpha_i > 0$ for each $i \in \mathbb{N}$ then every $\Gamma$-isotype subgroup of $G$ is neat in $G$. Suppose $G$ is torsion and if $\alpha_i = 0$ then $G_{\pi_i}$ is elementary. If $H$ is an $\Gamma$-isotype subgroup of $G$ then $H_{\pi_i}$ is $\alpha_i$-isotype in $G_{\pi_i}$ for each $i \in \mathbb{N}$ and hence neat in $G_{\pi_i}$ by [11]. Consequently, $H$ is neat in $G$.

LEMMA 6. Let $G$ be a group, $p$ a prime and $\beta$ an ordinal. Let $H$ be a $p^\alpha G$-high subgroup of $G$ and $a \in p^\beta G$. If $p^\alpha H \neq 0$ for each ordinal $\alpha < \beta$ then there is a subgroup $X$ of $G$ such that $p^\alpha X = X \cap p^\beta G$ for each ordinal $\alpha < \beta$ and $p^\beta X = \langle a \rangle$.

PROOF. If $(o(a), p) = 1$ then write $X = \langle a \rangle$ and all is well. Hence suppose that $p | o(a)$ or $o(a) = \infty$.

If $\beta$ is not a limit ordinal then there is an element $b \in p^\beta G$ such that $pb = a$. If $b \notin p^\beta G$ then write $a_\alpha = b$ for every ordinal $\alpha < \beta$ and $X_\beta = \langle b \rangle$. If $b \in p^\beta G$ and $0 \neq c \in p^{\beta-1} H[p]$ then $b' = b + c \in p^{\beta-1} G \setminus p^\beta G$, $pb' = a$; in this case write $a_\alpha = b'$ for every ordinal $\alpha < \beta$ and $X_\beta = \langle b' \rangle$. Obviously $X_\beta \cap p^\beta G = \langle a \rangle$.

Let $\beta$ be a limit ordinal. For each ordinal $\alpha < \beta$ there is an element $x \in p^\alpha G \setminus p^\beta G$ such that $a = px$. We use the transfinite induction to define the sets $X_\alpha, \alpha < \beta$: $X_0 = \langle a \rangle$; obviously $X_\alpha \cap p^\beta G = \langle a \rangle$ and $(G \cap X_\alpha)[p] \subset \langle a \rangle$. Further, $X_1 = \langle X_0, a_1 \rangle$, where $a_1 \in p^\alpha G \setminus p^\beta G$ and $p a_1 = a$; obviously $X_1 \cap p^\beta G = \langle a \rangle$ and $(pG \cap X_1)[p] \subset \langle a \rangle$. Suppose that $X_{\alpha-1}$ has been defined such that $X_{\alpha-1} \cap p^\beta G = \langle a \rangle$ and $(p^{\alpha-1} G \cap X_{\alpha-1})[p] \subset \langle a \rangle$, define $X_\alpha$. If there is an element $x \in X_{\alpha-1} \setminus p^\alpha G$ such that $px = a$ then let $a_\alpha = x$ and $X_\alpha = X_{\alpha-1}$. Otherwise let $X_\alpha = \langle X_{\alpha-1}, a_\alpha \rangle$, where $a_\alpha \in p^\alpha G \setminus p^\beta G$ and $p a_\alpha = a$. We show that $X_\alpha \cap p^\beta G = \langle a \rangle$. Let $y + za_\alpha \in p^\beta G$, where $y \in X_{\alpha-1}$ and $z$ is an integer. Obviously $py \in X_{\alpha-1} \cap p^\beta G = \langle a \rangle$; write $py = ma$, where $m$ is an integer. If $(p, m) = 1$ then there are integers $u, v$ such that $upa + vma = a$ and hence $a = p(ua + vy)$, $ua + vy \in X_{\alpha-1} \cap p^\alpha G$—a contradiction. Hence $m = pm'$, $p(y - m'a) = 0$ and $y - m'a \in (p^{\alpha-1} G \cap X_{\alpha-1})[p] \subset \langle a \rangle$. Therefore $y \in \langle a \rangle$, $y + za_\alpha \in \langle a_\alpha \rangle \cap p^\beta G = \langle a \rangle$. Further we show that $(p^\alpha G \cap X_\alpha)[p] \subset \langle a \rangle$. Let $y + za_\alpha \in (p^\alpha G \cap X_\alpha)[p]$, where $y \in X_{\alpha-1}$ and $z$ is an integer; hence $py = - za$.
If \((p, z) = 1\) then \(a = p(ua - vy)\), where \(u, v\) are integers, \(ua - vy \in X_{\alpha - 1} \cap \rho^\alpha G\) — a contradiction. Hence
\[z = pz', \quad y + z' a \in (\rho^{\alpha - 1} G \cap X_{\alpha - 1})[p] \subseteq \langle a \rangle\]
and therefore \(y \in \langle a \rangle\). Now, \(y + za \in \langle a \rangle\). Finally, if \(\alpha\) is a limit ordinal then let \(X_\alpha = \bigcup_{\gamma < \alpha} X_\gamma\).

Let \(X\) be a subgroup of \(G\) maximal with respect to the properties:
\(X \cap \rho^\alpha G = \langle a \rangle, \quad X^\beta \subseteq X\). We prove that \(\rho^\alpha X = X \cap \rho^\alpha G\) for every \(\alpha < \beta\). It is sufficient to show that if this equality holds for \(\alpha - 1\) then it holds for \(\alpha\). Let \(x \in X \cap \rho^\alpha G\), i.e. \(x = pq\), where \(g \in \rho^{\alpha - 1} G\).

If \(g \in X\) then \(g \in X \cap \rho^{\alpha - 1} G = \rho^{\alpha - 1} X\) and \(x \in \rho^\alpha X\). If \(g \notin X\) then there is an element \(y \in X\) and an integer \(z\) such that \(zg + y \in \rho^\alpha G \setminus \langle a \rangle\).

Obviously \(y \in \rho^{\alpha - 1} G\) and \((z, p) = 1\). Since \(pzg + py \in X \cap \rho^\beta G = \langle a \rangle\), \(zx + py = ra = rpa_\alpha - 1\) and \(zx = p(ra_\alpha - 1 - y)\). Now, \(ra_\alpha - 1 - y \in X \cap \rho^{\alpha - 1} G = \rho^{\alpha - 1} X\), \(zx \in \rho^\alpha X\) and \(x \in \rho^\alpha X\).

**Lemma 7.** Let \(G\) be a \(p\)-group and \(\beta\) an ordinal. The following are equivalent:

(i) Every \(\beta\)-isotype subgroup of \(G\) is isotype in \(G\).

(ii) Either \(G = D \oplus B\), where \(D\) is divisible and \(\rho^\beta B = 0\) for some ordinal \(\gamma < \beta\), or \(\rho^\beta G\) is elementary or \(\rho^{\beta - 1} G = B_\epsilon \oplus B_{\epsilon + 1}\) for some \(\epsilon \in \mathbb{N}\).

**Proof.** Assume (i). If \(\beta = 0\) then \(G\) is elementary by [4]. If \(\beta\) is a limit ordinal then write \(\alpha = \beta\), otherwise write \(\alpha = \beta - 1\). Let \(\rho^\alpha G = D \oplus R\), where \(D\) is divisible and \(R\) is reduced. If both \(D\) and \(R\) are nonzero, write \(R = \langle a \rangle \oplus R'\), where \(o(a) = p^k, \quad k \in \mathbb{N}\). The subgroup \(\rho^{\alpha + 1} G\) is not essential in \(\rho^\alpha G\), \(\rho^{\alpha + 1} G \neq 0\) and lemma 5 implies a contradiction. If \(\rho^\alpha G\) is reduced and \(\rho^\beta G = \langle a \rangle \oplus \langle b \rangle \oplus R'\), where \(o(a) = p^k, \quad o(b) = p^l\) and \(j - k > 2\), then \(\rho^{\alpha + k} G\) is not essential in \(\rho^\alpha G\), \(\rho^{\alpha + k + 1} G \neq 0\) and lemma 5 implies a contradiction. Consequently, either \(\rho^\alpha G\) is nonzero divisible or \(\rho^\alpha G = B_\epsilon \oplus B_{\epsilon + 1}\) for some \(\epsilon \in \mathbb{N}\). If \(\alpha = \beta - 1\) then we are through, since if \(\rho^\alpha G\) is divisible then \(G = \rho^\alpha G \oplus B\) and obviously \(\rho^\beta B = 0\). Hence suppose \(\alpha = \beta\). Let \(\rho^\beta G\) be nonzero divisible; write \(G = \rho^\beta G \oplus B\). If \(\rho^\beta B \neq 0\) for every ordinal \(\gamma < \beta\) and \(0 \neq a \in \rho^\beta G[p]\) then there is a \(\beta\)-isotype subgroup \(X\) of \(G\) such that \(\rho^\beta X = \langle a \rangle\) by lemma 6. Now, \(\rho^{\beta + 1} X = 0 \neq \langle a \rangle = X \cap \rho^{\beta + 1} G\) — a contradiction. Hence \(\rho^\beta B = 0\) for some ordinal \(\gamma < \beta\). Let \(\rho^\beta G = B_\epsilon \oplus B_{\epsilon + 1}\) and suppose that \(\rho^\beta G\) is not elementary.
If $H$ is $p^β G$-high subgroup of $G$ then $p^γ H \neq 0$ for every ordinal $γ < β$, since $β$ is a limit ordinal. Let $a \in p^{β+1} G[p]$ be a nonzero element. By lemma 6, there is a $β$-isotype subgroup $X$ of $G$ such that $p^{β} X = \langle a \rangle$. Now, $p^{β+1} X \neq X \cap p^{β+1} G$—a contradiction. Hence $p^β G$ is elementary.

Assume (ii). Let $H$ be a $β$-isotype subgroup of $G$. If $p^{β-1} G = B_0 \oplus B_{e+1}$ then

$$p(p^{β-1} H) = p^β H = H \cap p^β G = p^{β-1} H \cap p(p^{β-1} G),$$

hence $p^{β-1} H$ is neat in $p^{β-1} G$ and therefore $p^{β-1} H$ is pure in $p^{β-1} G$ by [9]. Consequently,

$$p^n(p^{β-1} H) = p^{β-1} H \cap p^n(p^{β-1} G) = H \cap p^n(p^{β-1} G)$$

for every natural number $n$ and moreover, if $n \geq e + 1$ then $p^n(p^{β-1} H) = 0$. If $G = D \oplus B$, where $D$ is divisible and $p^γ B = 0$ for some ordinal $γ < β$ then

$$p^γ H = H \cap p^γ G = H \cap p^β G = p^β H.$$

If $p^β G$ is elementary then

$$p^{β+1} H = H \cap p^{β+1} G = 0.$$

In all cases, $H$ is isotype in $G$.

**Theorem 5.** Let $G$ be a group and $Γ = (α_1, α_2, \ldots) \in \mathcal{K}$. The following statements are equivalent:

(i) Every $Γ$-isotype subgroup of $G$ is isotype in $G$.

(ii) For every $i \in \mathbb{N}$, either $G_{p_i} = D \oplus B$, where $D$ is divisible and $p^γ B = 0$ for some ordinal $γ < α_i$, or $p^α G$ is torsion and $p^α G_{p_i}$ is elementary or $p^α G$ is torsion and $p^α G_{p_i} = B_0 \oplus B_{e+1}$ for some $e \in \mathbb{N}$.

**Proof.** Assume (i). Every $α_i$-isotype subgroup of $G_{p_i}$ is isotype in $G_{p_i}$ and hence $G_{p_i}$ is as claimed in (ii) by lemma 7. Let $i \in \mathbb{N}$; write $β = α_i$ and $p = p_i$. Suppose that $p^β G$ is not torsion. If $p^{β-1} G = B_0 \oplus B_{e+1} \neq 0$ then $p^{β+1} G$ is not essential in $p^{β-1} G$ and $p^{β+1} G$ is not torsion. If $p^β G_{p_i}$ is nonzero elementary then $p^{β+1} G_{p_i}$ is not essential in $p^β G_{p_i}$ and $p^{β+1} G$ is not torsion. In these both cases, lemma 5 implies a contradiction.
Suppose that $p^\alpha G$ is not torsion, $p^\alpha G_\alpha = 0$ and $p^\gamma G_{\gamma \neq 0}$ for each ordinal $\gamma < \beta$. Let $a \in p^\alpha G$, $o(a) = \infty$ and $A$ be a $p^\alpha G$-high subgroup of $G$ containing $G_\alpha$. Hence $p^\gamma A_\alpha \neq 0$ for each ordinal $\gamma < \beta$. By lemma 6, there is a subgroup $X$ of $G$ such that $p^\nu X = X \cap p^\nu G$ for every ordinal $\gamma < \beta$ and $p^\sigma X = \langle p\alpha \rangle$. Let $H$ be a subgroup of $G$ maximal with respect to the properties: $X \subset H$, $a \notin H$. By lemma 6.1, $H$ is $q$-absorbing in $G$ for every $q \neq p$. We prove that $p^\nu H = H \cap p^\nu G$ for each ordinal $\gamma < \beta$. It is sufficient to show that if this equality holds for $\gamma - 1 < \beta$ then it holds also for $\gamma$. Let $h \in H \cap p^\nu G$; there is $g \in p^\nu - 1G$ such that $h = pg$. Obviously $h \in p^\nu - 1H$. If $g \in H$ then $h \in p^\nu H$. If $g \notin H$ then $a \in \langle g, H \rangle$, i.e. $a = zg + h'$, where $h' \in H$ and $z$ is an integer. Now, $(z, p) = 1$ and $h' \in H \cap p^\nu - 1G = p^\nu - 1H$. Further, $pa = zh + ph' \in p^\nu X \subset p^\nu X$, there is $x' \in p^\nu - 1X$ such that $zh + ph' = px'$. Hence $zh = p(x' - h')$, where $x' \in H \cap p^\nu - 1H$, and therefore $zh \in p^\nu H$. Now, $ph \in p^\nu H$, $zh \in p^\nu H$ and $(p, z) = 1$ imply $h \in p^\nu H$. Hence $H$ is $\Gamma$-isotype in $G$. Finally, $pa \in H \cap p^{\beta - 1} G \cap p^\beta H$. For, if $pa = py$, where $y \in p^\beta H$, then $a - y \in G_\beta \cap p^\beta G = 0$, $a \in H$ — a contradiction. Consequently, $H$ is not isotype in $G$.

Assume (ii). If $H$ is a $\Gamma$-isotype subgroup of $G$ then $H_\alpha$ is $\Gamma$-isotype in $G_\alpha$ and by lemma 1, each $H_\alpha$ is $\alpha_i$-isotype in $G_\alpha$. By lemma 7, each $H_\alpha$ is isotype in $G_\alpha$ and by lemma 1, $H_\alpha$ is isotype in $G_{\alpha \neq 0}$. Let $i \in \mathbb{N}$, write $\beta = \alpha_i$ and $p = p_i$. If $p^\alpha G$ is torsion then

$$p^\nu H = p^\nu H_\alpha = H_\alpha \cap p^\nu G_\alpha = H_\alpha \cap p^\nu G = H \cap p^\nu G$$

for every $\gamma \geq \beta$. Suppose that $G_\alpha = D \oplus B$, where $D$ is divisible and $p^\nu B = 0$ for some ordinal $\gamma < \beta$. Hence $p^\nu G_\alpha$ and $p^\nu H_\alpha$ are divisible. Write $p^\nu H = p^\nu H_\alpha \oplus Y$. Since $p^\nu G_\alpha \cap Y = 0$, $p^\nu G = p^\nu G_\alpha \oplus X$, where $Y \subset X$. We show that $p^\varepsilon Y = Y \cap p^\varepsilon X$ for each ordinal $\varepsilon$. It is sufficient to show that if this equality holds for $\varepsilon$ then it holds also for $\varepsilon + 1$. Let $y \in Y \cap p^{\varepsilon + 1} X$; there is $x \in p^\varepsilon X$ such that $y = px$. Now, $y \in p^{\varepsilon + 1} G \cap H = p^{\varepsilon + 1} H$, there is $h \in p^\nu H$ such that $y = ph$. Write $h = h' + y'$, where $h' \in p^\nu H_\alpha$ and $y' \in Y$. Since $y = ph' + py'$, $ph' \in Y \cap p^\nu H_\alpha = 0$. Hence $y = py'$, $x - y' \in X_\varepsilon = 0$, $x \in Y \cap p^\varepsilon X = p^\varepsilon Y$ and therefore $y \in p^{\varepsilon + 1} Y$. Finally,

$$p^\varepsilon (p^\nu H) = p^\nu H_\alpha \oplus p^\varepsilon Y = p^\nu H_\alpha \oplus (Y \cap p^\varepsilon X) =$$

$$= p^\nu H \cap (p^\nu G_\alpha \oplus p^\varepsilon X) = p^\nu H \cap p^\varepsilon (p^\nu G) = H \cap p^\varepsilon (p^\nu G)$$

for each ordinal $\varepsilon$. 

\[ \text{Jindřich Bečvář} \]

THEOREM 6. Let $G$ be a nonzero group and $I' = (\alpha_1, \alpha_2, \ldots) \in \mathcal{I}$. The following are equivalent:

(i) Every $I'$-isotype subgroup of $G$ is an absorbing subgroup of $G$.

(ii) Either $G$ is torsion-free and $\alpha_i > 0$ for each $i \in \mathbb{N}$ or $G$ is cocyclic and if $\alpha_i = 0$ for some $i \in \mathbb{N}$ then either $G_{p_i} = 0$ or $G = \mathbb{Z}(p_i)$.

PROOF. Every $I'$-isotype subgroup of $G$ is absorbing in $G$ iff every $I$-isotype subgroup of $G$ is isotype in $G$ and every isotype subgroup of $G$ is absorbing in $G$. Now, theorem 5 and theorem 6 [1] imply the desired result.

REFERENCES


Manoscritto pervenuto in redazione il 6 luglio 1979.