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On the ring of quotients of a noetherian commutative ring with respect to the Dickson topology

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On the Ring of Quotients of a Noetherian Commutative Ring with Respect to the Dickson Topology.

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The aim of this paper is to investigate the structure of the ring of quotients $R_{\mathfrak{D}}$ of a commutative Noetherian ring R with respect to the Dickson topology $\mathfrak{D}$. In particular we study under which conditions $R = R_{\mathfrak{D}}$ or $R_{\mathfrak{D}}$ is the total ring of fractions of R (§ 2), the structure of $R_{\mathfrak{D}}$ when R is a GCD-domain and when R is local and satisfies condition S_2 (§ 3), and the endomorphism ring of the R-module $R_{\mathfrak{D}}/R$ (§ 4).

1. Preliminaries.

The symbol R will be used consistently to denote a commutative Noetherian ring with an identity element.

Let $\mathfrak{D}$ be the Dickson topology on R, that is the Gabriel topology on R consisting of the ideals I of R such that R/I is an Artinian ring, *i.e.* the ideals I of R which contain the product of a finite number of maximal ideals of R (see [12], Chap. VIII, § 2). For every R-module M we put $X(M) = \{x \in M | Ix = 0 \text{ for some } I \in \mathfrak{D}\}$. $X(M)$ is a submodule of M said the $\mathfrak{D}$-torsion submodule of M. The functor X has been studied by E. Matlis ([6]). Let us define

$$M_{\mathfrak{D}} = \varinjlim_{I \in \mathfrak{D}} \operatorname{Hom}_R (I, M/X(M)) ,$$

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where the direct limit is taken over the downwards directed family $\mathfrak{D}$. $M_{\mathfrak{D}}$ is called the module of quotients of M with respect to the topology $\mathfrak{D}$. It is known that $R_{\mathfrak{D}}$ becomes a ring in a natural way and that $M_{\mathfrak{D}}$ becomes an $R_{\mathfrak{D}}$-module.

We shall always suppose that R has no $\mathfrak{D}$-torsion. This is equivalent to request that every maximal ideal of R be dense, that is to request that (since R is Noetherian) every maximal ideal of R contain a regular element. Under such a hypothesis R is a subring of $R_{\mathfrak{D}}$ and $R_{\mathfrak{D}}$ is a subring of Q, the total ring of fractions of R. A more convenient description of $R_{\mathfrak{D}}$ is that $R_{\mathfrak{D}} = \{x \in Q \mid xI \subseteq R \text{ for some } I \in \mathfrak{D}\}$. More precisely we have that

1.1. Lemma. *If R possesses maximal ideals of grade* 1, *then* $R_{\mathfrak{D}} = \{x \in Q \mid$ *there exist maximal ideals* $\mathcal{M}_1, \dots, \mathcal{M}_n$ *in* R *of grade* 1 *such that* $x\mathcal{M}_1 \dots \mathcal{M}_n \subseteq R\}$. *Otherwise* $R_{\mathfrak{D}} = R$.

Proof. It is clear that the products of a finite number of maximal ideals of R form a basis for $\mathfrak{D}$. Therefore $R_{\mathfrak{D}} = \{x \in Q \mid$ there exist maximal ideals $\mathcal{M}_1, \dots, \mathcal{M}_n$ in R such that $x\mathcal{M}_1 \dots \mathcal{M}_n \subseteq R\}$. Hence it is sufficient to prove that if $x \in Q$, $\mathcal{M}_1, \dots, \mathcal{M}_n$ are maximal ideals of R, $x\mathcal{M}_1 \dots \mathcal{M}_n \subseteq R$ and $\operatorname{gr}(\mathcal{M}_n) \neq 1$, then $x\mathcal{M}_1 \dots \mathcal{M}_{n-1} \subseteq R$. For this it is enough to show that if $y \in Q$, $\mathcal{M}$ is a maximal ideal of R, $\operatorname{gr}(\mathcal{M}) \neq 1$ and $y\mathcal{M} \subseteq R$, then $y \in R$. Now $y = s^{-1}r$ for some $r, s \in R$, s regular. Hence from $y\mathcal{M} \subseteq R$ it follows that $r\mathcal{M} \subseteq Rs$, that is $\mathcal{M} \subseteq (Rs : r)$. By the maximality of $\mathcal{M}$, $\mathcal{M} = (Rs : r)$ or $(Rs : r) = R$. In the first case $\operatorname{gr}(\mathcal{M}) = 1$. Therefore $R = (Rs : r)$, *i.e.* $y = s^{-1}r \in R$.

From lemma 1.1 we immediately have a complete description of the grade of all ideals in a local ring R such that $R \neq R_{\mathfrak{D}}$. For such a ring it is easy to prove that $\operatorname{gr}(I) = 1$ for every regular ideal I of R, and $\operatorname{gr}(J) = 0$ for every non-regular ideal J. From lemma 1.1 it is also clear that the study of $R_{\mathfrak{D}}$ is equivalent to the study of the maximal ideals of grade 1 in R.

It is also easy to describe the elements of $R_{\mathfrak{D}}$ in relation to the primary decomposition in R. In fact let $x \in Q$. Then $x \in R_{\mathfrak{D}}$ if and only if x is of the form $s^{-1}r$, where s is a regular element of R, and if $Rs = Q_1 \cap \dots \cap Q_n$ is a normal primary decomposition of the ideal Rs, where Q_i is associated to a prime non-maximal ideal for $i = 1, \dots, t$ and to a maximal ideal for $i = t + 1, \dots, n$, then $r \in Q_1 \cap \dots \cap Q_t$.

Now let us « count » the number of the generators of the ideals in the localization $R_{\mathfrak{D}}$. We have always supposed that the ring R is Noetherian. Of course we cannot hope that this implies the ring of quotients $R_{\mathfrak{D}}$ is Noetherian. There exist rings R such that $R_{\mathfrak{D}}$ possesses ideals which cannot be generated by a finite number of elements. Nevertheless there exists an upper bound for the number of elements needed to generate any ideal of $R_{\mathfrak{D}}$.

1.2. PROPOSITION. *Let R be a ring,* $\mathrm{Max}^{(1)}(R)$ *the set of all maximal ideals in R of grade* 1, *ξ the cardinality of* $\mathrm{Max}^{(1)}(R)$. *Then $R_{\mathfrak{D}}$ is the union of a directed family, of cardinality $\leqslant \aleph_0 \cdot \xi + 1$, of Noetherian subrings of Q. Every ideal of $R_{\mathfrak{D}}$ can be generated by at most $\aleph_0(\xi + 1)$ elements. In particular if R is local, every ideal of $R_{\mathfrak{D}}$ is countably generated.*

PROOF. If $\mathrm{Max}^{(1)}(R) = \emptyset$, by lemma 1.1 $\{R\}$ is the requested family. Hence let us suppose $\mathrm{Max}^{(1)}(R) \neq \emptyset$.

Let I be a regular ideal of R and set $R_{(I)} = R[\{x \in Q | xI \subseteq R\}]$. $R_{(I)}$ is a subring of Q. Let $s \in I$ be a regular element. Then if $x \in Q$, we have that $xI \subseteq R$ if and only if $xs \in R$ and $xs \in (Rs : I)$. Let $r_1, \dots, r_n$ be a set of generators of the ideal $(Rs : I)$ in R. It follows that $xI \subseteq R$ if and only if x is a linear combination of $s^{-1}r_1, \dots, s^{-1}r_n$ with coefficients in R. Therefore $R_{(I)} = R[s^{-1}r_1, \dots, s^{-1}r_n]$ is a Noetherian ring. Now let us consider the family $\mathcal{F}$ of the rings $R_{(I)}$ where I ranges over the set of all products of a finite number of elements of $\mathrm{Max}^{(1)}(R)$. The cardinality of $\mathcal{F}$ is $\leqslant \aleph_0 \cdot \xi$. $\mathcal{F}$ is directed because $R_{(I)} \cup R_{(J)} \subseteq R_{(IJ)}$ and by lemma 1.1 its union is $R_{\mathfrak{D}}$. Finally if $\mathcal{A}$ is an ideal of $R_{\mathfrak{D}}$, $\mathcal{A} = \bigcup_{R_{(I)} \in \mathcal{F}} (R_{(I)} \cap \mathcal{A})$ and hence there exists a set of generators of $\mathcal{A}$ of cardinality $\leqslant \aleph_0 \cdot |\mathcal{F}| \leqslant \aleph_0(\xi + 1)$.

If R and S are Noetherian rings and $R \subseteq S \subseteq R_{\mathfrak{D}}$, it may happen that $S_{\mathfrak{D}} \neq R_{\mathfrak{D}}$. This is not the case if S is integral over R.

1.3. PROPOSITION. *Let R, S be Noetherian rings, $R \subseteq S \subseteq R_{\mathfrak{D}}$, S integral over R. Then $R_{\mathfrak{D}} = S_{\mathfrak{D}}$.*

PROOF. First of all note that R and S have the same total ring of fractions Q. Furthermore if $\mathcal{N}$ is any maximal ideal in S, $\mathcal{N} \cap R$ is a maximal ideal in R and therefore it contains a regular element of R. Hence $\mathcal{N}$ contains a regular element (of S).

Let us show that $R_{\mathfrak{D}} \subseteq S_{\mathfrak{D}}$. Let $x \in R_{\mathfrak{D}}$. Then $x \in Q$ and $x\mathcal{M}_1 \dots \mathcal{M}_n \subseteq R$ for suitable maximal ideals $\mathcal{M}_i$ of R. Hence $x\mathcal{M}_1 \dots \mathcal{M}_n S \subseteq S$. To show that $x \in S_{\mathfrak{D}}$ it is then sufficient to show that $\mathcal{M}_1 \dots \mathcal{M}_n S = (\mathcal{M}_1 S) \dots (\mathcal{M}_n S)$ belongs to the Dickson topology of S. Hence it is enough to prove that $\mathcal{M}_i S$ belongs to the Dickson topology of S, and this is obvious because S is integral over R and hence every minimal prime ideal of $\mathcal{M}_i S$ is a maximal ideal of S.

Vice versa let us show that $S_{\mathfrak{D}} \subseteq R_{\mathfrak{D}}$. Let $x \in S_{\mathfrak{D}}$. Then $x\mathcal{N}_1 \dots \dots \mathcal{N}_n \subseteq S$ for suitable maximal ideals $\mathcal{N}_i$ of S. It follows that $x(\mathcal{N}_1 \cap R) \dots (\mathcal{N}_n \cap R) \subseteq S \subseteq R_{\mathfrak{D}}$. Now every $\mathcal{N}_i \cap R$ is maximal in R because S is integral over R; let $r_1, \dots, r_t$ be a set of generators of the ideal $(\mathcal{N}_1 \cap R) \dots (\mathcal{N}_n \cap R)$ of R. Then $xr_j \in R_{\mathfrak{D}}$, so that there exists an ideal $\mathcal{A}_j$ belonging to the Dickson topology of R such that $xr_j \mathcal{A}_j \subseteq R$. From this we have that $x(\mathcal{N}_1 \cap R) \dots (\mathcal{N}_n \cap R)\mathcal{A}_1 \dots \mathcal{A}_t \subseteq R$ and the ideal $(\mathcal{N}_1 \cap R) \dots (\mathcal{N}_n \cap R)\ \mathcal{A}_1 \dots \mathcal{A}_t$ belongs to the Dickson topology of R. Hence $x \in R_{\mathfrak{D}}$.

The preceding proposition may seem somewhat heavy due to the many hypotheses on R and S. However after proposition 2.2 we shall be able to prove that

1.4. Proposition. *Let R be a local ring. If $R_{\mathfrak{D}} \neq R$ and $R_{\mathfrak{D}} \neq Q$, then there always exists a Noetherian ring $S \subseteq R_{\mathfrak{D}}$, properly containing R and integral over R*

2. The two cases $R_{\mathfrak{D}} = R$ and $R_{\mathfrak{D}} = Q$.

Under our hypotheses (R is a Noetherian ring in which every maximal ideal contains a regular element) we know that $R \subseteq R_{\mathfrak{D}} \subseteq Q$. The first problem which naturally arises is studying under what conditions on R $R_{\mathfrak{D}}$ coincides with R and Q respectively. The case $R_{\mathfrak{D}} = Q$ is handled in theorem 2.1 and the case $R_{\mathfrak{D}} = R$ in theorem 2.2.

2.1. Theorem. *The following statements are equivalent:*

i) $R_{\mathfrak{D}} = Q$;

ii) *every maximal ideal of R has height* 1;

iii) *no proper ideal of $R_{\mathfrak{D}}$ is dense in $R_{\mathfrak{D}}$*;

iv) *R satisfies Stenström's* $\mathfrak{D}$-inv *condition* (see [11]);

v) $\mathfrak{D}$ *is a* 1-*topology* (see [12], Chap. VI, § 6, 1).

Furthermore if R is a reduced ring the preceding statements are also equivalent to:

vi) *$R_{\mathfrak{D}}$ is a (Von Neumann-) regular ring*;

vii) *$R_{\mathfrak{D}}$ is a semisimple ring.*

PROOF. i) $\Rightarrow$ ii). Let $\mathcal{M}$ be a maximal ideal of R. Let $s \in \mathcal{M}$, s regular. Then by i), $s^{-1} \in R_{\mathfrak{D}}$. It follows that $s^{-1} \cdot I \subseteq R$ for some $I \in \mathfrak{D}$, that is $I \subseteq Rs$. But then R/Rs is Artinian and $\mathcal{M}/Rs$ is a prime ideal in R/Rs, hence a minimal prime ideal. Therefore $\mathcal{M}$ is a minimal prime ideal of Rs in R. Since s is regular, $\mathcal{M}$ has height 1.

ii) $\Rightarrow$ i). Let us suppose $R_{\mathfrak{D}} \neq Q$. Then there exists some regular element $s \in R$ non invertible in $R_{\mathfrak{D}}$, *i.e.* such that $s^{-1} \cdot I \not\subseteq R$ for every ideal $I \in \mathfrak{D}$, that is $Rs \notin \mathfrak{D}$. Therefore R/Rs is not Artinian, and hence it has a maximal ideal of height $\geqslant 1$. It follows that R has a maximal ideal of height $\geqslant 2$.

i) $\Rightarrow$ iii). Obvious.

iii) $\Rightarrow$ i). Suppose iii) holds and let us show that if $s \in R$ is regular in R then it is invertible in $R_{\mathfrak{D}}$ (this will prove i)).

Now if $s \in R$ is regular in R, s is invertible in Q and hence regular in $R_{\mathfrak{D}}$. Therefore $R_{\mathfrak{D}} \cdot s$ is a dense ideal of $R_{\mathfrak{D}}$. By iii) $R_{\mathfrak{D}} s = R_{\mathfrak{D}}$, that is s is invertible in $R_{\mathfrak{D}}$.

ii) $\Rightarrow$ v). Let s be a regular element of R. Then every minimal prime ideal of Rs has height 1, and hence by ii) it is maximal. Therefore $Rs \in \mathfrak{D}$. It follows that the filter of all regular ideals is contained in $\mathfrak{D}$ Since every ideal of $\mathfrak{D}$ is regular, $\mathfrak{D}$ is exactly the filter of all regular ideals of R. Hence $\mathfrak{D}$ is a 1-topology.

v) $\Rightarrow$ iv). Trivial.

iv) $\Rightarrow$ ii). Suppose R satisfies Stenstrom's $\mathfrak{D}$-inv condition. Let $\mathcal{M}$ be a maximal ideal of R. Then $\mathcal{M} \in \mathfrak{D}$ and therefore there exists $I \in \mathfrak{D}$ such that $I \subseteq \mathcal{M}$ and I is a projective ideal. Let $I = Q_1 \cap \ldots \cap Q_n$ be a normal primary decomposition of I. Since $I \in \mathfrak{D}$ the minimal prime ideals of I are exactly the maximal ideals of R containing I. Let Q_1 be the $\mathcal{M}$-primary component of I. Localize with respect to the ideal $\mathcal{M}$. Then $IR_{\mathcal{M}}$ is a projective ideal (and hence it is principal generated by a regular element of $R_{\mathcal{M}}$), and $IR_{\mathcal{M}} = Q_1 R_{\mathcal{M}}$. Hence $IR_{\mathcal{M}}$ is a $\mathcal{M}R_{\mathcal{M}}$-primary principal ideal. From this it follows that the height of the ideal $\mathcal{M}R_{\mathcal{M}}$ in $R_{\mathcal{M}}$ is 1. Hence the height of $\mathcal{M}$ is 1.

Now suppose R is reduced, *i.e.* without non-zero nilpotent elements. Then

i) $\Rightarrow$ vii). Trivial, because the total ring of fractions of a reduced Noetherian ring is always semisimple.

vii) $\Rightarrow$ vi). Obvious.

vi) $\Rightarrow$ i). Let s be any regular element of R. Then $s = s^2x$ for some element $x \in R_{\mathfrak{D}}$. It follows that $1 = sx$, *i.e.* $x = s^{-1}$. Therefore $s^{-1} \in R_{\mathfrak{D}}$. Hence $Q = R_{\mathfrak{D}}$.

2.2. THEOREM. *The following statements are equivalent:*

i) $R = R_{\mathfrak{D}}$;

ii) $X\left(\bigoplus_{s\in\Sigma} R/Rs\right) = 0$, *where* Σ *is the set of all regular elements of* R;

iii) *No maximal ideal of* R *is* R*-reflexive* (see [8], § 7);

iv) *No maximal ideal of* R *is associated to an ideal* Rs *with* s *regular element of* R;

v) *No maximal ideal of* R *has grade* 1.

PROOF. i) $\Rightarrow$ ii). Suppose $X\left(\bigoplus_{s\in\Sigma} R/Rs\right) \neq 0$. Then there exists $s \in \Sigma$ such that $X(R/Rs) \neq 0$, that is such that R/Rs has a simple submodule. Let $r + Rs$ be a generator of such a submodule. Then $r \notin Rs$ and $(Rr + Rs)/Rs \cong R/\mathcal{M}$ for some maximal ideal $\mathcal{M}$ of R, and hence $r\mathcal{M} \subseteq Rs$, that is $s^{-1}r\mathcal{M} \subseteq R$. From this it follows that $s^{-1}r \in R_{\mathfrak{D}}$. But $s^{-1}r \notin R$, for otherwise $r \in Rs$. Hence $R \neq R_{\mathfrak{D}}$.

ii) $\Rightarrow$ iii). Let $\mathcal{M}$ be a R-reflexive maximal ideal of R. Then $\mathcal{M} = (Rs : r)$ for some $r, s \in R$, with s regular (see [3], theorem 1.5). Then $(Rr + Rs)/Rs \cong R/(Rs : r) \cong R/\mathcal{M}$ is a simple submodule of R/Rs. It follows that $X\left(\bigoplus_{s\in\Sigma} R/Rs\right) \neq 0$.

iii) $\Rightarrow$ iv). Let $\mathcal{M}$ be a maximal ideal of R associated to the ideal Rs with s a regular element of R. Then $\mathcal{M} = \text{rad}\,(Rs : r)$ for some $r \in R$ (see [1], theorem 4.5), and hence $\mathcal{M}^n \subseteq (Rs : r)$ for some natural number n. Suppose n is the least for which such relation holds. Then $n \geqslant 1$. Let $t \in \mathcal{M}^{n-1}$, $t \notin (Rs : r)$. Then from $\mathcal{M}^n \subseteq (Rs : r)$ it follows that $t\mathcal{M} \subseteq (Rs : r)$, i.e. $\mathcal{M} \subseteq (Rs : rt)$, and from $t \notin (Rs : r)$ it follows that $1 \notin (Rs : rt)$. Hence $\mathcal{M} = (Rs : rt)$ and so $\mathcal{M}$ is R-reflexive ([3], theorem 1.5).

iv) $\Rightarrow$ v). Obvious, because a maximal ideal of grade 1 is associated to an ideal Rs, with s a regular element of R.

v) $\Rightarrow$ i). By lemma 1.1.

Now we are ready to prove proposition 1.4:

Let $\mathcal{M}$ be the maximal ideal of R. Since $R_{\mathfrak{D}} \neq Q$, by theorem 2.1 the height of $\mathcal{M}$ is $\geqslant 2$, and since $R_{\mathfrak{D}} \neq R$ $\mathcal{M}$ is R-reflexive by theorem 2.2.

Therefore there exists $y \in Q$ such that $y\mathcal{M} \subseteq R$, $y \notin R$. From this it follows that $R_{(\mathcal{M})}$, the Noetherian subring of $R_{\mathfrak{D}}$ defined in the proof of proposition 1.2, properly contains R. In order to show that $S = R_{(\mathcal{M})}$ satisfies the thesis of the proposition it is sufficient to show that if $x \in Q$ and $x\mathcal{M} \subseteq R$, then x is integral over R. Since the height of $\mathcal{M}$ is $\geqslant 2$, there exists a regular prime ideal $\mathcal{T}$ in R properly contained in $\mathcal{M}$. Now if $x \in Q$ and $x\mathcal{M} \subseteq R$, then $x\mathcal{M}\mathcal{T} \subseteq \mathcal{T}$. Let us show that $x\mathcal{T} \subseteq \mathcal{T}$. If $x\mathcal{T} \nsubseteq \mathcal{T}$, there would exist $p \in \mathcal{T}$ such that $xp \notin \mathcal{T}$. But $xp \in R$, because $p \in \mathcal{M}$. Let $m \in \mathcal{M} \setminus \mathcal{T}$. Then $xpm \notin \mathcal{T}$, a contradiction because $x\mathcal{T}\mathcal{M} \subseteq \mathcal{T}$. Therefore $x\mathcal{T} \subseteq \mathcal{T}$. Furthermore $\mathcal{T}$ is a faithful R-module (because $\mathcal{T}$ is regular in R) and finitely generated. From this it follows that x is integral over R.

3. The structure of $R_{\mathfrak{D}}$ for some classes of rings R.

One of the classes of rings for which it is possible to give a complete description of the Dickson localization is the class of GCD-domains, that is of integral domains R such that for every $a, b \in R$ there exists a greatest common divisor $[a, b] \in R$ (see [4], page 32).

3.1. THEOREM. *Let R be a* GCD-*domain. Let S be the multiplicatively closed subset of R generated by all elements $s \in R$ such that Rs is a maximal ideal of R. Then $R_{\mathfrak{D}} = S^{-1}R$.*

PROOF. Let us show that in a GCD-domain R every maximal ideal $\mathcal{M}$ of grade 1 is principal. If $\mathcal{M}$ has grade 1, there exists $y \in Q$ such that $y\mathcal{M} \subseteq R$ and $y \notin R$. Let $y = s^{-1}r$ with $r, s \in R$, $r, s \neq 0$, and let $m_1, \ldots, m_n$ be a set of generators of $\mathcal{M}$. Then from $y\mathcal{M} \subseteq R$ it follows that $rm_i \in Rs$ for every $i = 1, \ldots, n$. Therefore s divides rm_i for every i, and hence s divides their greatest common divisor $[rm_1, \ldots, rm_n] = r[m_1, \ldots, m_n]$. Let $d = [m_1, \ldots, m_n]$. Then $\mathcal{M} \subseteq Rd$. Now if $Rd = R$, d would be a unit in R and hence from $s|rd$ it would follow $s|r$ in R, *i.e.* $y \in R$, a contradiction. Since $\mathcal{M}$ is maximal, $\mathcal{M} = Rd$, *i.e.* $\mathcal{M}$ is principal.

By lemma 1.1, $R_{\mathfrak{D}} = \{x \in Q \mid$ there exist maximal ideals $\mathcal{M}_1, \ldots, \mathcal{M}_r$ of grade 1 such that $x\mathcal{M}_1 \ldots \mathcal{M}_r \subseteq R\}$, from which $R_{\mathfrak{D}} = \{x \in Q \mid$ there exist principal maximal ideals $Rs_1, \ldots, Rs_r$ such that $xs_1 \ldots s_r \in R\} = \{x \in Q \mid xs \in R$ for some $s \in S\} = S^{-1}R$.

3.2. PROPOSITION. *Let R be a local ring and suppose that at least one of the following four conditions holds:*

i) *there exists a regular non invertible element $s \in R$ such that the ideal Rs is a radical of R;*

ii) *R satisfies condition S_2* (see [12], Ch. VII, § 6);

iii) *R is a Macaulay ring;*

iv) *R is an integrally closed domain.*

Then if $\dim R = 1$, $R_{\mathfrak{D}} = Q$, *and if* $\dim R > 1$, $R_{\mathfrak{D}} = R$.

PROOF. Suppose that in R there exists a regular non-invertible element s such that the ideal Rs is a radical in R. If $R_{\mathfrak{D}} \neq R$, there exists $x \in R_{\mathfrak{D}}$ such that $x \notin R$ and $x\mathcal{M} \subseteq R$, where $\mathcal{M}$ is the maximal ideal of R. Then $xs \in R$, and hence $x = s^{-1}r$ for some $r \in R$. From this we have that $r\mathcal{M} \subseteq Rs$ and $r \notin Rs$. Therefore in the ring R/Rs the maximal ideal $\mathcal{M}/Rs$ contains only zero-divisors, and since the ring R/Rs is reduced, $\mathcal{M}/Rs$ is contained in the union of the minimal prime ideals of 0 in R/Rs, and hence $\mathcal{M}/Rs$ itself is a minimal prime ideal of 0 in R/Rs. Therefore $\mathcal{M}$ is a minimal prime ideal of Rs in R and hence it has height 1. Thus we have proved that if $\dim R > 1$, then $R_{\mathfrak{D}} = R$.

If $\dim R = 1$, then $R_{\mathfrak{D}} = Q$ by theorem 2.1.

Next if R satisfies S_2, the prime ideals of height $\geqslant 2$ have grade $\geqslant 2$, from which the thesis follows by theorem 2.1 and 2.2.

Finally if R is a Macaulay ring or an integrally closed domain, R satisfies S_2.

4. The endomorphism ring of $R_{\mathfrak{D}}/R$.

We shall now study the R-module $K = R_{\mathfrak{D}}/R$. It is a $\mathfrak{D}$-torsion R-module (it is the $\mathfrak{D}$-torsion submodule of Q/R). Therefore (see [6], theorem 1), it splits in its $\mathcal{M}$-primary components:

$$K = \bigoplus_{\mathcal{M} \in \mathrm{Max}(R)} X_{\mathcal{M}}(K) .$$

$X_{\mathcal{M}}(K)$ is the submodule of K consisting of all elements $x \in K$ such that $\mathrm{Ann}_R(x)$ contains a power of $\mathcal{M}$.

Now let I be a regular ideal. Set $I^{-1} = \{q \in Q | qI \subseteq R\}$ and $I^{-1^{-1}} = \{q \in Q | qI^{-1} \subseteq R\}$. It is known that $I^{-1^{-1}}$ is an ideal of R canonically isomorphic to $\mathrm{Hom}_R(\mathrm{Hom}_R(I, R), R)$, the R-bidual of I. Let $\mathcal{B}$ be a basis of the filter of neighborhoods of zero for a ring topology over R consisting of regular ideals of R. Let us define the bidual topology over R as the topology having $\mathcal{B}^{-1-1} = \{I^{-1-1} | I \in \mathcal{B}\}$ as basis of the filter of neighborhoods of zero. The original topology is finer than the bidual topology.

4.1. THEOREM (see [5], theorem 3.4). *Let R be a ring, $\mathcal{M}$ a maximal ideal of R. For every natural number n let $A_{(\mathcal{M}),n} = \{x \in K | \mathcal{M}^n x = 0\}$. Then*

i) *$A_{(\mathcal{M}),n}$ is a submodule of $X_{\mathcal{M}}(K)$, $A_{(\mathcal{M}),n} \subseteq A_{(\mathcal{M}),n+1}$, $X_{\mathcal{M}}(K) = \bigcup_n A_{(\mathcal{M}),n}$ and $A_{(\mathcal{M}),n} \cong \mathrm{Ext}^1_R(R/\mathcal{M}^n, R)$;*

ii) $\mathrm{Ann}_R A_{(\mathcal{M}),n} = (\mathcal{M}^n)^{-1-1} \cong \mathrm{Hom}_R(\mathrm{Hom}_R(\mathcal{M}^n, R), R)$;

iii) *The non-zero elements of $A_{(\mathcal{M}),n+1}/A_{(\mathcal{M}),n}$ are exactly the elements of $K/A_{(\mathcal{M}),n}$ having annihilator $\mathcal{M}$;*

iv) *$A_{(\mathcal{M}),n+1}/A_{(\mathcal{M}),n}$ is in a natural way a finite dimensional vector space over the field $R/\mathcal{M}$;*

v) *$X_{\mathcal{M}}(K)$ is countably generated.*

PROOF. i) The only non obvious statement is that $A_{(\mathcal{M}),n} \cong \mathrm{Ext}^1_R(R/\mathcal{M}^n, R)$; this is proved by observing that $A_{(\mathcal{M}),n} = (\mathcal{M}^n)^{-1}/R$ and applying the functor $\mathrm{Hom}_R(-, R)$ to the short exact sequence

$$0 \to \mathcal{M}^n \to R \to R/\mathcal{M}^n \to 0 .$$

ii) $\mathrm{Ann}_R A_{(\mathcal{M}),n} = \{r \in R | r(\mathcal{M}^n)^{-1} \subseteq R\} = (\mathcal{M}^n)^{-1-1}$.

iii) Obvious.

iv) and v) From iii) it follows that $A_{(\mathcal{M}),n+1}/A_{(\mathcal{M}),n}$ is a vector space over the field $R/\mathcal{M}$. In order to show that it is finite dimensional it is enough to prove that $A_{(\mathcal{M}),n}$ is a finitely generated R-module (and this with i) immediately gives v)). To this end we only have to show that the R-submodule $(\mathcal{M}^n)^{-1}$ of Q is finitely generated.

Let $s \in \mathcal{M}^n$, s regular and let $q \in (\mathcal{M}^n)^{-1}$. Then $qs \in R$, whence $q = s^{-1}r$ for some $r \in R$. From this we have that

$$(\mathcal{M}^n)^{-1} = \{s^{-1}r | r \in R,\ r\mathcal{M}^n \subseteq Rs\} = \{s^{-1}r | r \in (Rs : \mathcal{M}^n)\} .$$

Since the ideal $(Rs : \mathcal{M}^n)$ of R is finitely generated, it follows that $(\mathcal{M}^n)^{-1}$ is finitely generated.

Let us consider $\mathrm{End}_R\left(X_{\mathcal{M}}(K)\right)$. For every positive integer n set $H_{(\mathcal{M}),n} = \{f \in \mathrm{End}_R\left(X_{\mathcal{M}}(K)\right) | f(A_{(\mathcal{M}),n}) = 0\}$. Let $\mathrm{End}_R\left(X_{\mathcal{M}}(K)\right)$ have the topology defined by the filtration $\{H_{(\mathcal{M}),n}\}_{n\in\mathbb{N}}$ and let $\mathrm{End}_R(K) \simeq \simeq \prod_{\mathcal{M}} \mathrm{End}_R\left(X_{\mathcal{M}}(K)\right)$ have the product topology. Let us call this topology the natural topology of $\mathrm{End}_R(K)$.

4.2. PROPOSITION. $\mathrm{End}_R(K)$ *endowed with its natural topology is a Hausdorff complete topological ring.*

PROOF. It is enough to prove that every $\mathrm{End}_R\left(X_{\mathcal{M}}(K)\right)$ is complete and Hausdorff. Clearly it is Hausdorff. Now if $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{End}_R\left(X_{\mathcal{M}}(K)\right)$ let us define $f \in \mathrm{End}_R\left(X_{\mathcal{M}}(K)\right)$ in the following way: if $x \in A_{(\mathcal{M}),n}$, there exists $r \in \mathbb{N}$ such that for every $r', r'' \in \mathbb{N}$, $r', r'' \geqslant r$, we have $f_{r'} - f_{r''} \in H_{(\mathcal{M}),n}$; let $f(x) = f_r(x)$. The proof that this f is a well-defined homomorphism and that it is limit of the sequence (f_n) is routine.

Now let R be a local ring, $\mathcal{M}$ its maximal ideal. Consider the canonical homomorphism $\varphi\colon R \to \mathrm{End}_R(K)$ which to every element of R associates the multiplication over K by that element. If $\mathrm{End}_R(K)$ has its natural topology and R has the bidual topology of the $\mathcal{M}$-adic topology, it is easy to prove that φ is a continuous ring homomorphism whose kernel is the closure of 0 in R. Therefore φ induces a continuous monomorphism $\tilde{\varphi}\colon R' \to \mathrm{End}_R(K)$, where R' is the « Hausdorffized » of R, that is R modulo the closure of 0 in R with the quotient topology. It is easy to see that $\tilde{\varphi}$ is a topological embedding. Therefore we have proved the following

4.3. PROPOSITION. *Let R be a local ring, $\mathcal{M}$ its maximal ideal and let R have the bidual of the $\mathcal{M}$-adic topology. Then R modulo the closure of zero with the quotient topology is in a natural way a topological subring of* $\mathrm{End}_R(K)$ *endowed with its natural topology.*

Therefore $\mathrm{End}_R(K)$ contains the Hausdorff completion of R.

The following theorem gives a sufficient condition for $\mathrm{End}_R(K)$ to be the Hausdorff completion of R (see [7]).

4.4. THEOREM. *Let R be a local ring, $\mathcal{M}$ its maximal ideal. Suppose that $\mathcal{M}^{-1}$ can be generated by two elements. Then* $\mathrm{End}_R(K)$ *is the Hausdorff completion of R with the bidual topology of the $\mathcal{M}$-adic topology.*

PROOF: If $\mathcal{M}^{-1} = R$, then $(\mathcal{M}^n)^{-1} = R$ for every n. Hence the closure of 0 in R is R and $K = 0$.

Therefore we may suppose $\mathcal{M}^{-1} \neq R$. Then $A_{(\mathcal{M}),1} = \mathcal{M}^{-1}/R \cong R/\mathcal{M}$ (see [7], lemma 2.3, true also when R is not an integral domain). Thus $E(K) \cong E(A_{(\mathcal{M}),1}) \cong E(R/\mathcal{M})$. It follows that if $f \in \mathrm{End}_R(K)$ and $n \in \mathbb{N}$, then f extends to a endomorphism of $E(R/\mathcal{M})$ and $A_{(\mathcal{M}),n} \subseteq B_n$, where B_n is the submodule of $E(R/\mathcal{M})$ consisting of all elements of $E(R/\mathcal{M})$ annihilated by $\mathcal{M}^n$. But f coincides over B_n with the multiplication by an element of R (see [10], lemma 5.11). It follows that $f|A_{(\mathcal{M}),n}$ is the multiplication by an element of R. Hence $\varphi(R)$ is dense in $\mathrm{End}_R(K)$. We conclude by proposition 4.2 and 4.3.

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