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μ -Minkowskianity of the Schwarzschild Universe.

GAETANO ZAMPIERI (*)

1. Brief introduction ⁽¹⁾.

In [1] the author defines the μ -Minkowskian space-time in order to describe precisely insular gravitating systems, by means of certain properties having a physical meaning. Moreover there it was possible to associate in a natural way each μ -Minkowskian space-time with an asymptotic Minkowski space and (the right number of) asymptotic inertial spaces.

In this note the validity of our hypothesis in the case of every space-time admitting the Schwarzschild exterior metric form is proved.

2. Existence of the Minkowski asymptotic space for the Schwarzschild universe.

Let S_4 be a space-time admitting the Schwarzschild exterior metric form ⁽²⁾

$$(1) \quad \Phi = (1 - 2m/r)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) - (1 - 2m/r) dt^2, \\ r \geq a > 2m,$$

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⁽¹⁾ Originally this note constituted an appendix of [1], and it has been printed separately by typographical reasons.

⁽²⁾ See e.g. [2] formula (184) in chapter VII. We refer to this chapter also for possible explanations on the coordinates in (1).

where a is a positive constant and m has the well known meaning. Let us denote by $\mathbf{S}_4^e$ the region of $\mathbf{S}_4$ where Φ_e is defined, and let (x') be a frame such that

$$(2) \quad x'^0 = t, \quad x'^1 = r, \quad x'^2 = \theta, \quad x'^3 = \varphi \quad \text{in } \mathbf{S}_4^e.$$

By (1) the metric tensor satisfies

$$(3) \quad g'_{i0} = 0, \quad g'_{\alpha\beta,0} = 0, \quad g'_{\alpha\beta,3} = 0, \\ \alpha, \beta \in \{0, 1, 2, 3\}, \quad i \in \{1, 2, 3\}, \quad \text{in } \mathbf{S}_4^e.$$

By (3)_{1,2} $\mathbf{S}_4^e$ is static. Moreover, from (3)_{2,3} we have at once that the vectors whose components in (x') are (δ_0^α) and (δ_3^α) respectively, are Killing vectors in $\mathbf{S}_4^e$, i.e. they satisfy the condition $v_{\alpha/\beta} + v_{\beta/\alpha} = 0$. By these equations, along any geodesic in $\mathbf{S}_4^e$, the scalar product of the tangent vector with each Killing vector is constant.

We are interested in space-like geodesics of $\mathbf{S}_4^e$. As is well known there is no loss of generality if we consider only those belonging to the hypersurface $\theta = \pi/2$. By the considerations above each of them satisfies the following equations involving the unit tangent vector $d\mathbf{x}/ds$ and the constants E and A joined to the aforementioned Killing vectors.

$$(4) \quad (1 - 2m/r)(dt/ds) = E,$$

$$(5) \quad r^2(d\varphi/ds) = A,$$

$$(6) \quad (dr/ds)^2 = E^2 + (1 - 2m/r)(1 - A^2/r^2).$$

Let us prove the μ -Minkowskianity—see [1], Def 5.1—of the frame (x) in which the spatial coordinates x^i are obtained from those x'^i in (x') by the usual transformations relating Cartesian to polar coordinates, and the temporal coordinate x^0 coincides with x'^0 .

Let b be a positive constant with $b \geq a$ —see (1)—, and let T_b be the set of all events x such that $r < b$. Observe that T_b is a world-bundle—see (d) in [1], N. 4. The clause (ii) in [1], Def 5.1, is immediately proved for the frame (x) by using world-bundles of the above kind. In fact we see at once that, for some constant k_1 , $|g_{\alpha\beta}(x) - m_{\alpha\beta}| < k_1 r^{-1}$ at any $x \in \mathbf{S}_4^e$ —where $(m_{\alpha\beta})$ is the diagonal matrix $(-1, 1, 1, 1)$.

We shall prove that the clause (i) in [1], Def 5.1, holds for (x) with

respect to $\mathcal{W}_0 = T_b$, where b satisfies

$$(7) \quad b > 3m \quad \text{and} \quad b \geq a.$$

We see at once that, for some positive constant k_2 ,

$$(8) \quad \left| \left\{ \begin{matrix} \alpha \\ \beta \quad \sigma \end{matrix} \right\} (x) \right| \leq k_2 r^{-2}, \quad x \in (\mathbf{S}_4^e)^c (= \mathbf{S}_4 - \mathbf{S}_4^e), \quad \alpha, \beta, \sigma \in \{0, 1, 2, 3\}.$$

Thus, to show that the subclause (i'') holds for (x) , it suffices to prove that for some positive function g , continuous in $\mathcal{W}_0^c$, along every space-like geodesic in $\mathcal{W}_0^c$,

$$(9) \quad g(\bar{\mathcal{E}})s \leq r(s),$$

where s is the arc parameter and $\bar{\mathcal{E}}$ is the origin of the geodesic.

Let us fix one of the aforementioned geodesics l . Along it, (4) to (6) hold for some constant E and $\mathcal{A}$. By (7), $(d^2r/ds^2) = mr^{-2} + (1 - 3m/r)\mathcal{A}^2r^{-3} > 0$, which shows that dr/ds is a strictly increasing function on l . If $(dr/ds)_{s=0} \geq 0$

$$(10) \quad s \leq [b/b - 3m]r(s).$$

This immediately follows from (6) in the case where $\mathcal{A} = 0$. In the contrary case let us set

$$(11) \quad \xi = r(E^2 + 1 - 2m/r)^{\frac{1}{2}}[\mathcal{A}^2(1 - 2m/r)]^{-\frac{1}{2}}.$$

By (7) we see at once that $\xi(r)$ is invertible and

$$(12) \quad 0 < \frac{dr}{d\xi} \leq \frac{r}{\xi} \frac{r - 2m}{r - 3m}.$$

By (6), (11), and (12)

$$(13) \quad s \leq \frac{b - 2m}{b - 3m} \frac{|\mathcal{A}|}{E^2 + 1 - 2m/b} \int_{\xi(0)}^{\xi(s)} \xi(\xi^2 - 1)^{-\frac{1}{2}} d\xi \leq \frac{b - 2m}{b - 3m} \frac{r(s)}{(E^2 + 1 - 2m/b)^{\frac{1}{2}}(1 - 2m/b)^{\frac{1}{2}}}.$$

This prove (10). Now let us consider the case where $(dr/ds)_{s=0} < 0$. In it, $r(s)$ strictly decreases in some interval $[0, \hat{s}]$, and, if the length of l is larger than $\hat{s}$, it is strictly increasing for $s > \hat{s}$. By reasoning as we did in the preceding case, we arrive at the inequality $\hat{s} < [b/b - 3m]r(0)$, which yields immediately $s < [2r(0)/b - 3m]r(s)$. By setting $g(x) = [b - 3m/2r]$, the preceding inequality and (10) prove (9).

Now let us prove that our frame satisfies the subclause (i') in [1], Def 5.1, with respect to the aforementioned choice of $\mathcal{W}_0$.

By [1]—Def 3.3—we must show that the following condition, on the typical space-like geodesic $l \subset \mathcal{W}_0^\circ$ having endpoints, holds. Let $\mathcal{E}_1$ be the origin of l , let $\mathcal{E}_2$ be the other endpoint, let $\boldsymbol{\gamma}$ be the 4-velocity of the ideal fluid joined to the frame (x) —and hence to (x') —, and let $\mathbf{u}(s)$ be the field obtained from $\boldsymbol{\gamma}(\mathcal{E}_1)$ by parallel transport along l ; then for some constant Γ independent of l ,

$$(14) \quad \gamma(l) \equiv -(\mathbf{u} \cdot \boldsymbol{\gamma})(\mathcal{E}_2) \leq \Gamma.$$

It is easily shown that in our case we have

$$d(-\mathbf{u} \cdot \boldsymbol{\gamma}) = mr^{-2}(1 - 2m/r)^{-1}u'^1\gamma'_0 dt.$$

Since $|u'^1/u'^0| < 1$, and $r > b$, we have

$$(15) \quad \ln \gamma(l) \leq \left| \int_i (1 - 2m/b)^{-1} mr^{-2} dt \right|.$$

By the preceding discussion of clause (i''), there is only one point, $\bar{\mathcal{E}} \in l$, such that $r(\bar{\mathcal{E}}) = \inf_i r$. Let $l_1[l_2]$ be the arc of l with origin $\bar{\mathcal{E}}$ and endpoint $\mathcal{E}_1[\mathcal{E}_2]$ ⁽³⁾. Then, by (4) and (15), for some constant E (dependent on the choice of l)

$$(16) \quad \ln \gamma(l) \leq (1 - 2m/b)^{-2}m|E| \sum_{i=1}^2 \int_{l_i} r^{-2} ds.$$

If $\Lambda \neq 0$, (13) holds for each l_i because $(dr/ds)_{s=0} \geq 0$ for them. But the first term in (13) is less than or equal to the third also in the case

⁽³⁾ Obviously one of these arcs can consist of a single point.

where $\Lambda = 0$. Therefore (16) yields

$$\ln \gamma(l) \leq 2(1 - 2m/b)^{-2} m |E| \cdot$$

$$\left[\int_0^{b/|E|} b^{-2} ds + \int_{b/|E|}^{\infty} \left(\frac{b-2m}{b-3m} \right)^2 \frac{s^{-2}}{(1-2m/b)(E^2 + 1 - 2m/b)} ds \right],$$

for $E \neq 0$. Thus

$$\ln \gamma(l) \leq 2(1 - 2m/b)^{-2} \frac{m}{b} \left[1 + \frac{b(b-2m)}{(b-3m)^2} \right],$$

which prove the assertion including (14).

We have thus proved that (x) satisfies all the clauses in [1], Def 5.1. Then, observing that Condition 5.1 in [1] is obviously fulfilled by it, we have that our frame is μ -Minkowskian. This implies the μ -Minkowskianity of every space-time admitting the Schwarzschild exterior metric form, and hence the existence of the Minkowski asymptotic space for such space-times—see [1], N. 8.

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