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JURGEN BIERBRAUER

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On a Certain Class of 2-Local Subgroups in Finite Simple Groups.

JURGEN BIERBRAUER (*)

The object of this paper is to study a class of special 2-groups which occur as the maximal normal 2-subgroups in 2-local subgroups of finite simple groups.

Among these simple groups are the Chevalleygroups $D_n(2)$, $n \geq 4$ and the Steinberg groups ${}^2D_n(2)$, $n \geq 4$ as well as the sporadic groups J_4 and $M(24)'$.

We consider a special group Q_0 of order 2^9 with elementary abelian center of order 8, which admits $\Sigma_3 \times L_3(2)$ as an automorphism group. Let Q^n, $n \geq 1$ denote the automorphism type of the central product of n copies of Q_0. We determine the automorphism group of Q^n and we show, that J_4 contains a maximal 2-local subgroup of the form $Q^2(\Sigma_5 \times L_3(2))$ and that $M(24)'$ contains a maximal 2-local subgroup of the form $Q^2(A_6 \times L_3(2))$. The groups $D_n(2)$ resp. ${}^2D_n(2)$ contain parabolic subgroups of the form $Q^{n-3}(D_{n-3}(2) \times L_3(2))$ resp. $Q^{n-3}({}^2D_{n-3}(2) \times \times L_3(2))$, which are maximal with the exception of the case $D_4(2)$.

These results and several characterizations of the groups Q^n by properties of groups of automorphisms are collected in the first part of the paper. The second part contains a characterization of $M(24)'$ by the 2-local subgroup mentioned above. In [19] Tran van Trung gives an analogous characterization of Janko's group J_4.

Standard notation is like in [6]. In addition D_8 resp. Q_8 denotes the dihedral resp. quaternion group of order 8 and D_8^n resp. Q_8^n the

(*) Indirizzo dell'A.: Mathematisches Institut der Universität Heidelberg - 69 Heidelberg, Im Neuenheimer Feld 288, W. Germany.

central product of n copies of D_8 resp. Q_8. The central product with amalgamated centers of groups H and K is denoted $H * K$.

For a regular matrix A, the transposed-inverse of A shall be written A^*.

If an element g of the group G operates on some vectorspace V with fixed basis, the symbol g_V denotes the matrix giving the operation of g with respect to the fixed basis.

1. Properties of some 2-groups.

(1.1) LEMMA. Let Q be a p-group of class 2 and let N be an automorphism-group of Q such that $(|Q|, |N|) = 1$. Assume further $[Q', N] = 1$. Set $A = C_Q(N)$ and $B = [Q, N]$.

Then we have $Q = A * (BZ(Q))$.

PROOF. This follows from the 3-subgroup-lemma like in the case that Q is an extraspecial 2-group and N a cyclic group of odd order [12, prop. 4].

(1.2) LEMMA. Let Q be a special group of order 2^9 with center Z of order 8. Let n be an element of order 7, which operates fixed-point-freely on Q. Assume further, that $\tilde{Q} = Q/Z$ is the direct sum of two isomorphic irreducible $\langle n \rangle$-modules. Then Q is isomorphic to one of the following groups:

(1) a Suzuki-2-group of type (B);

(2) a central product of two Suzuki-2-groups of type (A) and order 2^6.

(3) a group of type $L_3(8)$;

(4) a group Q_0 which has the following structure:

$$Q_0 = AB , \quad A \cong B \cong E_{64} , \quad A = A_0 \oplus Z \quad \text{and} \quad B = B_0 \oplus Z$$

as $\langle n \rangle$-modules,

$$Z = \langle z_1, z_2, z_3 \rangle , \quad A_0 = \langle x_1, x_2, x_3 \rangle , \quad B_0 = \langle y_1, y_2, y_3 \rangle$$

and

$$[x_i, y_j] = \begin{cases} 1 & \text{for } i = j \\ z_r & \text{for } \{i, j, r\} = \{1, 2, 3\} . \end{cases}$$

Q_0 contains exactly 3 elementary abelian subgroups of order 64, namely A, B and $A+B = Z\langle x_1y_1, x_2y_2, x_3y_3\rangle$.

PROOF. If Z is isomorphic as an $\langle n\rangle$-module to the irreducible submodules of Q, then it follows from [9, (2.5)], that Q is of type $L_3(8)$.. So we shall assume, that Z is not isomorphic to a submodule of Q.

(1) Assume, that $Q - Z$ doesn't contain involutions. Then Q is a Suzuki-2-group of type (B) or (C) in the sense of [7].

If $49 \mid |\mathrm{Aut}\,(Q)|$, there is an element m of order 7 in $\mathrm{Aut}\,(Q)$, which centralizes Z. Because of (1.1), we have $C_Q(m) = Z$. This contradicts a result of Beisiegel [1]. So we have $49 \nmid |\mathrm{Aut}\,(Q)|$ for every Suzuki-2-group of type (B) or (C) and order 2^9.

It follows now from [7], that the Suzuki-2-groups of type (B) and order 2^9 possess an automorphism n with the required properties, whereas the groups of type (C) and order 2^9 don't.

(2) Assume, that $Q - Z$ contains exactly 7×8 involutions. Let $H < Z$, $|H| = 4$. The number of cosets in $\tilde{Q}^{\#}$, which contain elements with square in H, is then $7 + 3\times 56/7 = 31$.

It follows $Q/H \simeq Z_2\times Z_4 * (Q_8)^2$ or $Q/H \simeq E_8\times Z_4 * Q_8$. Let A denote the unique elementary abelian subgroup of order 2^6 of Q.

Assume $Q/H \simeq E_8\times Z_4 * Q_8$. The maximal elementary abelian subgroups of Q/H have order 32 and we have $|A/H \cap Z(Q/H)| \geqslant 8$. It follows, that $|[x, Q]| = 2$ for every $x \in A - Z$. This shows, that $\tilde{A} \underset{\langle n\rangle}{\simeq} Z$, a contradiction. We have $Q/H \simeq Z_2\times Z_4 * Q_8 * Q_8$.

Set $Q/H = \langle x_1\rangle \times \langle v_1\rangle * Q_1 * Q_2H/H$, where $Q_1 \simeq Q_2 \simeq Q_8$.

Then $x_1 \in A - Z$, $v_1^2 \notin H$. Set $B = \langle v_1^Q\rangle$. Then B is isomorphic to the Suzuki-2-group of type (A) and order 2^6. As $[v_1, Q] \subseteq H$, we have $[v_1, Q] = [v_1, B] = H$.

Assume $[x_1, y_1] \neq 1$. Set $[x_1, y_1] = z_1$. We can choose bases $\{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3\}$, $\{\tilde{y}_1, \tilde{y}_2, \tilde{y}_3\}$ and $\{z_1, z_2, z_3\}$ of $\tilde{A}, \tilde{B}$ resp. Z, such that

$$n_{\tilde{A}} = n_{\tilde{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } n_Z = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

We have $[x_1, y_1] = z_1$ and further commutator relations follow by application of the automorphism n. Especially

$$z_1z_3 = [x_1x_2, y_1y_2] = z_1z_2[x_1, y_2][x_2, y_1]$$

and thus $z_2z_3 \in H$. It follows $H = \langle z_1, z_2z_3 \rangle$. On the other hand

$$z_2z_3 = [x_1x_3, y_1y_3] = z_1z_3[x_1, y_3][x_3, y_1]$$

and thus $z_1z_2 \in H$, a contradiction. We have $[x_1, y_1] = 1$. If $|[x, Q]| = 2$ for $x \in A - Z$, we get the contradiction $\tilde{A} \simeq Z$ again. It follows

$$[x_1, Q] = [x_1, B] = H = [y_1, Q] = [y_1, B] .$$

Set $x_2 = x_1^n$, $x_3 = x_2^n$, $y_2 = y_1^n$, $y_3 = y_2^n$. Then we can assume

$$n_{\tilde{A}} = n_{\tilde{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and we have $[x_i, y_i] = [x_ix_j, y_iy_j] = 1$ and thus $[x_i, y_j] = [x_j, y_i]$ for $i, j \in \{1, 2, 3\}$. Set $[x_1, y_2] = [x_2, y_1] = z_1$ and $z_1^n = z_2$, $z_2^n = z_3$. Then

$$n_Z = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

with respect to the basis $\{z_1, z_2, z_3\}$ and

$$[x_1x_3, y_1] = [x_1, y_2]^{n^{-1}} = z_1^{n^{-1}} = z_2z_3 .$$

Thus $[x_1, y_3] = z_2 \cdot z_3$ and $H = \langle z_1, z_2z_3 \rangle$. Further $[x_2, y_3] = z_2$. We have $y_1 \notin H = \langle z_1, z_2z_3 \rangle$. Assume $y_1^2 = z_1z_2$. Then $y_3^2 = z_1$, $(y_1y_3)^2 = = z_1z_2z_3$ and $[y_1, y_3] = y_1^2y_3^2(y_1y_3)^2 = z_1z_3 \notin H$, a contradiction.

Similar calculations show $y_1^2 \neq z_3$ and $y_1^2 \neq z_1z_3$. It follows $y_1^2 = z_2$, $y_2^2 = z_3$, $y_3^2 = z_1z_3$ and the group-table of Q is determined. We have $Q = \langle y_1, y_2, y_3 \rangle * \langle x_2 \cdot y_2, x_3y_3, x_1x_2y_1y_2 \rangle$ and Q is a central product of two Suzuki-2-groups of type (A) and order 2^6.

(3) Assume, that $Q - Z$ contains exactly 14×8 involutions. Then $Q = AB$, where A and B are the only subgroups of Q isomorphic to E_{64}. Let $A = A_0 \oplus Z$ and $B = B_0 \oplus Z$ as $\langle n \rangle$-modules. Choose $x_1 \in A_0^{\#}$, $y_1 \in B_0^{\#}$ and set $z_1 = [x_1, y_1]$. Then $z_1 \neq 1$. Set further $x_2 = x_1^n$,

$x_3 = x_2^n$, $y_2 = y_1^n$, $y_3 = y_2^n$, $z_2 = z_1^n$, $z_3 = z_2^n$. We can assume, that

$$n_{A_0} = n_{B_0} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad n_Z = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

with respect to the bases $\{x_1, x_2, x_3\}$, $\{y_1, y_2, y_3\}$, $\{z_1, z_2, z_3\}$ resp.

As $C_Q(a) = A$ and $C_Q(b) = B$ for elements $a \in A - Z$, $b \in B - Z$, we have $[x_1, y_2] \notin \{1, z_1, z_2\} \not\ni [x_2, y_1]$. As

$$z_1 z_3 = [x_1 x_2, y_1 y_2] = z_1 z_2 [x_1, y_2][x_2, y_1],$$

we have $[x_1, y_2] = z_2 z_3 [x_2, y_1]$ and thus $[x_1, y_2] \notin \{z_2 z_3, z_1 z_2 z_3, z_3\}$. It follows

$$\{[x_1, y_2], [x_2, y_1]\} = \{z_1 z_2, z_1 z_3\}.$$

Assume first, that $[x_1, y_2] = z_1 z_2$, $[x_2, y_1] = z_1 z_3$. Then $[x_2, y_3] = z_2 z_3$, $[x_3, y_2] = z_1 z_2 z_3$. As $z_1 = [x_3, y_1 y_2] = [x_3, y_1][x_3, y_2]$, we have $[x_3, y_1] = z_2 z_3$. From $[x_3, y_2] = z_1 z_2 z_3$ it follows $[x_1 x_2, y_3] = z_1 z_2$. Thus $[x_1, y_3] = z_1 z_3$. Identify A_0, B_0 and Z with the additive group of $GF(8)$, i.e. $A_0 = \{x(\alpha) | \alpha \in GF(8)\}$, $B_0 = \{y(\alpha) | \alpha \in GF(8)\}$, $Z = \{z(\alpha) | \alpha \in GF(8)\}$ with the obvious multiplication. Let λ be a generator of $GF(8)^\times$. Interpret the operation of n on A_0, B_0 resp. Z as multiplication with λ, λ^4 resp. λ^5. Choose $x_1 = x(1)$, $y_1 = y(1)$, $z_1 = z(1)$. It is then easy to check, that $[x(\alpha), y(\beta)] = z(\alpha\beta)$ for every $\alpha, \beta \in GF(8)$. Thus Q is of type $L_3(8)$ in this case.

If $[x_1, x_2] = z_1 z_3$, $[x_2, y_1] = z_1 z_2$, we get the same result by a similar calculation. In this case, the operation of n on A_0, B_0 resp. Z has to be interpreted as multiplication with λ, λ^2 resp. λ^3.

(4) Assume, that $Q - Z$ contains exactly 21×8 involutions. Like under (3), let $Q = AB$, where $A \cong B \cong E_{64}$, let $A = A_0 \oplus Z$, $B = B_0 \oplus Z$ be the decompositions as $\langle n \rangle$-modules, $A_0 = \langle x_1, x_2, x_3 \rangle$, $B_0 = \langle y_1, y_2, y_3 \rangle$, where $C_{B_0}(x_1) = \langle y_1 \rangle$ and

$$n_{A_0} = n_{B_0} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

with respect to the bases $\{x_1, x_2, x_3\}$ resp. $\{y_1, y_2, y_3\}$.

Assume $|[x_1, Q]| = 2$, let $[v_1, w_1] = z_1 \neq 1$ and $[v_2, w_2] = z_2 \neq 1$ for $w_1, w_2 \in B$, $v_1, v_2 \in A_0$, $v_1 Z \neq v_2 Z$. Then $z_1 \neq z_2$. Further $[v_1 v_2, w_1] \in \in z_1\langle z_2\rangle$ and thus $[v_1 v_2, w_1] = z_1 z_2$. This shows $\tilde{A} \underset{\langle n\rangle}{\cong} Z$, a contradiction. We have $|[x, Q]| = 4$ for every $x \in Q - Z$, $x^2 = 1$. Set $[x_1, y_2] = z_3$ and $[x_1, y_3] = z_2$. It follows

$$1 = [x_1 x_2, y_1 y_2] = [x_1, y_2][x_2, y_1]\,, \qquad 1 = [x_1, y_3][x_3, y_1]\,,$$

and thus $[x_2, y_1] = z_3$, $[x_3, y_1] = z_2$. Further

$$z_2^n = [x_3, y_1]^n = [x_1 x_2, y_2] = z_3\,, \qquad z_3^n = [x_2 x_3, y_3] = [x_2, y_3] \notin \langle z_2, z_3\rangle\,.$$

Set $[x_2, y_3] = z_1$. With respect to the basis $\{z_1, z_2, z_3\}$ we have

$$n_Z = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The structure of Q is now uniquely determined. Let $H < Z$, $|H| = 4$. Then Q contains exactly $21 + 3 \times 42/7 = 39$ cosets which contain elements with square in H. It follows $Q/H \cong E_4 \times (Q_8)^2$.

(5) Assume $Q - Z$ contains more than 21×8 involutions. Then $Q = AB$, $A \cong B \cong E_{64}$ and for $x \in A - Z$ we have $|C_B(x)| = 2^5$. It follows $|[x, Q]| = 2$ and $\tilde{A} \cong Z$ as $\langle n\rangle$-modules, a contradiction.

(1.3) Lemma. Let Q be a special 2-group of order 2^9 with elementary abelian center Z of order 8. Let F be a Frobenius-group of order 21 operating on Q, $F = \langle n, r\rangle$, $n^7 = r^3 = 1$, $n^r = n^2$. Assume, that n operates fixed-point-freely on Q and that $C_Q(r) \cong E_8$. Then Q is isomorphic to the group Q_0 in (1.2) (4) and the operation of F on Q is uniquely determined.

Proof. Let $\tilde{V}$ be an irreducible F-submodule of $\tilde{Q} = Q/Z$. The operation of r shows, that $V - Z$ contains involutions. Thus $\tilde{V} = E_{64}$. We have $Q = AB$, $A \cong B \cong E_{64}$, $A \cap B = Z$ and F normalizes A and B.

Set $C_Z(r) = \langle z_1\rangle$, $C_A(r) = \langle z_1, x_1\rangle$, $C_B(r) = \langle y_1, z_1\rangle$.

Assume $\tilde{A} \underset{\langle n\rangle}{\not\cong} \tilde{B}$. Then we can choose bases $\{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3\}$, $\{\tilde{y}_1, \tilde{y}_2, \tilde{y}_3\}$,

$\{z_1, z_2, z_3\}$ of $\tilde{A}, \tilde{B}$ resp. Z such that

$$n_{\tilde{A}} = n_Z = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad n_{\tilde{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

It follows

$$r_{\tilde{A}} = r_Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad r_{\tilde{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

We have

$$[x_1, y_1] = 1, \quad [x_1, \langle y_1 y_2, y_1 y_3 \rangle] = [y_1, \langle x_1, x_2 \rangle] = \langle z_2, z_3 \rangle .$$

Assume $[x_1, y_1 y_2] = z_2$. Then

$$[x_1, y_1 y_3] = z_3, \qquad [x_2, y_1] = [x_1, y_2 y_3]^n = (z_2 z_3)^n = z_1 z_2 z_3,$$

a contradiction. The same calculation shows $[x_1, y_1 y_2] \neq z_2 z_3$.

Thus $[x_1, y_1 y_2] = z_3$, $[x_1, y_1 y_3] = z_2 z_3$, $[x_2, y_1] = z_2^n = z_3$.

On the other hand

$$[x_1, y_2]^n = [x_1 x_3, y_1] = z_3^{n^{-1}} = z_2,$$

$$[x_1, y_3]^{n^{-2}} = [x_1 x_2 x_3, y_1] = (z_2 z_3)^{n^{-2}} = z_3$$

and thus $[x_2, y_1] = z_2 z_3$, a contradiction. We have $\tilde{A} \underset{\langle n \rangle}{\cong} \tilde{B}$. It follows from (1.2), that Q is isomorphic to the group Q_0 of (1.2) (4) and that the operation of n on Q is uniquely determined. Choose notation for Q_0 and for the operation of n line in (1.2) (4).

Then

$$r_{\tilde{A}} = r_{\tilde{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad r_Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = (r_{\tilde{A}})^* .$$

(1.4) EXAMPLE. Let D be the Dempwolffgroup, i.e. the unique nonsplit extension of E_{32} by $L_5(2)$ [3]. For a description of D see [11]. Let $V = O_2(D) \cong E_{32}$, $X < V$, $|X| = 4$. Then $N_D(X)/V$ has the structure $E_{64}(\Sigma_3 \times L_3(2))$. Let $R_1 = O_2(N_D(X))$. Then $\tilde{R}_1 = R_1/X$ is isomorphic to Q_0 and $N_D(X)/X$ is a split extension of $\tilde{R}_1$ by $\Sigma_3 \times L_3(2)$

From now on Q_0 denotes the group given in (1.2) (4). We shall now describe the automorphism group of Q_0.

(1.5) COROLLARY. Let $A = \mathrm{Aut}\,(Q_0)$, $B = \{a | a \in A, [a, Z] = 1\}$, $C = \{a | a \in A, [a, Q_0] \subseteq Z\}$. Then B and C are normal subgroups of A. We have $C < B$, $C \cong E_2 18$, $B/C \cong \Sigma_3$, $A/B \cong L_3(2)$, $A/C \cong \Sigma_3 \times L_3(2)$.

PROOF. It follows from (1.4), that $A/B \cong L_3(2)$. An automorphism of Q_0, which induces the identity on Z and operates on each of the three E_{64}-subgroups of Q_0 has to lie in C. Thus $B/C \cong \Sigma_3$ and C is the kernel of the representation of B on the set of E_{64}-subgroups of Q_0. Clearly $C \cong E_2 18$.

The following is probably well known

(1.6) LEMMA. Let $V \cong E_{64}$, $L \cong L_3(2)$. Let L operate on V, Z an irreducible L-submodule of V. Assume, that Z and V/Z are nonisomorphic natural L-modules. Then either V is a completely reducible L-module or V is a uniquely determined indecomposable L-module. Choose $\langle n, r \rangle < L$ such that $n^7 = r^3 = 1$, $n^r = n^2$, $Z = \langle z_1, z_2, z_3 \rangle$ and let $V = Z \oplus V_0$ as an $\langle n, r \rangle$-module. We can choose a basis $\{v_1, v_2, v_3\}$ of V_0, such that

$$n_{V_0} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad r_{V_0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and for every $x \in L$ we have $x_Z = (x_{V/Z})^*$. Choose $t \in L$ such that

$$t_Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $t_Z = t_{V/Z}$. If V is an indecomposable L-module we have $v_1^t = v_1$, $v_2^t = v_3 z_2$, $v_3^t = v_2 z_3$. Then $C_V(t) > C_V(r)$ and thus $|C_V(t)| = 16$.

(1.7) Let L operate on Q_0, where $L \cong L_3(2)$. Fix $F < L$, $F \cong F_{21}$. $F = \langle n, r \rangle$, $n^7 = r^3 = 1$, $n^r = n^2$. Then $C_{Q_0}(r) \cong E_8$ and one of the following holds:

(1) L operates completely reducibly on two of the E_{64}-subgroups of Q_0 and indecomposably on the third.

(2) L operates indecomposably on all of the E_{64}-subgroups of Q_0.

We shall refer to the operations under (1) resp. (2) as operations of « dihedral » resp. « quaternion » type.

PROOF. Clearly n operates fixed-point-freely on Q_0 and $C_{Q_0}(r) \cong E_8$. We choose notation like in (1.3). Let $t \in L$ such that

$$t_{\bar{A}} = t_{\bar{B}} = t_Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $(tr)^2 = (tn)^3 = 1$.

(α) Assume, that L operates completely reducibly on A, i.e. L operates on $A_0 = \langle x_1, x_2, x_3 \rangle$.

(α_1) Assume, that L operates on $B_0 = \langle y_1, y_2, y_3 \rangle$. The F-complement of Z in $A + B$ is $(A + B)_0 = \langle x_1 y_1 z_1, x_2 y_2 z_1 z_2, x_3 y_3 z_1 z_2 z_3 \rangle$. We have $(x_1 y_1 z_1)^t = x_1 y_1 z_1$, $(x_2 y_2 z_1 z_2)^t = (x_3 y_3 z_1 z_2 z_3) z_2$ and $(x_3 y_3 z_1 z_2 z_3)^t = x_2 y_2 z_1 z_2 \cdot z_3$. Thus $A + B$ is an indecomposable L-module.

(α_2) Assume, that B is an indecomposable L-module. Then we see like above, that $A + B$ is a completely reducible L-module.

(β) Let L operate indecomposably on A, B and $A + B$. With respect to the basis $\{x_1, x_2, x_3, z_1, z_2, z_3\}$ resp. $\{y_1, y_2, y_3, z_1, z_2, z_3\}$ we have then

$$t_A = t_B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Set $t_1 = t^{r^{-1}n^3}$, $t_2 = t^{r^{-1}n^5}$. Then

$$(t_1)_A = (t_1)_B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix},$$

$$(t_2)_A = (t_2)_B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We note, that in both cases, (α) and (β), $C_{Q_0}(t) \cong Z_2 \times (D_8)^2$, $C_{Q_0}(t, t_2) \cong$ $\cong (D_8)^2$ and $C_{Q_0}(t, t_1) \cong E_{16}$.

The following is easily verified:

(1.8) Lemma. Let the operation of L on Q_0 be of dihedral type, where $L \cong L_3(2)$. Choose notation like in (1.7) (α_1). Then $\langle F, x_1 y_2 t\rangle \cdot$ $\cdot Z/Z \cong L_3(2)$ and the operation of $\langle F, x_1 y_1 t\rangle Z/Z$ on Q_0 is of quaternion type.

(1.9) Notation. Let Q^n denote the isomorphism-type of the central product (with amalgamated centers) of n copies of Q_0 and let Q_i, $1 \leqslant i \leqslant n$ be groups which are isomorphic to Q_0. Further φ_i, $1 \leqslant i \leqslant n$ are isomorphisms from Q_0 on Q_i. Consider $Q = Q_1 * Q_2 * \dots * Q_n \cong Q^n$. We can assume $Z = \boldsymbol{Z}(Q_0) = \boldsymbol{Z}(Q_i) = \boldsymbol{Z}(Q)$, $1 \leqslant i \leqslant n$, $\varphi_i|_Z = 1_Z$ and we set $\varphi_i(x_j) = v_i^{(j)}$, $\varphi_i(y_j) = v_{-i}^{(j)}$, $1 \leqslant i \leqslant n$, $j = 1, 2, 3$. Set $A = \mathrm{Aut}\,(Q)$, $B = \{a | a \in A, [a, Z] = 1\}$, $C = \{a | a \in A, [a, Q] \subseteq Z\}$. Here the index « n » is omitted as no confusion will occur. We set $\tilde{Q} = Q/Z$ and identify $\tilde{Q}$ with a subgroup of A. Then $\tilde{Q} < C < B < A$ and the groups B, C are normal subgroups of A. Further $A/B \cong L_3(2)$.

Set

$$V_i = \varphi_i(A)\,, \quad V_i^{(0)} = \varphi_i(A_0)\,, \quad V_{-i} = \varphi_i(B)\,, \quad V_{-i}^{(0)} = \varphi_i(B_0)\,, \quad 1 \leqslant i \leqslant n\,.$$

For elements α_i, α_{-i} of $GF(2)$, not all zero, set

$$\sum_1^n \alpha_i V_i + \alpha_{-i} V_{-i} = Z\Big\langle \prod_1^n (v_i^{(k)\alpha_i} \cdot v_i^{(k)\alpha_{-i}}),\ k = 1, 2, 3\Big\rangle \simeq E_{64}.$$

Set $\sum_1^n O V_i + O V_{-i} = 0$.

Consider the set $\mathfrak{V} = \left\{V | V = \sum_1^n (\alpha_i V_i + \alpha_{-i} V_{-i}) | \alpha_i, \alpha_{-i} \in GF(2)\right\}$. Then $\mathfrak{V}$ is a $GF(2)$-vectorspace with respect to the addition

$$\Big(\sum_1^n \alpha_i V_i + \alpha_{-i} V_{-i}\Big) + \Big(\sum_1^n \beta_i V_i + \beta_{-i} V_{-i}\Big) = $$
$$= \sum_1^n (\alpha_i + \beta_i) V_i + (\alpha_{-i} + \beta_{-i}) V_{-i}.$$

The set $\{V_i | 1 \leqslant i \leqslant n\} \cup \{V_{-i} | 1 \leqslant i \leqslant n\}$ is a basis of $\mathfrak{V}$. We consider further the non-singular scalar product $(,)$ on $\mathfrak{V}$ given by $(V, W) = 0$ if $V = 0$ or $W = 0$ or $[V, W] = \langle 1 \rangle$ and $(V, W) = 1$ otherwise.

For $x \in A_0 = \langle x_1, x_2, x_3\rangle$ set $E_x = Z\langle \varphi_i(x), \varphi_i(C_{B_0}(x)) | 1 \leqslant i \leqslant n\rangle$. Then $E_x \simeq E_2 2n + 3$. Set $E_j = E_{x_j} = Z\langle v_1^{(j)}, v_{-1}^{(j)}, \ldots, v_n^{(j)}, v_{-n}^{(j)}\rangle$, $j = 1, 2, 3$. We have $[E_j, E_k] = \langle z_r\rangle$, where $\{j, k, r\} = \{1, 2, 3\}$.

Set $E_j^{(0)} = \langle v_1^{(j)}, v_{-1}^{(j)}, \ldots, v_n^{(j)}, v_{-n}^{(j)}\rangle$, $j = 1, 2, 3$ and

$$E_1^{(1)} = E_1^{(0)}\langle z_1\rangle, \quad E_2^{(1)} = E_2^{(0)}\langle z_1 z_2\rangle, \quad E_3^{(1)} = E_3^{(0)}\langle z_1 z_2 z_3\rangle.$$

For $i \in \{1, 2, \ldots, n\}$ let $B_i = \langle b_i, b_{-i}\rangle < B$ with $[B_i, Q_j] = 1$ for $j \neq i$, $[b_i, V_{-i}] = 1 = [b_{-i}, V_i]$ and

$$v_i^{(1)b_i} = v_i^{(1)} v_{-i}^{(1)} z_1, \quad v_i^{(2)b_i} = v_i^{(2)} v_{-i}^{(2)} z_1 z_2, \quad v_i^{(3)b_i} = v_i^{(3)} v_{-i}^{(3)} z_1 z_2 z_3,$$

$$v_{-i}^{(1)b_{-i}} = v_i^{(1)} v_{-i}^{(1)} z_1, \quad v_{-i}^{(2)b_{-i}} = v_i^{(2)} v_{-i}^{(2)} z_1 z_2, \quad v_{-i}^{(3)b_{-i}} = v_i^{(3)} v_{-i}^{(3)} z_1 z_2 z_3.$$

Then $b_i^2 = b_{-i}^2 = 1$, $B_i \simeq \Sigma_3$ and $[B_i, B_j] = 1$ for $i \neq j$.

Let L be a complement of B in $A = \operatorname{Aut}(Q)$.

(1.10) LEMMA. Let $q \in Q - Z$. Then $[q, Q] \neq Z$ is equivalent to $q \in E_x$ for an $x \in A_0^{\#}$. Especially, $\bigcup E_x$, $x \in A_0$, is a characteristic subset of Q. We have $[E_x, Q] = [q, Q]$ for every $q \in E_x - Z$. It follows $B \leqslant N(E_x)$ for every $x \in A_0^{\#}$.

Proof. Let $q \in Q - Z$, $\tilde{q} = \tilde{q}_1 \dots \tilde{q}_n$, $\tilde{q}_i \in \tilde{Q}_i$. If $q_i^2 = 1$ for an inverse image q_i of $\tilde{q}_i$, we have $Z = [q_i, Q_i] \subseteq [q, Q_i] \subseteq [q, Q]$.

Assume $[q, Q] \neq Z$. Then $q_i^2 \neq 1$, $1 \leqslant i \leqslant n$ and $[q_i, Q_i] = [q_j, Q_j]$ wherever $q_i \notin Z$ and $q_j \notin Z$. This shows $q_j \in E_x$ for an $x \in A_0^\#$.

(1.11) Lemma. $C \cong E_2 18n$, $B = CB_0$, $B_0 \cong Sp(2n, 2)$, $B_0 \cap C = 1$, $A/B \cong L_3(2)$.

Proof. It is clear, that $C \cong E_2 18n$ and $A/B \cong L_3(2)$. Let $X \in \in \mathfrak{V} - \{0\}$. Then X satisfies the following conditions:

(α) $Z < X \cong E_{64}$.

(β) $C_Q(X) = X_0 \times R$, where $X \supset X_0 \cong E_8$, $R \cong Q^{n-1}$.

(γ) $Q = R_1 * R$, where $X \subset R_1 \cong Q_0$.

(δ) $|E_x : E_x \cap C(X)| = 2$ for each $x \in A_0^\#$.

(ε) For every $x_1, x_2 \in X - Z$ such that $x_1 Z \neq x_2 Z$, we have

$$C_Q(X) = C_Q(x_1) \cap C_Q(x_2) \neq C_Q(x_1) .$$

Consider the set $M = \{X | X < Q,\ X \text{ satisfies conditions } (\alpha)\text{-}(\varepsilon)\}$.

(1) $M = \mathfrak{V} - \{0\}$: let $X \in M$. For $x \in X$ write $xZ = \prod_1^n x_i Z$, $x_i \in Q_i$. Assume $[x, Q] = Z$. Then $|Q : C_Q(x)| = 8$ and $C_Q(x) = C_Q(X)$ by (β), a contradiction to (ε). Thus $x \in E_y$, $y \in A_0^\#$ by (1.10). It follows from (γ), that we can write $X = Z \times X_0$, $X_0 = \langle q, r, s\rangle = E_8$, $q \in E_1$, $r \in E_2$, $s \in E_3$. By (δ) we have $C(X) \cap \langle v_i^{(j)}, v_{-i}^{(j)}\rangle \neq \langle 1\rangle$ for every $j \in \{1, 2, 3\}$. Choose $i \in \{1, 2, \dots, n\}$.

Assume $\langle v_i^{(j)}, v_{-i}^{(j)}\rangle \leqslant C_Q(X)$ for a $j \in \{1, 2, 3\}$. Without restriction we can choose $j = 1$. It follows $\langle r_i, s_i\rangle < Z$ and from (ε) we get $Q_i \leqslant C_Q(r) \cap C_Q(s) = C_Q(X)$ and thus $q_i \in Z$. We now choose $i \in \in \{1, 2, \dots, n\}$ such that $\langle q_i, r_i, s_i\rangle \nleqslant Z$. By the above we have $|\langle v_i^{(j)}, v_{-i}^{(j)}\rangle \cap C(X)| = 2$ for every $j \in \{1, 2, 3\}$ and $\langle x, y\rangle \nleqslant Z$ for every $\{x, y\} \subseteq \{q_i, r_i, s_i\}$ such that $x \neq y$. Without loss $v_i^{(1)} \in C(V)$. It follows $r_i \in \langle v_i^{(2)}\rangle Z$, $s_i \in \langle v_i^{(3)}\rangle Z$. Assume $s_i \in Z$. Then $C_{Q_i}(X) = C_{Q_i}(r_i) = C_{Q_i}(q_i)$ by (ε). It follows $\langle r_i, q_i\rangle \leqslant Z$, a contradiction.

We have $r_i \in v_i^{(2)} Z$ $s_i \in v_i^{(3)} Z$. It follows $q_i \in \langle v_i^{(1)}\rangle Z$ and by the same operation as above $q_i \in v_i^{(1)} Z$. This holds for every $i \in \{1, 2, \dots, n\}$ such that $\langle q_i, r_i, s_i\rangle \nleqslant Z$. This shows $X \in \mathfrak{V} - \{0\}$. We have shown $M = \mathfrak{V} - \{0\}$.

(2) It follows from (1), that the automorphism-group B operates on $\mathfrak{V}$. Further B respects the linear structure and the symplectic scalar product of $\mathfrak{V}$. The kernel of this representation of B is exactly C, as $\langle X|X\in\mathfrak{V}-\{0\}\rangle = Q$ and $A/B\cong L_3(2)$. Hence B/C is isomorphic to a subgroup of $Sp(2n, 2)$.

(3) Define a symplectic non-singular scalar-product on $E_i^{(0)}$ over $GF(2)$ by $(v_k^{(i)}, v_r^{(i)}) = 1$ exactly if $k = -r$ (and 0 otherwise). Let $B_0\cong Sp(2n, 2)$ and let B_0 be represented in the natural way on $E_1^{(0)}$, $E_2^{(0)}$ and $E_3^{(0)}$. Let $q\in Q$. Then q possesses a unique representation of the form $q = q_1q_2q_3z$, $q_i\in E_i^{(0)}$, $z\in Z$, $i = 1, 2, 3$. We extend the operation of B_0 on Q by setting $q^b = q_1^bq_2^bq_3^bz$ for $b\in B_0$. It is now easy to see, that B_0 is a group of automorphisms of Q.

(1.12) LEMMA. Let Q be a special 2-group, $Z(Q) = Z\cong E_8$ and let a group L, $L\cong L_3(2)$, operate nontrivially on Q. Suppose $\tilde{Q}= Q/Z = \tilde{V}_1\oplus\dots\oplus\tilde{V}_m$ as an L-module such that $\tilde{V}_i\cong\tilde{V}_j\not\cong Z$ for $i, j\in\{1, \dots, n\}$. Here $\tilde{V}_i$ and Z are natural L-modules.

Then $m = 2n$ and $Q\cong Q^n$. If r is an element of order 3 in L, then $C_Q(r)\cong E_22n + 1$.

PROOF. (1) Let $\tilde{V}\subset\tilde{Q}$ be an irreducible L-submodule and V be the inverse image of $\tilde{V}$. Then V cannot be a Suzuki-2-group of (A)-type as $L_3(2)$ operates on V. As $\tilde{V}\not\cong Z$ as an L-module, we must have $V\cong E_{64}$. It follows $C_Q(r)\cong E_2m + 1$, when r is an element of order 3 in L.

(2) Consider $\tilde{Q}$ as a $GF(2)$-vectorspace. Then it is easy to see, that $X = C(L)\cap\mathrm{Aut}(\tilde{Q})\cong L_m(2)$ and $|\{\tilde{V}_1^x|x\in X\}| = 2^m-1$. Set $\mathfrak{V}' = \{V_1^x|x\in X\}$.

(3) $\mathfrak{V}' = \{V|V<Q,\ \tilde{V}$ is an irreducible L-submodule of $\tilde{Q}\}$. Let $\mathfrak{W}' = \{V|V<Q,\ \tilde{V}$ an irreducible L-submodule of $\tilde{Q}\}$.

Clearly $\mathfrak{V}'\subseteq\mathfrak{W}'$. Let $V\in\mathfrak{W}'$ and let τ be an L-isomorphism of $\tilde{V}$ on $\tilde{V}_1$. Then τ can be extended to an L-isomorphism of $\tilde{Q}$, that is $\tau\in X$.

(4) Set $\mathfrak{V} = \mathfrak{V}'\cup\{0\}$. Then $\mathfrak{V}$ is an $GF(2)$-vectorspace by the following definition: $0 + V = V + 0 = V$, $V + V = 0$ for $V\in\mathfrak{V}$. Let $V, W\in\mathfrak{V}'$ such that $V\neq W$. Then $\widetilde{V + W}$ is defined as the unique irreducible L-submodule of $\langle\tilde{V}, \tilde{W}\rangle$ which is different from $\tilde{V}$ and $\tilde{W}$. Then clearly $V + W = W + V$. The associativity of the so defined addition is easily proved with the help of the fact, that a 9-dimensional L-invariant subspace of $\tilde{Q}$ contains exactly 7 irreducible L-submodules.

(5) We define a symplectic non-singular $GF(2)$-scalar product $(\ ,\ )$ on $\mathfrak{V}$ by $(0,0)=(0,V)=(V,0)=0$ and, for $V, W\in\mathfrak{V}'$, $(V,W)=0$ if and only if $[V,W]=1$.

Clearly $(V,W)=(W,V)$ and $(V,V)=0$. We show $(A+B,C)=$ $=(A,C)+(B,C)$ for all $A, B, C\in\mathfrak{A}$. We can assume $0\notin\{A,B,C\}$ and $A\neq B$.

If $[A,C]=1=[B,C]$, we have $[A+B,C]=1$, as $A+B\leqslant$ $\leqslant\langle A,B\rangle$. If $[A,C]=1\neq[B,C]$, we have $[A+B,C]\neq 1$.

So we can assume $[A,C]\neq 1\neq[B,C]$, and we have to show $[A+B,C]=1$. This however follows directly from the structure of Q_0, as $\langle A,C\rangle\cong\langle B,C\rangle\cong Q_0$. As $Q=\langle V|V\in\mathfrak{V}'\rangle$, it is clear that $(\ ,\ )$ is non-singular.

(6) It follows $m=\dim\mathfrak{V}=2n$. Let $\mathfrak{V}=\mathfrak{V}_1\oplus\mathfrak{V}_2\oplus\ldots\oplus\mathfrak{V}_n$ be a decomposition of $\mathfrak{V}$ in hyperbolic planes with respect to $(\ ,\ )$. Then $Q=Q_1*Q_2*\ldots*Q_n$, where $Q_i=\langle V|V\in\mathfrak{V}_i-\{0\}\rangle$. The group Q_i is special of order 2^9 with center Z. The operation of an element of order 7 and (1.2) show $Q_i\cong Q_0$, $1\leqslant i\leqslant n$. Thus $Q\cong Q^n$.

The following lemma gives further motivation for the term « dihedral type » resp. « quaternion type ». introduced in (1.7).

(1.13) Lemma. Let $Q=Q_1*Q_2\cong Q^2$ like in (1.9) for $n=2$. Let $L\cong L_3(2)$ and assume the operation of L on Q_1 and Q_2 is of quaternion type like in (1.7) (2). Then we can choose $R_1, R_2<Q$, $R_1\cong R_2\cong Q_0$, $Q=R_1*R_2$, such that L operates on R_i, $i=1,2$, and the operation of L on R_i is of dihedral type.

Proof. Let φ_i be the isomorphism from Q_0 on Q_i, $i=1,2$, and let the operation of L on Q_i be like in (1.7) (2).

Set $R_1=\langle V_1, V_{-1}+V_{-2}\rangle$, $R_2=\langle V_1+V_2, V_{-2}\rangle$.

From (1.12) and (1.13) we get the following

(1.14) Corollary. Let L be a complement of B in $A=\mathrm{Aut}(Q)$ such that the operation of L on Q satisfies the hypothesis of (1.12). Then L is conjugate in A to one of the following two automorphism-groups of Q (notation like in (1.9)).

(1) L_+, where the operation of L_+ on Q_i, $1\leqslant i\leqslant n$ is of dihedral type like given in (1.7) (1).

(2) L_-, where the operation of L_- on Q_i is of dihedral type like above for $2\leqslant i\leqslant n$ and of quaternion type like in (1.7) (2) for $i=1$,

(1.15) Lemma. Consider the subgroups L_+ and L_- of A as introduced in (1.14). Then $L_+ \cap L_- = F \cong F_{21}$ and $F = \langle n, r \rangle$, where $n^7 = r^3 = 1$, $n^r = n^2$ and the operation of F on Q_i is described in (1.7), $1 \leqslant i \leqslant n$. We have $\boldsymbol{C}_B(F) = \boldsymbol{C}_B(n) = B^* \cong Sp(2n, 2)$ and B^* is a complement of C in B.

Further $E_i^{(1)}$ is an indecomposable B^*-module, $i = 1, 2, 3$ and $\boldsymbol{C}_A(L_\varepsilon) = \boldsymbol{C}_B(L_\varepsilon) = B_\varepsilon^* \cong O^\varepsilon(2n, 2)$, where $\varepsilon \in \{+, -\}$.

Proof. It follows from (1.7), that $L_+ \cap L_- = F \cong F_{21}$. As it is easy to see, that n operates fixed-point-freely on C, we have that $\boldsymbol{C}_B(n)$ is isomorphic to a subgroup of $Sp(2n, 2)$. The group B/C is represented in the natural way on the vector-space E_1/Z and the complement B_0 of C in B is represented on the complement $E_1^{(0)}$ of Z in E_1 (1.11). Fix the basis $\{v_1^{(1)}, v_2^{(1)}, \dots, v_n^{(1)}, v_{-1}^{(1)}, \dots, v_{-n}^{(1)}\}$ for $E_1^{(0)}$ and the analogous basis for E_1/Z. Identify each element of B/C resp. B_0 with the matrix representing the operation of the element with respect to the above basis. Then B/C resp. B_0 is generated by the matrices of the following form (see [15]):

(1) $I + e_{ij} + e_{-j,-i}$ (2) $I + e_{-i,-j} + e_{ji}$

(3) $I + e_{i,-j} + e_{j,-i}$ (4) $I + e_{-i,j} + e_{-j,i}$

(5) $I + e_{i,-i}$ (6) $I + e_{-i,i}$, $0 < i < j \leqslant n$.

Here I denotes the $(2n, 2n)$-unit matrix and e_{kr} denotes the matrix with entry 1 at the intersection of row k and column r and 0 otherwise. Let $B_+^{*\prime}$ be the subgroup of B_0 generated by the elements which correspond to the matrices of forms (1)-(4). Then $B_+^{*\prime} \cong \Omega^+(2n, 2)$. Set $B^* = \langle B_+^{*\prime}, B_1 \times B_2 \times \dots \times B \rangle$. The involutoric generators of B_i, $1 \leqslant i \leqslant n$ (1.9) are elements of B, which are not contained in B_0. They correspond to the matrices of forms (5) and (6). It follows from (1.9), that B^* operates on $E_i^{(1)}$. This operation is clearly indecomposable. Further it is a matter of direct calculation, that $B^* \leqslant \boldsymbol{C}(F)$. As $\boldsymbol{C}_C(n) = 1$, we have $\boldsymbol{C}_B(n) = \boldsymbol{C}_B(F) = B^* \cong Sp(2n, 2)$.

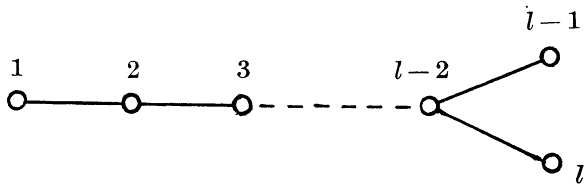
Let q_ε, $\varepsilon \in \{+, -\}$ be defined on the vector space $\mathfrak{V}$ with values in $GF(2)$ by $q_\varepsilon(0) = 0$ and $q_\varepsilon(V) = 0$ if and only if L_ε operates completely reducibly on V for $V \in \mathfrak{V}'$.

It is then easy to see with the help of (1.7), that q_ε are quadratic forms on $\mathfrak{V}$ with respect to the scalar product $(\,,\,)$. Let $V \in \mathfrak{V}$, $V = \sum_1^n (x_i V_i + x_{-i} V_{-i})$. Then $q_+(V) = \sum_1^n x_i x_{-i}$ and $q_-(V) = \sum_2^n (x_i x_{-i}) +$

$+ x_1 + x_1 x_{-1} + x_{-1}$. Thus the indices of q_+, q_- are n resp. $n-1$. Let B_ε^* be the subgroup of B^* respecting the form q_ε. Then B_ε^* is isomorphic to $O^\varepsilon(2n, 2)$. Clearly $\boldsymbol{C}_B(L_\varepsilon) \subseteq B_\varepsilon^*$. The equality $\boldsymbol{C}_A(L_\varepsilon) = B_\varepsilon^*$ will follow from the examples (1.15) (i) and (ii).

(1.15) EXAMPLES.

(i) Consider the Chevalleygroup $D_l(2)$, $l \geqslant 4$. We use the notation of [2]. So let $e_1, \dots, e_l$ be an orthonormal basis for an euclidean vector space. Then $\Phi = \{\pm e_i \pm e_j | i \neq j;\ i, j = 1, 2, \dots l\}$ is a root-system of type D_1. The vectors r_i, $1 \leqslant i \leqslant l$ with $r_i = e_i - e_{i+1}$ for $i < l$ and $r_l = e_{l-1} + e_l$ form a system of fundamental roots. This choice corresponds to the following labelling of the Dynkin-diagram



Let $P = P_3$ for $l > 4$ and $P = P_{\{3,4\}}$ for $l = 4$. Then P is a parabolic subgroup of $D_l(2)$. Set $Q = \boldsymbol{O}_2(P)$. Then

$$Q = \langle X_{e_i - e_j}, X_{e_i + e_j} | 0 < i \leqslant 3,\ 4 \leqslant j \leqslant l \rangle ,$$

$$Z = \boldsymbol{Z}(Q) = \langle X_{e_i + e_j} | 0 < i < j \leqslant 3 \rangle .$$

For $4 \leqslant j \leqslant l$ set

$$Q_{j-3} = \langle X_{e_i - e_j}, X_{e_i + e_j} | 0 < i \leqslant 3 \rangle .$$

Then $Q_i \simeq Q_0$ with $\boldsymbol{Z}(Q_i) = Z$ for $1 \leqslant i \leqslant l - 3$. Further

$$L = \langle X_{\pm e_i \pm e_j} | i \neq j,\ 0 < i, j \leqslant 3 \rangle$$

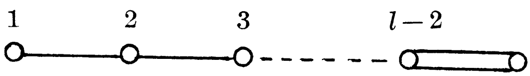
is isomorphic to $L_3(2)$ and

$$X = \langle X_{\pm e_i \pm e_j} | i \neq j,\ 4 \leqslant i, j \leqslant l \rangle \simeq D_{l-3}(2) .$$

Then $[L, X] = 1$ and $P = Q(X \times L)$. The group L operates on each of the subgroups $\langle X_{e_i + e_j} | 0 < i \leqslant 3 \rangle$ and $\langle X_{e_i - e_j} | 0 < i \leqslant 3 \rangle$, $4 \leqslant j \leqslant l$.
Set $n = l - 3$. Then $Q \simeq Q^n$ and the operation of L on each of

the subgroups Q_i of Q, $1 \leqslant i \leqslant n$ is of dihedral type. The group P is a maximal parabolic subgroup of $D_l(2)$ with the exception of the case $l = 4$, where P is contained in a maximal parabolic subgroup, which is a split extension of E_{64} by $\Omega^+(6, 2)$.

(ii) Consider the Steinberg group ${}^2D_l(2)$ for $l \geqslant 4$. Let ϱ be the symmetry of the Dynkin diagram for D_l interchanging r_{l-1} and r_l and fixing the fundamental roots r_i, $1 \leqslant i \leqslant l-2$. The Dynkin-diagram for ${}^2D_l(2)$ is then



Let $P = P_3$ be a maximal parabolic subgroup of ${}^2D_l(2)$, $Q = \boldsymbol{O}_2(P)$. If r is a root and $r \neq r^\varrho$, set $D_{r,r^\varrho} = \{x_r(\alpha)x_{r^\varrho}(\alpha^\sigma)|\alpha \in K\}$, where $K = GF(4)$ and σ is the non-trivial field-automorphism of K. Then $D_{r,r^\varrho} \cong E_4$. If $r = r^\varrho$, set $X_r = \{x_r(\alpha)|\alpha \in K_0\}$, where K_0 is the prime field of K. Then

$$Q = \langle X_{e_i - e_j}, X_{e_i+e_j}, D_{e_i-e_l, e_i+e_l} | 0 < i \leqslant 3,\ 4 \leqslant j \leqslant l\rangle ,$$

$$Z = \boldsymbol{Z}(Q) = \langle X_{e_i+e_j} | 0 < i < j \leqslant 3\rangle \cong E_8 .$$

Set

$$Q_1 = \langle D_{e_i-e_l, e_i+e_l} | 0 < i \leqslant 3\rangle , \qquad Q_{j-2} = \langle X_{e_i-e_j, e_i+e_j} | 0 < i \leqslant 3\rangle$$

for $4 \leqslant j \leqslant l-1$. Then $Q = Q_1 * Q_2 * \ldots * Q_n$, where

$$n = l-3 , \quad Q_1 \cong Q_2 \cong \ldots \cong Q_n \cong Q_0 , \quad \boldsymbol{Z}(Q_i) = Z , \quad 1 \leqslant i < n .$$

Let $L = \langle X_{\pm e_i \pm e_j} | i \neq j,\ 0 < i, j \leqslant 3\rangle \cong L_3(2)$,

$$X = \langle X_{e_i \pm e_j}, D_{e_k - e_l, e_k + e_l} | i \neq j,\ 4 \leqslant i, j \leqslant l-1,\ 4 \leqslant k \leqslant l-1\rangle \cong {}^2D_n(2)$$

We have $[L, X] = 1$, $P = Q(X \times L)$, $Q \cong Q^n$ and the operation of L on Q_i is of dihedral type for $i \geqslant 2$, of quaternion type for $i = 1$.

(iii) Janko has shown [8, Prop. 13], that J_4 contains an elementary abelian subgroup V of order 8 such that $L = \boldsymbol{O}_2(\boldsymbol{N}(V))$ is a special group of order 2^{15} with center V, $\overline{\boldsymbol{N}(V)} = \boldsymbol{N}(V)/L = \bar{J} \times \bar{C}$, where $\bar{J} \cong \Sigma_5$ and $\bar{C} \cong L_3(2)$. Here $J = \boldsymbol{C}(V)$, an element of order 5 of $\bar{J}$ operates fixed-point-freely on L/V and an element of order 7 in C operates fixed-point-freely on L. Further an element of order 3 in J

operates fixed-point-freely on L/V. Tran Van Trung has characterized the simple group J_4 by a maximal 2-local subgroup having the above structure [19]. It is easy to see and follows from Tran Van Trung's proof, that the operation of $\bar{C}$ on L/V and V satisfies the conditions of (1.12). It follows then from (1.12), that $L \cong Q^2$.

It should be noted, that $O^-(10, 2)$ possesses a maximal 2-local subgroup M such that $\boldsymbol{O}_2(M) \cong Q^2$ and $M/\boldsymbol{O}_2(M) = \bar{J} \times \bar{C}$, $\bar{J} \cong \Sigma_5$, $\bar{C} \cong L_3(2)$, where an element of order 5 in M operates fixed-point-freely on $\boldsymbol{O}_2(M)/\boldsymbol{O}_2(M)'$ and an element of order 7 operates fixed-point-freely on $\boldsymbol{O}_2(M)$. In this case, however, an element d of order 3 in J will not operate fixed-point-freely on $\boldsymbol{O}_2(M)/\boldsymbol{O}_2(M)'$. In fact we have $\boldsymbol{O}_2(M) = \big(\boldsymbol{C}(d) \cap \boldsymbol{O}_2(M)\big) * [d, \boldsymbol{O}_2(M)]$, where

$$\boldsymbol{C}(d) \cap \boldsymbol{O}_2(M) \cong [d, \boldsymbol{O}_2(M)] \cong Q_0 .$$

(iv) Let $G = M(24)'$ be one of the Fischer-groups, let z be a 2-central involution of G and set $H = \boldsymbol{C}_G(z)$, $K = H'$, $J = \boldsymbol{O}_2(H)$ like in [13]. Then $J \cong (D_8)^6$, $|\boldsymbol{O}_{2,3}(H)/J| = 3$, $K/\boldsymbol{O}_{2,3}(H) \cong U_4(3)$ and $|H:K| = 2$. Let j_2 be an involution in $J - \langle z\rangle$ such that $\boldsymbol{C}_K(j_2)/\boldsymbol{C}_J(j_2) \cong$ $\cong E_{16}/A_6$. Then $\boldsymbol{C}_H(j_2)/\boldsymbol{C}_J(j_2) \cong E_{32}A_6$. Let $R = \boldsymbol{O}_2\big(\boldsymbol{C}_K(j_2)\big)$, $\bar{H} = H/J$, $\tilde{H} = H/\langle z\rangle$ and use the « bar convention ». Then $\tilde{F} = \boldsymbol{C}_{\bar{J}}(\bar{R}) \cong E_{64}$ and $F \cong E_{128}$. Further $V = \boldsymbol{C}_G(F) \subseteq R$, $V \cong E_2 11$ and $\boldsymbol{N}_G(V)/V \cong M_{24}$. Set $M = \boldsymbol{N}_G(V)$. Like in [8, Prop. 13] consider the inverse image U in M of a maximal 2-local-subgroup of M/V, which is a faithful and splitting extension of E_{64} by $\Sigma_3 \times L_3(2)$. Set $Z = \boldsymbol{Z}\big(\boldsymbol{O}_2(U)\big)$, let P be a subgroup of order 3 in $\boldsymbol{O}_{2,3}(U)$ and let C be a subgroup of order 7 in U. Similarly like in [8, Prop 13], we get $\boldsymbol{C}_V(P) = Z \cong E_8$. Further $Z - \langle 1\rangle$ consists of 2-central involutions of G. We can choose $\langle z, j_2\rangle < Z$. Set $B = \boldsymbol{C}_H(Z)$ and $Q = \boldsymbol{O}_2(B)$. The operation of P shows $Z < F$. Further $Q = \boldsymbol{C}_R(Z)$, $|Q| = 2^{15}$ and R operates fixed-point-freely on Q/Z. We have $B/Q \cong A_6$ and $\boldsymbol{N}_G(Z)/Q \cong A_6 \times L_3(2)$. As elements of order 7 of $L_3(2)$ and elements of order 5 in A_6 operate fixed-point-freely on Q/Z, the group Q has to be special with center Z. Let S be an element of order 3 in B, which doesn't operate fixed-point-freely on Q/Z. Then by (1.1) we have $Q = Q_1 * Q_2$, where $Q_1 = \boldsymbol{C}_Q(S)$, $Q_2 = Z[Q, S]$ and Q_1, Q_2 are $L_3(2)$-admissible special groups of order 2^9 with center Z. It follows from (1.2), that $Q_1 \cong Q_2 \cong Q_0$ and thus $Q \cong Q^2$. Obviously, $\boldsymbol{N}_G(Z)$ is a maximal 2-local subgroup of $M(24)'$. Further

$$\boldsymbol{N}(Z) \cap M(24)/Q \cong \Sigma_6 \times L_3(2) .$$

(1.16) LEMMA. For every $n \geqslant 1$ there is a group X with the following properties:

(i) $\boldsymbol{O}_2(X) = Q \cong Q^n$.

(ii) $\boldsymbol{C}_X(Q) = \boldsymbol{Z}(Q) = Z$.

(iii) $X/Q = (B/Q) \times (L/Q)$, where $B = \boldsymbol{C}_X(Z)$, $B/Q \cong \mathrm{Sp}\,(2n, 2)$, $L/Q \cong L_3(2)$.

(iv) The operation of L/Q on Q/Z and Z satisfies the hypothesis of (1.12).

Then the sequence $0 \to Q/Z \to X/Z \to X/Q \to 0$ is non-split for $n > 1$, split for $n = 1$. Further the structure of X/Z is uniquely determined.

PROOF. It follows from (1.14), (1.8) and (1.5), that there is a group X satisfying (i)-(iv). Further X/Z is isomorphic to a subgroup of Aut (Q) and its structure is uniquely determined by the same lemmas mentioned above. It follows from (1.15), that the above sequence splits only for $n = 1$.

2. A 2-local characterization of Fischer's simple group $M(24)'$.

THEOREM. Let G be a finite simple group possessing a 2-local subgroup M with the following properties:

(i) $Q = \boldsymbol{O}_2(M)$ is a special group of order 2^{15} with elementary abelian center Z of order 8.

(ii) $M = \boldsymbol{N}_G(Z)$, $\boldsymbol{Z}(M) = \langle 1 \rangle$.

(iii) $Z = \boldsymbol{C}_G(Q)$.

(iv) $\bar{M} = M/Q = \bar{B} \times \bar{L}$, $\bar{B} \cong A_6$, $\bar{L} \cong L_3(2)$.

Then G has a 2-local subgroup of the form $E_2 11 \cdot M_{24}$.

COROLLARY. Under the additional assumption, that $\boldsymbol{O}(\boldsymbol{C}_G(z)) = 1$ for a 2-central involution z in G, it follows from [16] and [14], that G is isomorphic to $M(24)'$.

PROOF. Let G be a group which satisfies the assumptions of the theorem. Set $\bar{M} = M/Q$, $\tilde{M} = M/Z$ and use the « bar convention ». Let B resp. L be the inverse images of $\bar{B}$ resp. $\bar{L}$. Then $B = \boldsymbol{C}(Z)$. Let $F = \langle n, r \rangle$ be a Frobeniusgroup of order 21 contained in L, where $n^7 = r^3 = 1$, $n^r = n^2$. Clearly, elements of order 5 and 7 of M have to operate fixed-point-freely on $\tilde{Q}$. As n operates fixed-point-freely on Q, we have $B_1 = \boldsymbol{C}_B(n) = \boldsymbol{C}_B(F) \cong A_6$.

We use the symbol $\leftrightarrow$ to denote the correspondence of elements in the isomorphism $B_1 \cong A_6$. Let $D \in \boldsymbol{Syl}_3(B_1)$, $D = \langle d_1, d_2 \rangle$, where $d_1 \leftrightarrow (1, 2, 3)$, $d_2 \leftrightarrow (4, 5, 6)$. As d_1 and $d_1 d_2$ are conjugate under Aut (A_6), we can assume $|C_{\tilde{Q}}(d_1)| = |C_{\tilde{Q}}(d_2)| = 2^6$.

Assume $\boldsymbol{C}_{\tilde{Q}}(d_1) = \boldsymbol{C}_{\tilde{Q}}(d_2)$. Then $[d_1, \tilde{Q}] = [d_2, \tilde{Q}]$. As $\tilde{Q} = \langle \boldsymbol{C}_{\tilde{Q}}(d) | d \in D \rangle$, there is $\varepsilon \in \{+1, -1\}$ such that $\boldsymbol{C}_{\tilde{Q}}(d_1 d_2^{\varepsilon}) \cap [d_1, Q] \neq 1$. The operation of $\bar{n}$ shows then $[d_1 d_2^{\varepsilon}, \tilde{Q}] = 1$, a contradiction. Thus $\boldsymbol{C}_{\tilde{Q}}(d_1) = [d_2, \tilde{Q}]$, $\boldsymbol{C}_{\tilde{Q}}(d_2) = [d_1, \tilde{Q}]$, the elements $d_1 d_2$ and $d_1 d_2^{-1}$ operate fixed-point-freely on $\tilde{Q}$.

Set $Q_1 = \boldsymbol{C}_Q(d_2) = Z[d_1, Q]$ and $Q_2 = \boldsymbol{C}_Q(d_1) = Z[d_2, Q]$. It follows from (1.1), that $Q = Q_1 * Q_2$. Further $Z = \boldsymbol{Z}(Q_1) = \boldsymbol{Z}(Q_2)$ and the groups Q_i are special groups of order 2^9, $i = 1, 2$.

As B_1 operates on $\boldsymbol{C}_Q(r)$, we have $\boldsymbol{C}_Q(r) \cong E_{32}$. By (1.3) $Q_1 \cong Q_2 \cong Q_0$ and thus $Q \cong Q^2$. Further $L < \boldsymbol{N}(Q_i)$, $i = 1, 2$.

Set $L_0 = \boldsymbol{C}_L(d_1 d_2)$. Then $L_0/Z \cong L_3(2)$ and it follows from the structure of Aut (Q^2), that L_0 is conjugate to L_+ as a subgroup of Aut (Q^2) in the sense of (1.14). Especially, $\tilde{L}_0$ is a uniquely determined subgroup of Aut (Q) and thus $\tilde{M}$ is uniquely determined. It follows, that M has the structure given in the following lemma:

(2.1). LEMMA. $Q = Q_1 * Q_2 \cong Q^2$. For elements of Q we use the notation of (1.9). $L = QL_0$, $L_0 = Z\langle F, t \rangle$, $L_0/Z \cong L_3(2)$. With respect to the bases $\{v_i^{(1)}, v_i^{(2)}, v_i^{(3)}\}$ of $V_i^{(0)}$, $i \in \{\pm 1, \pm 2\}$ and $\{z_1, z_2, z_3\}$ of Z we have

$$n_{v_i^{(0)}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad r_{v_i^{(0)}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$t_{v_i^{(0)}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and $g_Z = (g_{v_i^0})^*$ for every $g \in L_0 - Z$.

Further $\boldsymbol{C}_B(F) = B_1 \cong \mathrm{Sp}_4(2)'$ and the elements of B_1 are represented in the natural way on the elementary abelian groups $\tilde{E}_i$, $i = 1, 2, 3$. The operation of B_1 on $E_i^{(1)}$ has been given in (1.15) with respect to the bases $\{v_1^{(i)}, v_2^{(i)}, v_{-1}^{(i)}, v_{-2}^{(i)}, z\}$, where $z = z_1$ for $i = 1$, $z = z_1 z_2$ for $i = 2$ and $z = z_1 z_2 z_3$ for $i = 3$.

We have $C_B(L_0) = K\langle v\rangle = N(K) \cap B_1$, where $K = \langle k_1, k_2\rangle \in \in Syl_3(B_1)$, $k_1 \leftrightarrow (1, 3, 5)$, $k_2 \leftrightarrow (2, 4, 6)$, $v \leftrightarrow (1, 2)(3, 6, 5, 4)$ and

$$k_1|_{E_i^{(1)}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad k_2|_{E_i^{(1)}} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$v|_{E_i^{(1)}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Set $v_0 \in B_1$ such that $v_0 \leftrightarrow (3, 4)(5, 6)$. Then

$$v_0|_{E_i^{(1)}} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\langle v, v_0\rangle \in Syl_2(B_1), \qquad C_Q(v) = C_Q(v, v_0) = V_{-1} \cong E_{64},$$

$C_Q(v^2) = V_{-1}(V_2 + V_{-2}) \cong E_2{}^9$. B_1 contains exactly two conjugacy-classes of elementary abelian groups of order 4 with representatives X_1 and X_2, where

$$X_1 = \langle v^2, v_0\rangle \leftrightarrow \langle (3, 5)(4, 6), (3, 4)(5, 6)\rangle,$$

$$X_2 = \langle v^2, vv_0\rangle \leftrightarrow \langle (3, 5)(4, 6), (1, 2)(3, 5)\rangle.$$

We have $C_Q(X_1) = V_{-1} \cong E_{64}$ and $C_Q(X_2) = V_{-1}(V_2 + V_{-2}) \cong E_2{}^9$. Set $V = X_2 C_Q(X_2)$. Then $V \cong E_2{}^{11}$.

PROOF. The bulk of the lemma follows from the fact, that we can choose Q_1, Q_2 so, that $Q = Q_1 * Q_2 \cong Q^2$ and that the operation of L_0 on Q_1 and Q_2 is of dihedral type in the sense of (1.7). It is a matter of direct calculation, that $C(\tilde{L}_0) \cap \tilde{B} = \tilde{K}\langle\tilde{v}\rangle$. It follows from the 3-subgroup-lemma, that $[(K\langle v\rangle)', L_0] = [K, L_0]$. As v centralizes L_0/Z and Z, we get $[K\langle v\rangle, L_0] = 1$.

(2.2) LEMMA. Let $V \subset T \in \boldsymbol{Syl}_2(M)$. Then V is the only elementary abelian subgroup of order 2^{11} of T.

PROOF. We have $T/Q = \bar{D}_1 \times \bar{D}_2$, where $\bar{D}_1 \in \boldsymbol{Syl}_2(\bar{B})$, $\bar{D}_2 \in \boldsymbol{Syl}_2(\bar{L})$. Denote by D_i the inverse image in T of $\bar{D}_i$, $i = 1, 2$.

Let $A < T$, $A \cong E_{2^{11}}$.

(1) $A \cap D_2 \subseteq Q$: Assume $A \cap D_2 \nsubseteq Q$, let $a \in (A \cap D_2) - Q$. Then $A \cap Q$ is contained in the inverse image U of $C_{\bar{Q}}(\bar{a})$. But $\bar{a} \sim \bar{t}$ and so $U \sim Z\langle v_j^{(1)}, v_j^{(2)}, v_j^{(3)} | j \in \{\pm 1, \pm 2\}\rangle$, $U \cong E_4 \times (D_8)^4$ and $Z(U) = Z$. Further $C(a) \not\geqslant Z$. Thus $C_Q(a)$ doesn't contain an elementary abelian subgroup of order 2^7. It follows $|\bar{A}| \geqslant 32$, a contradiction.

(2) $A \subseteq D_1$: If $A \nsubseteq D_1$, there is an involution $a \in A - Q$ such that $a \notin D_1$, $a \notin D_2$. As $\bar{a}$ inverts an element of order 5 in $\bar{B}$, we have $|C_{\bar{Q}}(\bar{a})| = 2^6$. Further $Z \nleqslant C(a)$ and thus $|C_Q(a)| \leqslant 2^8$. It follows $|\bar{A}| \geqslant 8$ and $\bar{A} \cap \bar{D}_2 \neq \langle 1\rangle$, a contradiction to (1).

We have $A \subseteq D_1$, $A \cap Q \cong E_{2^9}$, $|\bar{A}| = 4$. All the involutions in the coset v^2Q are contained in $v^2 C_Q(v^2) = v^2(V \cap Q)$. Thus $V \cap Q \subset A$ and $A = C(V \cap Q) \cap D_1 = V$.

$$\text{(2.3)} \qquad \boldsymbol{Syl}_2(M) \subseteq \boldsymbol{Syl}_2(G)\,.$$

PROOF. Set $J = \{x | x \in Q,\ x^2 = 1,\ |Q : C_Q(x)| = 4\}$. We have $W = V \cap Q = \langle W \cap J\rangle$. Assume $T \in \boldsymbol{Syl}_2(M)$, $T \notin \boldsymbol{Syl}_2(G)$. Then $T < T_1$, $|T_1/T| = 2$. Choose $x \in T_1 - T$. Then $Q^x \neq Q$, $Z^x \neq Z$, but $Z^x < Q$, as $|C_T(z)| \geqslant 2^{19}$ for $z \in Z$. Thus $\bar{Q}^x$ is elementary abelian and $|\bar{Q}^x| \leqslant 16$.

(1) $\bar{Q}^x \cap \bar{V} = \langle 1\rangle$: Assume the contrary. We have $Q^x \cap V = (Q \cap V)^x = W^x$. So there is an element $y \in J \cap W$ such that $y^x \in (Q^x \cap V) - Q$. As $y^x \in V - Q$, we have $C_Q(y^x) = W \cong E_{2^9}$. On the other hand $|Q \cap Q^x| \geqslant 2^{11}$ and so $|C(y^x) \cap Q \cap Q^x| \geqslant 2^9$, as $y^x \in J^x$. So $C(y^x) \cap Q \cap Q^x = W \subseteq Q$ and $1 = \bar{W} = \bar{Q}^x \cap \bar{V} \neq 1$, a contradiction.

Clearly $|\bar{Q}^x| < 16$.

(2) Assume $|\bar{Q}^x| = 8$. Then $\bar{Q}^x \cap \bar{D}_1 \neq 1$. But $\bar{Q}^x \lhd \bar{T}$ and $Z(\bar{D}_1) < \bar{Q}^x$. It follows $\bar{Q}^x \cap \bar{V} \neq \langle 1\rangle$, a contradiction.

(3) We have $|\bar{Q}^x| \leqslant 4$, $|Q \cap Q^x| \geqslant 2^{13}$. Let $y^x \in (J^x \cap Q^x) - Q$. Then $|C(y^x) \cap Q \cap Q^x| \geqslant 2^{11}$, another contradiction.

We consider now the involutions contained in $M - Q$.

(2.4) We may and will take t to be an involution in $L - Q$. Let t' be an involution in $L - Q$. Then $C_Q(t') \cong Z_2 \times (Q_8)^4$. Further $t' \underset{Q}{\sim} t'z$ if and only if $z \in \langle z_2 z_3 \rangle$.

Proof. As $L_3(2)$ contains only one class of involutions, all the involutions in $L - Q$ are conjugate to an involution in the coset tQ. If L_0 is a non-split extension of E_8 by $L_3(2)$, the Sylow-2-subgroup of L_0 is of type M_{12}. Thus in any case $L_0 - Z$ contains involutions and we can take t to be an involution. We have

$$C_Q(t) = \langle z_1 \rangle \langle v_1^{(1)}, v_{-1}^{(2)} v_{-1}^{(3)} \rangle \langle v_{-1}^{(1)}, v_1^{(2)} v_1^{(3)} \rangle \cdot$$
$$\cdot \langle v_2^{(1)}, v_{-2}^{(2)} v_{-2}^{(3)} \rangle \langle v_{-2}^{(1)}, v_2^{(2)} v_2^{(3)} \rangle \cong Z_2 \times (Q_8)^4 .$$

Let U be the inverse image of $C_{\bar{Q}}(\bar{t})$. Then $U \cong E_4 \times (D_8)^4$. Thus tQ contains exactly 2^{10} involutions. They have one of the following forms:

(1) tx, $x \in C_Q(t)$, $x^2 = 1$.

(2) $tz_2 y$, $y \in C_Q(t)$, $y^2 = z_2 z_3$.

By direct calculation we see $C_Q(t') \cong Z_2 \times (D_8)^4$ for every involution $t' \in tQ$. Obviously $t' \underset{Z}{\sim} t' z_2 z_3$ for all these involutions t'.

Assume $t' = tx$, $x \in C_Q(t)$, $x^2 = 1$, $t'^q = t' z_1$, $q \in Q$. Then $\langle q, x \rangle < U$, $t^q \in t z_1 \langle z_2 z_3 \rangle$, a contradiction.

Assume

$$t' = t z_2 y\ , \quad y \in C_Q(t)\ , \qquad y^2 = z_2 z_3\ , \qquad t'^q = t' z_1, \quad q \in Q\ .$$

Then $\langle q, z_2 y \rangle < U$, $t^q = t z_1$, a contradiction like above.

(2.5) Lemma. All the involutions in $B - Q$ are conjugate to v^2 or to $v^2 z_1$. We have $C_Q(v^2) = W = V \cap Q \cong E_{2^9}$.

Proof. We have $C_Q(v^2) = W \cong E_{2^9}$ and $[v^2, Q] < W$, $|[v^2, Q]| = 2^6$. It follows, that the involutions in $v^2 Q$ are all contained in $v^2 W$. As $|C_{\bar{Q}}(\bar{v}^2)| = 2^6$, there are exactly 8 classes of involutions in $B - Q$ under the operation of Q and the elements $v^2 z$, $z \in Z$, are representatives of these classes. If $z, z' \in Z - \{1\}$, we have $v^2 z \sim v^2 z'$ under L_0, as $[v^2, L_0] = 1$. But $v^2 \underset{M}{\not\sim} v^2 z$ if $z \in Z - \{1\}$.

(2.6) LEMMA. All the involutions in $M-Q$, which are not contained in B or L, are conjugate to v^2t. We have $C_Q(v^2t)\cong E_{16}\times Q_8$:

PROOF. By (2.1) v^2t is an involution. Clearly all the involutions in $M-(B\cup L)$ are conjugate to an involution in v^2tQ. Let U be the inverse image of $C_{\tilde{Q}}(\bar{v}^2\bar{t})$. Then $U=Z\langle x_1, x_2, \ldots, x_6\rangle$, where

$$x_1=v_{-1}^{(1)},\ x_2=v_2^{(1)}v_{-2}^{(1)},$$

$$x_3=v_1^{(2)}v_1^{(3)}v_2^{(3)}v_{-1}^{(3)}v_{-2}^{(3)},\qquad x_4=v_2^{(2)}v_2^{(3)}v_{-1}^{(3)},$$

$$x_5=v_{-1}^{(2)}v_{-1}^{(3)},\qquad x_6=v_{-2}^{(2)}v_{-1}^{(3)}v_{-2}^{(3)},$$

$$|U|=2^9,\ x_i^2=1 \text{ for } i\neq 3,\ x_3^2=z_1,$$

$$C_Q(v^2t)=\langle z_1, x_1, x_5, x_4x_6\rangle\times\langle x_2, x_4\rangle\cong E_{16}\times D_8\,,\qquad C_Q(v^2t)'=\langle z_2z_3\rangle\,.$$

Set $U_1=C_Q(v^2t)$. We have $z_2^{v^2t}=z_3$, $x_3^{v^2t}=x_3^{-1}=x_3z_1$.

By direct calculation we see, that v^2tQ contains exactly 2^8 involutions, namely 96 in v^2tU_1, 32 in $v^2tz_2U_1$, 64 in $v^2tx_3U_1$ and 64 in $v^2tx_3z_2U_1$. As $|Q:U_1|=2^8$, the lemma is proved.

(2.7) LEMMA. Every involution in Q is conjugate under M to an involution contained in V.

PROOF. There are exactly $3\times5\times7^2$ nontrivial cosets in $\tilde{Q}$, which consist of involutions. Consider the operation of $\bar{M}$ on $\tilde{Q}$. Let $\bar{t}_1\in\bar{L}$ like in (1.7). Then $C_{\tilde{Q}}(\bar{t},\bar{t}_1)\cong E_{16}$ and $\bar{B}$ induces a natural representation as $\mathrm{Sp}(4,2)'$ on $C_{\tilde{Q}}(\bar{t},\bar{t}_1)$. Let $\tilde{q}_1\in C_{\tilde{Q}}(\bar{t},\bar{t}_1)$. Then $q_1^2=1$ and $C_{\bar{M}}(\tilde{q}_1)=C_{\bar{B}}(\tilde{q}_1)\times N_{\bar{L}}(\langle\bar{t},\bar{t}_1\rangle)\cong\Sigma_4\times\Sigma_4$, $|\tilde{q}_1^{\bar{M}}|=3\times5\times7$. Let $\tilde{q}_2\in\tilde{Q}$ $q_2^2=1$, $\tilde{q}_2\notin\tilde{q}_1^{\bar{M}}$. Then $2^6\nmid|C_{\bar{M}}(\tilde{q}_2)|$.

Assume $9\,||C_{\bar{M}}(\tilde{q}_2)|$. Then $\tilde{q}_2$ has to be centralized by an element of order 3 in $\bar{B}$. We can assume $q_2\in Q_2$, where $Q=Q_1*Q_2$. But then $\tilde{q}_2$ is centralized by an element of order 3 in $\bar{L}$. It follows $\tilde{q}_2\underset{\bar{M}}{\sim}\tilde{q}_1$, a contradiction. We have $9\nmid|C_{\bar{M}}(\tilde{q}_2)|$ and thus $|\tilde{q}_2^{\bar{M}}|\geqslant 2\times3^2\times5\times7$. It follows $|\tilde{q}_2^{\bar{M}}|=2\times3^2\times5\times7$.

So there are exactly two conjugacy-classes of nontrivial cosets in Q/Z, which contain involutions. These classes then have to consist of those cosets which contain involutions $q\in Q-Z$ such that $|Q\colon C_Q(q)|=4$ resp. $|Q\colon C_Q(q)|=8$. As $W-Z$ contains involutions of both types, the lemma is proved.

(2.8) LEMMA. $N_G(V)\not\subseteq M$.

PROOF. (1) If $N_G(V) \subseteq M$, then Z is strongly closed in B with respect to G: Let $z \in Z - \{1\}$, $z^g \in Q$, $g \in G$. Then $z^{gm} \in W \subset V$, $m \in M$. By (2.2), (2.3) we can assume $gm \in N(V)$. By assumption $gm \in M$, $g \in M$ and thus $z^g \in Z$.

Assume $z^g \in B - Q$, $g \in G$. Then $z^{gm} \in xQ$, $m \in B$, $x \in V$. As $z^2 = 1$, we have $z^{gm} \in xC_Q(x) \subset V$. Thus we can assume $gm \in N(V)$, $g \in M$, a contradiction.

(2) If $N_G(V) \subseteq M$, then no element of Z is conjugate in G to an involution $x \in M - (B \cup L)$: Assume $x^g = z$, $g \in G$. We have $C_Q(x) \cong E_{16} \times D_8$ by (2.6). Let $E < C_Q(x)$, $E \cong E_{64}$. Then, by (2.6), x is conjugate under Q to all of the elements of the coset xE. Choose $g \in G$ such that $C_T(x)^g \subseteq T$. Then $E^g \cong E_{64}$, $E^g < T$, $z \notin E^g$ and z is conjugate to every element of the coset zE^g. We have $|E^g/E^g \cap D_1| \leqslant 4$, a contradiction to (1).

(3) If $N_G(V) \subseteq M$, then Z is strongly closed in $N_G(V)$ with respect to G: assume the contrary. Then by (1), (2) $z^g \in L - Q$, $g \in G$, $z \in Z$. We can choose $z^g \in tQ \subset N(W)$. Set $X = [z^g, W]$. By (2.4) either $|X| = 8$, $|X \cap Z| = 2$ or $|X| = 16$, $|X \cap Z| = 4$. Set $Z_0 = X \cap Z$. Again, z^g is conjugate to every element of the coset $z^g X$, but $X \cap z^G = Z_0 - \{1\}$ by (1), (2). We have $X^{g^{-1}} < C(z)$ and we can assume $X^{g^{-1}} < T$. Further $z \notin X^{g^{-1}}$, $z \sim zx^{g^{-1}}$ for all $x \in X$. It follows $X^{g^{-1}} \cap Z = Z_0^{g^{-1}}$. Let $x \in X - Z_0$. Then $C_Z(x^{g^{-1}}) \geqslant \langle Z_0^{g^{-1}}, z\rangle$. Thus $Z_0 = \langle z_0\rangle$, $|Z_0| = 2$, $|X| = 8$, $C_Z(x^{g^{-1}}) = \langle z, z_0^{g^{-1}}\rangle$ for every $x \in X - \langle z_0\rangle$. There is then an $y \in X - \langle z_0\rangle$ such that $y^{g^{-1}} \underset{Z}{\sim} y^{g^{-1}} \cdot z \underset{G}{\sim} z$, a contradiction to $X \cap z^G = \{z_0\}$.

We have proved, that Z is strongly closed in T, where $T \in Syl_2(G)$, in case $N_G(V) \subseteq M$. This contradicts Goldschmidt's result [5].

(2.9) LEMMA. Set $N = N_G(V)$, $\bar{N} = N/V$. Then $\bar{N} \cong M_{24}$ and the lengths of the orbits of $V^\#$ under the operation of N are 1771 and 276.

PROOF. We have $O(N) \leqslant C(V) \leqslant V$ and so $O(N) = \langle 1\rangle$. As $C_G(V) = V$, the group $\bar{N}$ is isomorphic to a subgroup of $GL(11, 2)$.

Further $|\bar{N}|_2 = 2^{10}$ and $\bar{N} > N \cap M/V$. It is clear from the structure of $GL(11, 2)$, that $O(\bar{N}) = \langle 1\rangle$. We have $V \leqslant O_2(N) \leqslant O_2(N \cap M) = VQ$. As Z char $Q = \langle x | x \in VQ, x^2 = 1, |C_{VQ}(x)| \geqslant 2^{11}\rangle$ char VQ, we get $O_2(N) \neq VQ$. Because of the irreducibility of $N \cap M/VQ$ on VQ/V, we have $O_2(N) = V$ and $O_2(\bar{N}) = \langle 1\rangle$.

Let $\bar{X}$ be a minimal normal subgroup of $\bar{N}$. Then $\bar{X} = \bar{X}_1 \times \ldots \times \bar{X}_s$, where the $\bar{X}_i$ are isomorphic non-abelian simple groups. Further

$O_2(\overline{N \cap M}) < \bar{X}$ and $\overline{N \cap M}/\overline{VQ} \simeq \Sigma_3 \times L_3(2)$. It follows $\bar{L} < \bar{X}$ and $|\tilde{X}|_2 \in \{2^9, 2^{10}\}$. Assume $s > 1$.

If $s = 2$, then $|\bar{X}|_2 = 2^{10}$, but the center of a Sylow-2-subgroup of $\overline{N \cap M}$ has order 2, a contradiction.

If $s \geqslant 3$, the center of a Sylow-2-subgroup of $\bar{X}$ has order at least 8, but this is impossible for the same reason.

Hence $s = 1$, $\bar{X}$ is a simple group and $\bar{N} \leqslant \mathrm{Aut}(\bar{X})$.

The lengths of the orbits of $V - \{1\}$ under the operation of $N \cap M$ are 7/336-84-84/1344-192. Here, the orbit of length 7 is $Z - \{1\}$, the orbits of lengths 336 and 84 are contained in $W - Z$.

(1) $N(V) \not\subseteq N(W)$: Assume $W \lhd N$. Let X be the inverse image of $\bar{X}$ in N, set $\tilde{X} = X/W$. Then $\tilde{X} = \tilde{V} \times \tilde{Y}$, where $\tilde{Y} \simeq \bar{X}$. The simple group $\tilde{Y}$ is isomorphic to a subgroup of $GL(9, 2)$ and is generated by involutions of type J_2 in the sense of [4]. Further $\tilde{Y}$ operates irreducibly on W. The length of the Y-orbit containing $Z - \{1\}$ is $5^2 \times 7$ or 7^3. We get then a contradiction from [4, Theorem A], [10] and [18].

(2) The lengths of the orbits of $V - \{1\}$ under N and under X are 1771 and 276: We use (1) and the fact, that $N \not\subseteq M$. The only other possibility for the lengths of orbits under N is 1519-528. Here $1519 = 7 + 1344 + 84 + 84 = 7^2 \times 31$,

$$528 = 336 + 192 = 2^4 \times 3 \times 11 .$$

Consider $V/\langle z \rangle$, where $1 \neq z \in Z$. We see then, that the homomorphic images in $V/\langle z \rangle$ of the elements in V contained in the $N \cap M$-orbits of length 336 are the only ones which don't contain an involution conjugate to z under N. Thus $W \lhd C_N(z)$. This contradicts the fact, that $11 \mid |C_N(z)|$.

We have

$$1771 = 7 + 1344 + 336 + 84 = 7 \times 11 \times 23 ,$$

$$276 = 192 + 84 = 2^2 \times 3 \times 23 .$$

Obviously, a Sylow-23-normalizer has to be a Frobeniusgroup of order 23×11 in $\bar{X}$ as well as in $\bar{N}$. It follows from the Frattiniargument, that $\bar{N} = \bar{X}$ and $\bar{N}$ is a simple group.

Further $\bar{N}$ possesses a 2-local subgroup, which is an extension of E_{64} by $\Sigma_3 \times L_3(2)$. The element of order 3 in Σ_3 operates fixed-point-freely and so the extension is split. As $L_3(2)$ operates completely reducibly on E_{64}, this 2-local subtroup is uniquely determined and a Sylow-2 subgroup of $\bar{N}$ is isomorphic to a Sylow-2-subgroup of M_{24}. It follows from [17], that $\bar{N}$ is isomorphic to M_{24}.

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