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Abelian Groups in which Every Pure Subgroup is an Isotype Subgroup.

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All groups in this paper are assumed to be abelian groups. Concerning the terminology and notation we refer to [2]. In addition, if G is a group then G_t and G_p are the torsion part and the p-component of G_t respectively. Let G be a group and p a prime. Following Rangaswamy [11], a subgroup H of G is said to be p-absorbing, resp. absorbing in G if $(G/H)_p = 0$, resp. $(G/H)_t = 0$. Obviously, every p-absorbing subgroup of G is p-pure in G. Recall that a subgroup H of G is isotype in G if $H \cap p^\alpha G = p^\alpha H$ for all primes p and all ordinals α. For example, G_t and every G_p are isotype in G, every basic subgroup of G_p is isotype in G_p. If H is an isotype subgroup of G and A a subgroup of G containing H then H is isotype in A. If G is torsion then a subgroup H of G is isotype in G iff every H_p is isotype in G_p. Each absorbing subgroup of G is isotype in G (see lemma 103.1, [2]).

The notion of isotype subgroups has been introduced by Kulikov [7] and investigated by Irwin and Walker [4]. It is well-known that there are groups in which not every pure subgroup is isotype (see e.g. [4] or ex. 6, 7, § 80, [2]).

The purpose of this paper is to describe the classes of all groups in which every pure subgroup is an isotype subgroup, every isotype subgroup is a direct summand, every isotype subgroup is an absolute direct summand, every neat subgroup is an isotype subgroup and every isotype subgroup is an absorbing subgroup.

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Note that the classes of all groups in which every subgroup is a neat, resp. a pure subgroup, resp. a direct summand, resp. an absolute direct summand have been described in [12] and [5], resp. [3], resp. [6], resp. [12]; these classes coincide with the class of all elementary groups. The classes of all groups in which every neat subgroup is a pure subgroup, resp. a direct summand, resp. an absolute direct summand have been described in [9], [12] and [14] (see theorem 4), resp. [12] and [8], resp. [12] (see theorem 3). The classes of all groups in which every pure subgroup is a direct summand, resp. an absolute direct summand have been described in [15] and [3] (see theorem 2), resp. [12] (see theorem 3). The class $\mathcal{A}$ of all groups in which every direct summand is an absolute direct summand has been described in [12]; $G \in \mathcal{A}$ iff either G is a torsion group in which each p-component is either divisible or a direct sum of cyclic groups of the same order or G is divisible or $G = G_t \oplus R$, where G_t is divisible and R is indecomposable. The class $\mathcal{B}$ of all groups in which every absorbing subgroup is a direct summand has been described in [11] and [12]; $G \in \mathcal{B}$ iff $G = T \oplus D \oplus N$, where T is torsion, D is divisible and N is a direct sum of a finite number mutually isomorphic rank one torsion free groups. Finally, in [12] have been described the classes of all groups in which every neat, resp. pure subgroup is an absorbing subgroup (see theorem 6).

Definition. Let $\mathfrak{C}$ be the class of all groups in which every pure subgroup is an isotype subgroup.

Lemma 1. The class $\mathfrak{C}$ is closed under pure subgroups.

Proof. Obvious.

Lemma 2. Let G be a group, p a prime and S a p-pure subgroup of G. If G_p is a direct sum of a divisible and a bounded group then $p^\alpha S = S \cap p^\alpha G$ for every ordinal α.

Proof. Let S be a p-pure subgroup of G. Since S_p is pure in G_p, $S_p = D \oplus B$, where D is divisible and B is bounded (see e.g. lemma 4.2, [1]). Now, $S = S_p \oplus H$, $p^\omega S = p^\omega S_p \oplus p^\omega H = D \oplus p^\omega H$ and obviously $p^\omega S$ is p-divisible. Consequently, $p^\alpha S = S \cap p^\alpha G$ for every ordinal α.

Lemma 3. Let G be a group, p a prime and k a natural number. If $p^{\omega+k} G_p$ is not essential in $p^\omega G_p$ and either $p^{\omega+k+1} G_p$ is nonzero or $p^{\omega+k+1} G$ is not torsion then $G \notin \mathfrak{C}$.

PROOF. There is a nonzero element $n \in p^{\omega}G[p]$ such that $\langle n\rangle \cap \cap p^{\omega+k}G_p = 0$. Let $g \in p^{\omega+k}G$ and either $0 \neq pg \in G_p$ or $o(g) = \infty$. Write $X = \langle p^{\omega+k}G[p], pg, n+g\rangle$. Now, $\langle n\rangle \cap X = 0$. For, if $n = x + apg + b(n+g)$, where a, b are integers and $x \in p^{\omega+k}G[p]$, then

$$(1-b)n = x + apg + bg \in \langle n\rangle \cap p^{\omega+k}G = 0 .$$

Hence $p|1-b$, $(ap+b)pg = 0$—a contradiction. Let H be an $\langle n\rangle$-high subgroup of G containing X. Since $\langle n\rangle \subset p^{\omega}G$, H is pure in G (see [10]). Now, $pg \in p^{\omega+k+1}G \cap H \setminus p^{\omega+k+1}H$. For, if $pg = ph$ for some $h \in p^{\omega+k}H$ then $g - h \in p^{\omega+k}G[p] \subset H$, $g \in H$ and $n \in H$—a contradiction. Consequently, $G \notin \mathcal{C}$.

LEMMA 4. If G is a p-group then $G \in \mathcal{C}$ iff either G is a direct sum of a divisible and a bounded group or G^1 is elementary.

PROOF. If G is a direct sum of a divisible and a bounded group then $G \in \mathcal{C}$ by lemma 2. If G^1 is elementary and S is a pure subgroup of G then $p^{\omega}S = S \cap p^{\omega}G$ and $p^{\omega+1}S = S \cap p^{\omega+1}G = 0$. Hence $G \in \mathcal{C}$.

Conversely, let $G \in \mathcal{C}$. Let $G^1 = D \oplus R$, where D is divisible and R is reduced. If both D and R are nonzero then write $R = \langle a\rangle \oplus R'$, where $o(a) = p^k$, $k > 0$. Now, p^kG^1 is not essential in G^1, $p^{k+1}G^1 \neq 0$ and lemma 3 implies a contradiction. If G^1 is reduced and not bounded then $G^1 = \langle a\rangle \oplus \langle b\rangle \oplus R'$, where $o(a) = p^k$, $o(b) = p^j$, $j - k \geqslant 2$. Now, p^kG^1 is not essential in G^1, $p^{k+1}G^1 \neq 0$ and lemma 3 implies a contradiction. Consequently, G^1 is either divisible or bounded. Let G^1 be nonzero divisible, write $G = G^1 \oplus H$. By [13], if H is not bounded then for any nonzero element $a \in G^1[p]$ there is a pure subgroup P of G such that $P \cap G^1 = \langle a\rangle$. Obviously, P is not isotype in G. Hence in this case, G is a direct sum of a divisible and a bounded group. Let G^1 be bounded; suppose that $pG^1 \neq 0$. If H is any high subgroup of G then H is not bounded. By [13], if a is a nonzero element of $pG^1[p]$ then there is a pure sugroup P of G such that $P \cap G^1 = \langle a\rangle$. Obviously, P is not isotype in G. Hence in this case, G^1 is elementary.

LEMMA 5. Let G be a torsion group. Then $G \in \mathcal{C}$ iff $G_p \in \mathcal{C}$ for every prime p.

PROOF. Obvious.

LEMMA 6. Let G be a group, p a prime and a an element of G such that $o(a) = \infty$ or $p|o(a)$. If H is a subgroup of G maximal with respect to the conditions $pa \in H$, $a \notin H$, then H is q-absorbing in G for each prime $q \neq p$.

PROOF. Let $g \in G \setminus H$ and $qg \in H$, where q is a prime, $q \neq p$. Evidently, $a \in \langle H, g\rangle$, i.e. $a = h + ng$, where $h \in H$ and n is an integer, $(n, q) = 1$. Now, $pa = ph + png$ and hence $png \in H$. Therefore q/pn—a contradiction. Consequently, H is q-absorbing in G.

THEOREM 1. Let G be a group. The following are equivalent:

(i) Every pure subgroup of G is isotype in G (i.e. $G \in \mathfrak{C}$).

(ii) For every prime p either G_p is a direct sum of a divisible and a bounded group or G_p is unbounded, $(G_p)^1$ is elementary and $p^\omega G$ is torsion.

PROOF. Suppose that (ii) holds. Let S be a pure subgroup of G. By lemmas 1, 4 and 5, S_t is isotype in G_t. Let p be any prime. If $p^\omega G$ is torsion and α an ordinal, $\alpha \geqslant \omega$, then

$$p^\alpha S = p^\alpha S_t = S_t \cap p^\alpha G_t = S \cap p^\alpha G_t = S \cap p^\alpha G .$$

If G_p is a direct sum of a divisible and a bounded group then by lemma 2, $p^\alpha S = S \cap p^\alpha G$ for every ordinal α. Consequently, the subgroup S is isotype in G.

Conversely suppose that $G \in \mathfrak{C}$. By lemmas 1 and 4, for every prime p either G_p is a direct sum of a divisible and a bounded group or $(G_p)^1$ is elementary. If for some prime p $(G_p)^1$ is a nonzero elementary group and $p^\omega G$ is not torsion then $p(G_p)^1$ is not essential in $(G_p)^1$, $p^{\omega+2}G$ is not torsion and lemma 3 implies a contradiction. Consequently, if $(G_p)^1$ is a nonzero elementary group then $p^\omega G$ is torsion.

To finish the proof it is sufficient to show that if G_p is unbounded, $(G_p)^1 = 0$ and $p^\omega G$ is not torsion then $G \notin \mathfrak{C}$. In this case, there is a linearly independent set $\{b_1, b_2, \dots\}$ in G such that $o(b_i) = p^i$. Let $g \in p^\omega G$ be an element of infinite order; there are elements $g_1, g_2, g_3, \dots$ such that $p^{i-1}g_i = g$ for every $i = 1, 2, 3, \dots$. Put $X = \langle pg, g_1 + b_1, g_2 + b_2, \dots\rangle$. We show that $g \notin X$. Suppose $g \in X$, i.e.

$$g = z_0 pg + z_1(g_1 + b_1) + \dots + z_k(g_k + b_k) ,$$

where $z_0, \dots, z_k$ are integers. Then

$$(*) \qquad -(z_1 b_1 + \dots + z_k b_k) = z_0 pg - g + z_1 g_1 + \dots + z_k g_k .$$

From $(*)$ follows

$$-p^{k-1} z_k b_k = p^{k-1}(z_0 pg - g + z_1 g_1 + \dots + z_k g_k) \in G_p \cap p^\omega G = 0$$

and hence $p|z_k$. From $(*)$ follows

$$-p^{k-2} z_{k-1} b_{k-1} - p^{k-2} z_k b_k = p^{k-2}(z_0 pg - g + \dots + z_k g_k) \in G_p \cap p^\omega G = 0$$

and hence $p|z_{k-1}$ and $p^2|z_k$. Finally we have $p^{k-1}|z_k$, $p^{k-2}|z_{k-1}, \dots, p|z_2$. Now, from $(*)$ follows

$$z_1 b_1 + \dots + z_k b_k \in G_p \cap p^\omega G = 0$$

and hence $p^k|z_k, \dots, p|z_1$. Write $z_2 = pz_2', \dots, z_k = p^{k-1} z_k'$; from $(*)$ follows

$$(z_0 p - 1 + z_1 + z_2' + \dots + z_k')g = 0$$

—a contradiction, since $p|z_1, p|z_2', \dots, p|z_k'$.

Let H be a subgroup of G maximal with respect to the properties $X \subset H$, $g \notin H$. By lemma 6, H is q-pure in G for every prime $q \neq p$. Moreover, H is p-pure in G. For, the inclusion $p^i G \cap H \subset p^i H$ holds for $i = 0$, suppose that holds for i. Let $p^{i+1} a \in H$ for some $a \in G$, we may suppose that $p^i a \notin H$. Now, $g \in \langle p^i a, H \rangle$, i.e. $g = rp^i a + h$, where $h \in H$ and r is an integer, and evidently, $(r, p) = 1$. Further, $rp^i a \in \langle g, H \rangle$, $pp^i a \in H$ and therefore $p^i a \in \langle g, H \rangle$, i.e. $p^i a = kg + h'$ for some $h' \in H$ and some integer k. Hence

$$p^i a - kp^i g_{i+1} = h' \in p^i G \cap H .$$

By induction hypothesis, $h' = p^i h''$ for some $h'' \in H$. Now,

$$p^{i+1} a = pkg + p^{i+1} h'' = p^{i+1}(kg_{i+1} + kb_{i+1} + h'')$$

and hence $p^{i+1} a \in p^{i+1} H$. Finally, the subgroup H is not isotype in G. For, if $pg = ph$ for some $h \in p^\omega H$ then $g - h \in G_p \cap p^\omega G = 0$—a contradiction; hence $pg \in H \cap p^{\omega+1} G \setminus p^{\omega+1} H$. Consequently, $G \notin \mathfrak{C}$.

THEOREM 2. Let G be a group. The following statements are equivalent:

(i) Every isotype subgroup of G is a direct summand of G.

(ii) Every pure subgroup of G is a direct summand of G.

(iii) $G = T \oplus D \oplus N$, where T is a torsion group in which each p-component is bounded, D is divisible and N is a direct sum of a finite number mutually isomorphic torsion-free rank one groups.

PROOF. Obviously, (ii) implies (i). Assume (i). Since every absorbing subgroup of G is isotype in G, every absorbing subgroup of G is a direct summand of G. By [11], $G = T \oplus D \oplus N$, where T is torsion reduced, D divisible and N is a direct sum of a finite number mutually isomorphic torsion free groups of rank one. Moreover, T_p is bounded for every prime p. Otherwise, T_p contains a proper basic subgroup B, B is isotype in T_p and hence in G. Consequently, $T_p = B \oplus C$, where C is divisible—a contradiction. By theorem 1, every pure subgroup of G is isotype in G and hence a direct summand of G. Consequently, (ii) holds. The equivalence (ii) $\leftrightarrow$ (iii) is proved in [15].

THEOREM 3. Let G be a group. Then the following are equivalent:

(i) Every isotype subgroup of G is an absolute direct summand of G.

(ii) Every pure (neat) subgroup of G is an absolute direct summand of G.

(iii) Either G is a torsion group each p-component in which is either divisible or a direct sum of cyclic groups of the same order or $G = G_t \oplus R$, where G_t is divisible and R is a group of rank one or G is divisible.

PROOF. The equivalence (ii) $\leftrightarrow$ (iii) is proved in [12]. Obviously, (ii) implies (i). If every isotype subgroup of G is an absolute direct summand of G then each isotype subgroup of G is a direct summand of G and every direct summand of G is an absolute direct summand of G. Now, theorem 2 and [12] imply (iii).

THEOREM 4. Let G be a group. The following are equivalent:

(i) Every neat subgroup of G is isotype in G.

(ii) Every neat subgroup of G is pure in G.

(iii) Either G is a torsion group in which every p-component is either divisible or a direct sum of cyclic groups of orders p^i and p^{i+1} or G_t is divisible.

PROOF. The equivalence (ii) $\leftrightarrow$ (iii) is proved in [9], the implication (i) $\leftrightarrow$ (ii) is trivial. Suppose that every neat subgroup of G is pure in G; hence (iii) holds. By theorem 1, every pure subgroup of G is isotype in G. Consequently, (i) holds.

THEOREM 5. Let G be a group. The following are equivelent:

(i) Every subgroup of G is isotype in G.

(ii) G is elementary.

PROOF. It follows from [3] and [6].

THEOREM 6. Let G be a group. The following statements are equivalent:

(i) Every isotype subgroup of G is an absorbing subgroup of G.

(ii) Every pure (neat) subgroup of G is an absorbing subgroup of G.

(iii) Either G is torsion free or G is cocyclic.

PROOF. Since the equivalence (ii) $\leftrightarrow$ (iii) is proved in [12], it is sufficient to show that (i) implies (iii). If G is torsion then G is indecomposable and hence cocyclic. If G is mixed then G_t is cocyclic, G splits—a contradiction.

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