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G-domains and pseudo-valuations

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1. Introduction.

In this note we show that there is a one to one correspondence between the equivalence classes of pseudovaluations on a field $K$ and the equivalence classes of $G$-domains contained in $K$ and having $K$ as their field of quotient (theorem 3). We also show that if a $G$-domain is completely integrally closed, then it gives rise to a homogeneous pseudo-valuation and conversely (theorems 5 and 6). We recall all the necessary definitions and basic results to make this note reasonably self contained.

2. Definitions.

Let $R$ be an integral domain and $K$ be its field of quotients. We say $R$ is a $G$-domain if $K$ is finitely generated as a ring over $R$. That is to say $R[a_1, \ldots, a_n] = K$ where $a_i \in K$. It is easy to see that if $R$ is a $G$-domain then $K = R[u^{-1}]$ where $u$ belongs to $R$. See [2] for details. In case $R \neq R$, then we have

(i) $u^{-1} \notin R$,

(ii) $\ldots u^2 \cdot R \subset u \cdot R \subset u^{-1} \cdot R \subset u^{-2} \cdot R \subset \ldots$,

(iii) $K = \bigcup_{n=1}^{\infty} u^{-n} R$,

(iv) $0 = \bigcap_{n=1}^{\infty} u^n R$.

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Only the last needs to be checked as the other three are evident. Suppose $0 \neq a \in \bigcap u^n \cdot R$. Then if we take $a^{-1}$ it cannot belong to any $u^{-n}R$, in contradiction to (iii), as otherwise $a^{-1} \in u^{-n}R$, $a \in u^{n+1}R \Rightarrow a^{-1}a = 1 \in uR$, in contradiction to the fact that $u$ is not a unit.

We recall a pseudo valuation $\omega$ on a field $K$ is a real valued function such that

(i) $\omega(x) > 0$ for all $x \in K$ with equality holding where $x = 0$;

(ii) $\omega(x \cdot y) < \omega(x) \cdot \omega(y)$ and

(iii) $\omega(x - y) < \omega(x) + \omega(y)$ for all $x$ and $y$ in $K$.

In case we have

(iii') $\omega(x - y) < \operatorname{Max} \{\omega(x), \omega(y)\}$

then $\omega$ is said to be a non-archimedean.

If $R$ is a $G$-domain with $K = R[u^{-1}]$, then by setting

$$
\omega(x) = \begin{cases} 
2^{-r(x)} & \text{if } x \neq 0, 1 < \alpha < 2, \\
0 & \text{if } x = 0,
\end{cases}
$$

where $r(x) = n$ whenever $x \in u^n \cdot R \setminus u^{n+1} \cdot R$, we find that $\omega$ satisfies all the conditions of a pseudo valuation with

$$
\omega(x - y) < \alpha \cdot \operatorname{Max} \{\omega(x), \omega(y)\}.
$$

We have the following:

3. Results.

**Theorem 1.** Let $R$ be a $G$-domain with its quotient field $K = R[u^{-1}]$. Then there exists a pseudo valuation $\omega_u$ on $K$ such that

$$
R = \{x \in K \mid \omega_u(x) < 1\}.
$$

Moreover, if $t$ is any other element in $R$ such that $K = R[t^{-1}]$ and $\omega_t$ is the pseudo valuation arising out of $t$ then $\omega_u$ and $\omega_t$ are equivalent in the sense that they define the same topology.
PROOF. Setting \( \omega = \omega_u \) in the discussion in 2 and using theorem 4.1 of Cohn [1], we get that \( \omega_u \) is a pseudo valuation.

Since \( v(x) > 0 \) for all \( x \in R \) and \( \omega_u(x) = 2^{-v(x)} < 1 \) for these \( x \) we get that

\[
R = \{x \in K | \omega_u(x) < 1\}.
\]

The topology arising out of \( \omega_u \) has \( \{u^n \cdot R\} \) as basis of neighbourhoods of 0. Similarly \( \{t^n \cdot R\} \) is a basis of neighbourhoods of 0 with respect to the pseudovaluation \( \omega_t \) and as these topologies are dependent on \( R \) and not on the gauge elements \( u \) and \( t \) we have the desired conclusion.

Thus to each \( G \)-domain \( R \) we have associated an equivalence class of pseudo valuations. Next we show that given a pseudovaluation \( \omega \) on \( K \), there exists a \( G \)-domain \( R_\omega \) associated with \( \omega \) such that the pseudovaluation arising out of \( R_\omega \) is equivalent to \( \omega \).

**THEOREM 2.** Let \( \omega \) be a non-trivial pseudovaluation on a field \( K \) and \( D = \{x \in K | \omega(x) < 1\} \). If \( R_\omega = \{x \in K | x \cdot D \subseteq D\} \) then \( R_\omega \) is a \( G \)-domain having \( K \) as its field of quotients.

**PROOF.** That \( R_\omega \) is a subring of \( K \) can be verified easily. As \( \omega \) is non-trivial there exist elements \( u \) in \( K \) such that \( 0 < \omega(u) < 1 \). Then as \( K = \bigcup u^{-n} \cdot R_\omega \) it is easily seen that \( K = R[u^{-1}] \). Now \( \{u^n \cdot R_\omega\} \) is a basis of neighbourhoods of 0 with respect to the topology of the pseudovaluation \( \omega \). If \( \omega_u \) is the pseudo valuation on \( K \) arising out of \( R_\omega \) with \( u \) as a gauge element, then the topology induced by \( \omega_u \) and \( \omega \) are equal. Thus \( \omega \) and \( \omega_u \) are equivalent pseudo valuations.

Next we define two \( G \)-domains \( R_1 \) and \( R_2 \) having the same quotients field \( K \) to be equivalent if there exist non-zero element \( a_1 \) and \( a_2 \) in \( K \) such that \( a_1 \cdot R_1 \subseteq R_2 \) and \( a_2 \cdot R_2 \subseteq R_1 \).

The following theorem establishes a one-one correspondence between the equivalence class of pseudo valuations and equivalent \( G \)-domains.

**THEOREM 3.** Let \( R_i \) (\( i = 1, 2 \)) be two \( G \)-domains having the same field of quotients \( K \). Then \( R_1 \) and \( R_2 \) are equivalent if and only if both these give rise to the same equivalence class of pseudo valuations.

**PROOF.** Suppose \( K = R_i[u_i^{-1}] \) for \( i = 1, 2 \). If \( R_1 \) and \( R_2 \) are equivalent, then the topologies induced by the pseudovaluations are equivalent and hence the two pseudo valuations belong to the same class.
On the other hand, if \( \psi_1 \) and \( \psi_2 \) are two equivalent pseudo valuations and \( R_i = R_{\psi_i} \) (\( i = 1, 2 \)) then \( R_i \) is a \( G \)-domain by theorem 2. If \( u_1, u_2 \in K \) such that \( K = R_i[u_i^{-1}] \), then a basis of neighbourhoods of 0 under the topology induced by \( \psi_i \) is given by \( \{u_i^n \cdot R_i\} \) for \( i = 1, 2 \). As these topologies are equivalent we find that \( R_2 \subset u_1^{-m} \cdot R_1 \) and \( R_1 \subset u_2^{-m} \cdot R_2 \). Thus \( u_1^n \cdot R_2 \subset R_1 \) and \( u_2^n \cdot R_1 \subset R_2 \) so that \( R_1 \) and \( R_2 \) are equivalent.

We recall that a pseudovaluation is called homogeneous if \( \omega(x^n) = (\omega(x))^n \) for all integers \( n > 0 \), and all \( x \) in \( K \).

As examples of homogeneous pseudovaluations we cite the usual valuations and \( \text{Min} \{v_i(x)\} = \omega(x) \) for any finite set of valuations on a given field.

We need the notion of complete integral closures. We begin with the definition of almost integral elements. Let \( R \subset S \) be two commutative rings with the same identity. An element \( s \) in \( S \) is called almost integral over \( R \) if \( \{s^n\} \), for all \( n > 0 \) belongs to a finite \( R \)-submodule of \( S \).

If \( R = R^* = \{x \in S \mid x \) is almost integral over \( R\} \), then we say that \( R \) is completely integrally closed in \( S \). If \( R \subset R^* \) then \( R^* \) is called the complete integral closures of \( R \) in \( S \). In case \( S \) is taken as the total quotient ring of \( R \) and \( R \) is completely integrally closed in \( S \), then we say that \( R \) is completely integrally closed.

The complete integral closure \( R^* \) of a ring \( R \) with total quotient ring \( K \) is given by

\[
R^* = \{x \in K \mid \text{there exists a regular element } r \text{ in } R \text{ such that } r \cdot x^n \text{ belongs to } R \text{ for all positive integers } n\}.
\]

**Theorem 4.** If two \( G \)-domains \( R_1 \) and \( R_2 \) having the same field of quotients \( K \) are equivalent, then their complete integral closures are equal.

**Proof.** Let \( R_i^* \) be the complete integral closure of \( R_i \) for \( i = 1, 2 \). As \( R_1 \) is equivalent to \( R_2 \), we have an element \( a_1 \neq 0 \) such that \( a_1 R_1 \subset R_2 \). If \( x \in R_i^* \) then, from the definition of complete integral closure, we have a regular element \( r \) in \( R_1 \) such that \( r \cdot x^n \in R_1 \) for all \( n \). Therefore, \( (a \cdot r) \cdot x^n \in R_2 \) for all \( n \). As \( R_1 \) and \( R_2 \) are both domains and \( (r \cdot a_1) \) is also regular we find that \( x \in R_2^* \). Thus \( R_1^* \subset R_2^* \) and similarly \( R_2^* \subset R_1^* \).

The next theorem connects the homogeneous pseudo valuation with completely integrally closed \( G \)-domains.
THEOREM 5. Let $\omega$ be a homogeneous pseudo valuation on $K$. Then the set of $\omega$-integers, namely

$$R = \{x \in K | x \cdot D \subseteq D\}$$

where

$$D = \{x \in K | \omega(x) < 1\},$$

is a completely integrally closed $G$-domain in $K$ having $K$ as its field of quotients.

PROOF. From theorem 2, we find that $R$ is a $G$-domain having $K$ as its field of quotients. We need to show that $R$ is completely integrally closed. For this, let $x \in K$ and $a, a \cdot x, a \cdot x^2, ...$ belong to $R$ for some non-zero element $a$ in $R$. As $\omega$ is homogeneous

$$\omega(x)^n = \omega(x^n) = \omega(a^{-1} \cdot a \cdot x^n) \leq \omega(a^{-1}) \cdot \omega(a \cdot x^n) \leq \omega(a^{-1})$$

since $\omega(a \cdot x^n) \leq 1$ as $a \cdot x^n \in R$. Thus $\omega(x) < \sqrt[n]{\omega(a^{-1})}$ and this holds for every integer $n$. Therefore $\omega(x) < 1$ so that $x \in R$.

The following is converse to the above.

THEOREM 6. Let $R$ be a completely integrally closed $G$-domain with $K$ as its quotient field. Then in the equivalence class of pseudo valuations arising out of $R$, there is a homogeneous pseudo valuation.

PROOF. Let $u \in R$ be such that $K = R[u^{-1}]$. This $u$ enables us to define the integer valued function $v$ on $K$. Now set $\mu(x) = \lim_{n \to \infty} \frac{1}{n} \cdot v(x^n)$. This limit exists since

$$v(x) < \frac{1}{n} \cdot v(x^n) < v(x) + 1.$$  

We can used $\mu$ to define a gauge function in the sense of Cohn [1] and use this gauge function to define a pseudovaluation by stipulating that

$$\omega(x) = 2^{-\mu(x)}.$$  

This $\omega$ is homogeneous as $\mu(x^n) = n \cdot \mu(x)$.

This we see that there is a one-one correspondence between the equivalence classes of homogeneous pseudo valuations on a field $K$.  

G-domains and pseudo-valuations
and completely integrally closed $G$-domains having $K$ as their field of quotients.

Surjit Singh, in his thesis, has shown that any pseudo valuation on an $A$-field (number or an algebraic function field in one variable over a finite field) can be expressed as supremum of a finite number of valuations. Now, given a valuation $v$ on an $A$-field, its valuation ring is evidently a $G$-domain with any uniformizing parameter playing the role of $u$ whose inverse generates the quotients field. If $\omega$ is any pseudovaluation on an $A$-field then the $\omega$-integers form a $G$-domain which is moreover a completely integrally closed ring. On the other hand, every $G$-domain in an $A$-field gives rise to a pseudovaluation which can be realized as the supremum of a finite number of valuations. Thus we get a complete description of all completely integrally closed $G$-domains contained in an $A$-field.

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