Periodic solutions of a differential delay equation of Rayleigh type

Rendiconti del Seminario Matematico della Università di Padova, tome 61 (1979), p. 115-124

<http://www.numdam.org/item?id=RSMUP_1979__61__115_0>
Periodic Solutions
of a Differential Delay Equation of Rayleigh Type.

S. Invernizzi - F. Zanolin (*)

1. Introduction.

It is well-known that the ordinary differential equation of Rayleigh type

\[
x''(t) + f(x'(t)) + g(x(t)) = h(t)
\]

is physically significant. For instance, in the problem of vibrations of a suspended wire subjected to disturbances as wind (like an electrical transmission line), the periodic solutions of

\[
x'' + |x'x' + qx + x - P^2x^3 = r \sin \omega t
\]

are of interest (see Cecconi [1]). This suggests to study the existence of p-periodic solutions of the differential delay equation

\[
x''(t) + f(x'(t + \sigma(t))) + g(x(t + \tau(t))) = h(t, x(t + r(t)), x'(t + s(t)))
\]

where the deviations \sigma, \tau, r, s are p-periodic, and \( h \) is a bounded function, \( p \)-periodic in \( t \). We assume that \( g \) is differentiable and we allow \( g' \) to change sign; hence we need some « Lyapunov-Schmidt »

(*) Indirizzo degli AA.: Istituto di Matematica, Università di Trieste - P.le Europa 1 - I-34100 Trieste, Italy.
Lavoro eseguito nell’ambito del G.N.A.F.A.; i contributi degli autori sono equivalenti.
technique. In particular, we shall use a theorem from the coincidence degree theory (see Mawhin [3]). A particular feature of our existence result for (D) (Theorem 1) is that we require only the continuity of \( f \), according to the fact that the differentiability of a damping term is not a reasonable physical requirement (see Utz [6]).

As a corollary of Theorem 1, we have an existence theorem of periodic solutions of ordinary differential equations (Corollary 1), which contains a result due to Reissig (see [5]).

At the end of the paper, we get an existence-uniqueness theorem (Theorem 2) for periodic solutions of (R) under a monotonicity condition for \( g \) and a regularity condition for \( f \).

2. Preliminaries.

We call \( x : R \to R \) a \( p \)-periodic function \((p > 0)\) if, for every \( t \in R \), \( x(t + p) = x(t) \). We denote \( C^i(p, R) \) \((i = 0, 1, 2)\) the Banach space of all \( p \)-periodic functions \( x : R \to R \) of class \( C^i \), with the norm \( x \to \sum_{k=0}^{i} |x^{(k)}|_{\infty} \), where \( | \cdot |_{\infty} \) denotes the supremum norm. Moreover, if \( x \in C^0(p, R) \), the symbol \( |x|_2 \) denotes the \( L^2(0, p) \)-norm of \( x \), i.e. \( |x|_2 = \left( \int_0^p |x(t)|^2 \, dt \right)^{1/2} \), and the symbol \( \delta(x) \) denotes the diameter of the set \( x(R) \cup \{0\} \). Observe that \( \delta \) is an equivalent norm for \( C^0(p, R) \).

In [2] the following technical lemma is proved:

**Lemma 1.** Let \( \tau \in C^0(p, R) \). Then the formula

\[
x(\cdot) \to \int_0^{\tau(\cdot)} x(\cdot + s) \, ds
\]

defines a linear operator \( G(\tau) : C^0(p, R) \to C^0(p, R) \) such that for every \( x \)

\[
|G(\tau)x|_2 < \delta(\tau)|x|_2.
\]

3. Main results.

We denote

\[
L_\tau = \sup_{\xi, \eta \in R: \xi \neq \eta} \left| \frac{f(\xi) - f(\eta)}{\xi - \eta} \right| \quad \text{(possibly } L_\tau = +\infty\text{)}.
\]
and we define similarly $L_g$. We assume the convention that

$$0 \cdot (+\infty) = 0.$$  

**Theorem 1.** Let us consider the following equation

$$x''(t) + f(x'(t + \sigma(t))) + g(x(t + \tau(t))) = h(t, x(t + r(t)), x'(t + s(t)))$$

where $f \in C^0(R, R)$, $g \in C^1(R, R)$, $h \in C^0(R^3, R)$ and it is $p$-periodic in the first variable, and the delays $\sigma, \tau, r, s \in C^0(p, R)$. Assume that

(i) $h$ is bounded, \(|h(t, x, x')| < M\),

(ii) the derivative $g'$ is bounded above, and the frequency $\omega = 2\pi/p$ satisfies $g'(\cdot) < K < \omega^2$ for some $K \in R$.

If the norms $\delta(\sigma)$ and $\delta(\tau)$ are so small that

(iii) $\omega^2 L_\sigma \delta(\sigma) + \omega L_\tau \delta(\tau) + K < \omega^2$,

and if

(iv) $\lim_{|x| \to +\infty} g(x) \text{ sign } x = +\infty \text{ (or } -\infty)$

then (1) has a least one $p$-periodic solution.

**Remark 1.** In the ordinary case, i.e. when $\sigma = \tau = 0$, we do not require any Lipschitz condition on $f$ or on $g$, since in this case the hypothesis (iii) means simply $K < \omega^2$. For instance, if $\sigma = \tau = 0$, we can assume $g(x) = a$ polynomial in $x$ of odd order with negative leading coefficient, as in the classical Rayleigh equation where $g(x) = -x - P^2x^3$. In fact for a polynomial of this kind, the hypothesis (ii) and the hypothesis (iv) with the limit equal to $-\infty$, are always satisfied, for suitable $p$.

**Corollary 1.** If $g \in C^1(R, R)$ has its derivative bounded above by a constant $K < \omega^2$ ($\omega = 2\pi/p$), if $h \in C^0(R^3, R)$ is a bounded function, $p$-periodic in the first variable, and if

$$\lim_{|x| \to +\infty} g(x) \text{ sign } x = -\infty \text{ (or } +\infty),$$

then the ordinary equation

$$x'' + f(x') + g(x) = h(t, x, x')$$

has at least one $p$-periodic solution, whatever the function $f \in C^0(R, R)$ may be.
Proof. Put $\sigma = \tau = r = s = 0$ in Theorem 1, and use the convention $0 \cdot (\pm \infty) = 0$.

Corollary 2 (Reissig [5], Theorem 5). The ordinary equation

$$x'' + f(x') + Kx + \gamma(x) = e(t),$$

where $f, \gamma, e$ are continuous and $e$ is $p$-periodic, has at least one $p$-periodic solution when $0 < K < \omega^2, |\gamma(x)| \leq P$.

Proof. Put $Kx = g(x), e(t) - \gamma(x) = h(t, x)$, and use Corollary 1.

Proof of Theorem 1. We use a result of coincidence degree theory. Let $X_i (i = 0, 1, 2)$ be Banach spaces, $X_2 \subseteq X_1 \subseteq X_0$ with completely continuous embeddings. Let $L: X_2 \to X_0$ be a continuous linear Fredholm map of index zero. This means that $\text{im } L$ is closed and $\dim \ker L \equiv \dim \text{coker } L < \infty$. As a consequence, we can find two continuous projections $P: X_1 \to \ker L, (I - Q): X_0 \to \text{im } L$. The restriction $L: X_2 \cap \ker P \to \text{im } L$ is bijective: we call $K$ its inverse. Let $N: X_1 \to X_0$ be an $L$-completely continuous map: this means that $QN: X_1 \to \ker P \to \text{im } L$ is continuous and maps bounded sets into bounded sets, and that $K(I - Q)N: X_1 \to X_1$ is completely continuous. Actually the map $A: X_1 \to X_0, Ax = Px$, is $L$-completely continuous. In fact, $QA: X_1 \to X_0$ and $K(I - Q)A: X_1 \to X_2$ are linear bounded (and the embedding $X_2 \to X_1$ is completely continuous). Moreover,

$$\ker (L - A) = \{0\}.$$  

Then it follows directly from a theorem by Mawhin (see [3]) that if there exists $\rho > 0$ such that $|x|_{X_1} < \rho$ whenever $(\lambda, x) \in ]0,1[ \times X_2$ satisfies

$$Lx = (1 - \lambda)Ax + \lambda Nx,$$

then the equation $Lx = Nx$ has at least one solution $x \in X_2$.

We shall apply this result with $X_i = C^i(p, R) (i = 0, 1, 2)$. We define $L: C^0(p, R) \to C^0(p, R), (Lx)(t) = -x''(t)$. It is well known that $L$ is a continuous linear Fredholm map of index zero. Moreover the projections

$$P: C^1(p, R) \to \ker L = \{\text{constants maps } R \to R\}$$
and
\[ Q: C^0(\mathbb{R}, \mathbb{R}) \to \{ \text{constants maps } \mathbb{R} \to \mathbb{R} \} \]

can be chosen as follows:
\[
(Px)(t) = \left( \frac{1}{p} \right) \int_0^p x(\xi) \, d\xi, \quad (Qx)(t) = \left( \frac{1}{p} \right) \int_0^p x(\xi) \, d\xi.
\]

We define \( N: C^1(\mathbb{R}, \mathbb{R}) \to C^0(\mathbb{R}, \mathbb{R}) \)
\[
(Nx)(t) = f(x'(t + \sigma(t))) + g(x(t + \tau(t)) - h(t, x(t + r(t)), x'(t + s(t))).
\]

Since \( f, g, h \) are continuous, and \( Q \) is linear bounded, we have easily that the composite map \( QN: C^1(\mathbb{R}, \mathbb{R}) \to C^0(\mathbb{R}, \mathbb{R}) \) is continuous and maps bounded sets into bounded sets. Moreover \( K(I - Q): C^0(\mathbb{R}, \mathbb{R}) \to C^2(\mathbb{R}, \mathbb{R}) \) is linear bounded; hence \( K(I - Q)N: C^1(\mathbb{R}, \mathbb{R}) \to C^1(\mathbb{R}, \mathbb{R}) \) is completely continuous. It follows that \( N \) is \( L \)-completely continuous.

Now equation (1) has a \( p \)-periodic solution \( x \) if and only if the coincidence equation \( Lx = Nx \) has a solution \( x \in C^2(\mathbb{R}, \mathbb{R}) \). So, to prove the existence of a \( p \)-periodic solution of (1), in virtue of the Mawhin's theorem, we need only to show that there exists a constant \( q > 0 \) such that, if \( \lambda \in ]0, 1[ \) and \( x \in C^2(\mathbb{R}, \mathbb{R}) \) verify

\[
(2) \quad Lx = (1 - \lambda) Ax + \lambda Nx
\]

(where \( Ax = \left( \frac{1}{p} \right) \int_0^p x(\xi) \, d\xi \)), then we have \( |x'|_\infty + |x|_\infty < q \).

First we prove the existence of a bound for \( |x'|_\infty \). If we multiply (2) by \( -x'' \) and we integrate on \([0, p]\), we have easily
\[
|x''|^2 = -\lambda \int_0^p (Nx)x'' \, dt.
\]

We shall use now the definition of \( N \), the boundedness of \( h \) (condition (i)), the upper bound of \( g' \) (condition (ii)), and, possibly, the
Lipschitz constants of \( f \) and \( g \):

\[
- \int_0^p (N x) x'' \, dt = - \int_0^p f(x'(t)) x''(t) \, dt - \int_0^p \left( f(x'(t + \sigma(t))) - f(x'(t)) \right) x''(t) \, dt - \int_0^p g(x(t)) x''(t) \, dt - \int_0^p \left( g(x(t + \tau(t))) - g(x(t)) \right) x''(t) \, dt + \int_0^p h(t, x(t + \tau(t)), x'(t + s(t))) x''(t) \, dt < 0 + L_r |x'(\cdot + \sigma)| < |x'|_2 + K |x'|_2^2 + L_x |x(\cdot + \tau) - x|_2 |x'|_2 + M p^4 |x''|_2.
\]

It follows from Lemma 1 that

\[
|x'(\cdot + \sigma) - x'|_2 < |x''|_2 = \int_0^{|x''|_2} d\xi_2 < \delta(\sigma)|x''|_2, \quad \int_0^{|x'(\cdot + \tau) - x|_2} d\xi_2 < \delta(\tau)|x'|_2.
\]

Using the Wirtinger inequality \( \omega |x'|_2 < |x''|_2 \), we obtain, since \( 0 < \lambda < 1 \),

\[
|x''|_2 < \int_0^p (N x) x'' \, dt < \left( L_r \delta(\sigma) + \frac{1}{\omega} L_x \delta(\tau) + \frac{1}{\omega^2} K \right) |x''|_2^2 + M p^4 |x''|_2.
\]

It follows from condition (iii) that \( |x''|_2 < \text{const} \), and this implies, by an elementary argument, that there exists a constant \( \alpha > 0 \) such that

\[
|x'|_\infty < \alpha.
\]

In order to show the existence of a bound for \( |x|_\infty \), we shall use the condition (iv). There is no loss of generality if we assume that \( g(x) \) sign \( x \to + \infty \) (as \( |x| \to + \infty \)). In fact, if \( g(x) \) sign \( x \to - \infty \), we have only to define the map \( A : X \to X_0 \) in the « abstract » part by \( Ax = - P x \) instead of \( Ax = P x \), that is, for the « concrete » case, \( (Ax)(t) = - (1/p) \int_0^p x'(\xi) d\xi \). It is easy to see that, with this sign modification,
the a priori bound $|x'|_\infty < \alpha$ is still true, and that the a priori bound for $|x|_\infty$ we shall prove for the case $g(x)$ sign $x \to + \infty$ can be obtained, in the case $g(x)$ sign $x \to - \infty$, with the same argument.

We compute the average for both terms of (2): we have $-Qx^s = (1 - \lambda)AQx + \lambda QNx$, that is

$$0 = (1 - \lambda)Ax + \lambda QNx.$$  

Claim. There exists $\beta > 0$ such that, for any $x \in C^2(p, R)$ which satisfies (3) with some $\lambda \in ]0, 1[$,

$$|Ax| < \beta.$$  

This statement guarantees the existence of a bound for $|x|_\infty$. In fact, for each $x \in C^1(p, R)$, for every $t \in [0, p]$, there exist two points $\xi, \eta$ such that $x(t) = Ax + x'(\xi)(t - \eta)$. It follows that if $x$ is a solution of (2) then $|x - Ax|_\infty \leq \alpha p$, and so, if the claim is true, we obtain $|x|_\infty \leq \alpha p + \beta$.

Let us assume our claim is false. We can find a suitable sequence of pairs $(\lambda_n, x_n) \in ]0, 1[ \times C^2(p, R)$ such that

(j) for every $n$, $0 = (1 - \lambda_n)Ax_n + \lambda_n QNx_n$,

(jj) the sequence $\lambda_n$ is convergent to some point of the closed interval $[0, 1]$,

(jjj) $Ax_n \to + \infty$ or $Ax_n \to - \infty$.

By definition, $QNx_n$ is equal to the sum of the sequence

$$a_n = (1/p) \int_0^p g(x_n(t + \tau(t))) dt$$

and of another sequence of the form

$$b_n = (1/p) \int_0^p (f(x')(\ldots) - h(\ldots)) dt.$$  

Clearly $b_n$ is bounded (by $\sup_{|x'| \leq \alpha} |f(x')| + M$). Let us consider $a_n$. We assume that the function $g$ reaches its minimum, on the interval
\[ [Ax_n - \alpha p, Ax_n + \alpha p], \] at the point \( u_n \), and its maximum on the same interval at the point \( v_n \). Since
\[
a_n = \left( \frac{1}{p} \right) \int_0^p g(x_n(t + \tau(t)) - Ax_n + Ax_n) \, dt,
\]
and since
\[
\sup_{t \in [0,p]} |x_n(t + \tau(t)) - Ax_x| \leq \sup_{t \in [0,p]} |x_n(t) - Ax_n| \leq \alpha p,
\]
we obtain easily that \( g(u_n) \leq a_n \leq g(v_n) \). Thus, if \( Ax_n \to +\infty \), we must have \( u_n \to +\infty \). It follows from condition (iv) that \( g(u_n) \to +\infty \) and hence \( a_n \to +\infty \). This is a contradiction with (j), since we have simultaneously \( Ax_n \to +\infty \) and \( QNv_n \to +\infty \). On the other hand, if \( Ax_n \to -\infty \), we obtain \( g(v_n) \to -\infty \) and \( a_n \to -\infty \), which is again a contradiction with (j).

This proves our claim and completes the proof of the theorem.

As a consequence of Corollary 1 we obtain the result that the non-linear ordinary differential equation
\[
x'' + f(x') + g(x) = h(t),
\]
where \( f \in C^0(R, R) \), \( g \in C^1(R, R) \), \( h \in C^0(p, R) \) has at least one \( p \)-periodic solution if \( g'(\cdot) \leq K < 0 \). In fact this condition implies that (ii) and (iv) hold.

A natural question arises: do the monotonicity condition \( g'(\cdot) \leq K < 0 \) imply the uniqueness of the periodic solution of (4)? We are able to give an affirmative answer provided that \( f \) satisfies only a regularity condition: \( f \) is of class \( C^1 \). For instance, all the viscous dampings \( f(x') = \beta |x'|^\sigma \text{sign}(x') \), with \( \beta > 0 \), \( \sigma > 1 \), can be considered.

**Theorem 2.** The ordinary differential equation
\[
x'' + f(x') + g(x) = h(t)
\]
where \( h \) is continuous and \( p \)-periodic, \( g \in C^1(R, R) \), and \( g'(\cdot) \leq K < 0 \), has exactly one \( p \)-periodic solution whatever \( f \in C^1(R, R) \) may be.

**Proof.** The existence follows from Corollary 1. Let us assume that \( x, y \) are \( p \)-periodic solutions of (5). Then the difference \( z = x - y \) is a \( p \)-periodic function which satisfies the linear homogeneous equa-
tion
\[ z''(t) + a(t)z'(t) + b(t)z(t) = 0 , \]
where
\[
 a(t) = \int_0^1 f'(sx(t) + (1 - s)y(t)) \, ds , \quad b(t) = \int_0^1 g'(sx(t) + (1 - s)y(t)) \, ds ,
\]
are continuous coefficients with \( b(\cdot) < 0 \). Let us define the auxiliary function \( w = e^A(z') \), where \( A(t) = \int_0^t a(s) \, ds \). We have
\[
 w' = 2e^A(z'^2 + z(z'' + az')) = 2e^A(z'^2 - bz^2) \geq 0 ,
\]
hence \( w \) is increasing. We consider the set \( N = \{ t \in R : z'(t) = 0 \} \).

REMARK 2. Theorem 2 can be proved using the Caccioppoli global inversion method (see [4]). In fact we can define a map
\[
 T : x \in C^2(p, R) \to x'' + f(x') + g(x) \in C^0(p, R)
\]
and we need only to prove that \( T \) is proper and that at each point \( x \) the differential \( DT(x) \) is bijective. The differential \( DT(x) \) is a linear map defined by
\[
 DT(x)[v] = v'' + f(x')v' + g'(x)v.
\]
Since \( g'(x) < 0 \), the argument of Theorem 2 shows that it is one-to-one, hence it is onto by the Fredholm Alternative. To prove the properness of \( T \), we take the \( L^2 \)-inner product of \( Tx = h \) with \( x'' \): we have
\[
 |x''|_2^2 - \int_0^p g'(x(t))x''(t) \, dt = \int_0^p h(t)x''(t) \, dt .
\]
It follows \( |x''|_2^2 \leq K|x'|_2^2 + |h|_2 |x''|_2 \leq p^\delta |h|_\infty |x''|_2 \). The usual technique yields that \( |x'|_\infty \) and consequently \( |f(x')|_\infty \) is bounded in terms of \( |h|_\infty \). Using \( Tx = h \), we deduce that \( |g(x)|_2 \) is bounded. Now it is
easy to see that $|x|_s$ is bounded: in fact, for $s \neq 0$, we have $(g(s) - g(0))/s < K < 0$, and so $(g(s) - g(0))^2/s^2 > K^2 > 0$, that is $s^2 < (1/K)^2 \cdot (g(s) - g(0))^2$, or $s^2 < e_1|g(s)|^2 + e_2|g(s)| + e_3$, with $e_1 > 0$, $e_2$, $e_3 > 0$. This last inequality holds for every $s$. In particular, for $s = x(t)$, we can deduce that $|x|_s$ is bounded. An elementary argument shows that $|x|_\infty$ is bounded in terms of $|h|_\infty$. This implies that $T$ is a proper map.

In this way we obtain the further result that the unique $p$-periodic solution $x$ of the equation (5) $C^1$-depends upon the forcing term $h$.

REFERENCES


Manoscritto pervenuto in redazione il 25 maggio 1978.