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**Periodic solutions of a differential delay equation of Rayleigh type**


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# Periodic Solutions of a Differential Delay Equation of Rayleigh Type.

S. INVERNIZZI - F. ZANOLIN (*)

## 1. Introduction.

It is well-known that the ordinary differential equation of Rayleigh type

$$(\mathrm{R}) \qquad x''(t) + f(x'(t)) + g(x(t)) = h(t)$$

is physically significant. For instance, in the problem of vibrations of a suspended wire subjected to disturbances as wind (like an electrical transmission line), the periodic solutions of

$$x'' + |x'|x' + qx' + x - P^2x^3 = r \sin \omega t$$

are of interest (see Cecconi [1]). This suggests to study the existence of $p$-periodic solutions of the differential delay equation

$$(\mathrm{D}) \quad x''(t) + f(x'(t + \sigma(t))) + g(x(t + \tau(t))) = h(t, x(t + r(t)), x'(t + s(t)))$$

where the deviations $\sigma$, $\tau$, $r$, $s$ are $p$-periodic, and $h$ is a bounded function, $p$-periodic in $t$. We assume that $g$ is differentiable and we allow $g'$ to change sign: hence we need some « Lyapunov-Schmidt »

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technique. In particular, we shall use a theorem from the coincidence degree theory (see Mawhin [3]). A particular feature of our existence result for (D) (Theorem 1) is that we require only the continuity of $f$, according to the fact that the differentiability of a damping term is not a reasonable physical requirement (see Utz [6]).

As a corollary of Theorem 1, we have an existence theorem of periodic solutions of ordinary differential equations (Corollary 1), which contains a result due to Reissig (see [5]).

At the end of the paper, we get an existence-uniqueness theorem (Theorem 2) for periodic solutions of (R) under a monotonicity condition for $g$ and a regularity condition for $f$.

## 2. Preliminaries.

We call $x\colon R \to R$ a $p$-periodic function ($p > 0$) if, for every $t \in R$, $x(t+p) = x(t)$. We denote $C^i(p, R)$ ($i = 0, 1, 2$) the Banach space of all $p$-periodic functions $x\colon R \to R$ of class $C^i$, with the norm $x \to \to \sum_{k=0}^{i} |x^{(k)}|_\infty$, where $|\cdot|_\infty$ denotes the supremum norm. Moreover, if $x \in C^0(p, R)$, the symbol $|x|_2$ denotes the $L^2(0, p)$-norm of $x$, i.e. $|x|_2 = = \left(\int_0^p |x(t)|^2\, dt\right)^{\frac{1}{2}}$, and the symbol $\delta(x)$ denotes the diameter of the set $x(R) \cup \{0\}$. Observe that $\delta$ is an equivalent norm for $C^0(p, R)$.

In [2] the following technical lemma is proved:

LEMMA 1. *Let $\tau \in C^0(p, R)$. Then the formula*

$$x(\cdot) \to \int_0^{\tau(\cdot)} x(\cdot + s)\, ds$$

*defines a linear operator $G(\tau)\colon C^0(p, R) \to C^0(p, R)$ such that for every $x$*

$$|G(\tau)x|_2 \leqslant \delta(\tau)|x|_2\,.$$

## 3. Main results.

We denote

$$L_f = \sup_{\xi, \eta \in R;\, \xi \neq \eta} \left| \frac{f(\xi) - f(\eta)}{\xi - \eta} \right| \qquad \text{(possibly } L_f = +\infty\text{)}\,,$$

and we define similarly $L_g$. We assume the convenction that

$$0 \cdot (+\infty) = 0 .$$

THEOREM 1. *Let us consider the following equation*

$$(1) \qquad x''(t) + f(x'(t+\sigma(t))) + g(x(t+\tau(t))) = \\ = h(t, x(t+r(t)), x'(t+s(t)))$$

*where* $f \in C^0(R, R)$, $g \in C^1(R, R)$, $h \in C^0(R^3, R)$ *and it is* $p$*-periodic in the first variable, and the delays* $\sigma, \tau, r, s \in C^0(p, R)$. *Assume that*

(i) $h$ *is bounded,* $|h(t, x, x')| \leqslant M$,

(ii) *the derivative* $g'$ *is bounded above, and the frequency* $\omega = 2\pi/p$ *satisfies* $g'(\cdot) \leqslant K < \omega^2$ *for some* $K \in R$.

*If the norms* $\delta(\sigma)$ *and* $\delta(\tau)$ *are so small that*

(iii) $\omega^2 L_f \delta(\sigma) + \omega L_g \delta(\tau) + K < \omega^2$,

*and if*

(iv) $\lim_{|x| \to +\infty} g(x) \operatorname{sign} x = +\infty$ (*or* $-\infty$)

*then* (1) *has a least one* $p$*-periodic solution.*

REMARK 1. In the ordinary case, i.e. when $\sigma = \tau = 0$, we do not require any Lipschitz condition on $f$ or on $g$, since in this case the hypothesis (iii) means simply $K < \omega^2$. For instance, if $\sigma = \tau = 0$, we can assume $g(x) =$ a polynomial in $x$ of odd order with negative leading coefficient, as in the classical Rayleigh equation where $g(x) = x - P^2 x^3$. In fact for a polynomial of this kind, the hypothesis (ii) and the hypothesis (iv) with the limit equal to $-\infty$, are always satisfied, for suitable $p$.

COROLLARY 1. *If* $g \in C^1(R, R)$ *has its derivative bounded above by a constant* $K < \omega^2$ ($\omega = 2\pi/p$), *if* $h \in C^0(R^3, R)$ *is a bounded function,* $p$*-periodic in the first variable, and if* $\lim_{|x| \to +\infty} g(x) \operatorname{sign} x = +\infty$ (*or* $-\infty$), *then the ordinary equation*

$$x'' + f(x') + g(x) = h(t, x, x')$$

*has at least one* $p$*-periodic solution, whatever the function* $f \in C^0(R, R)$ *may be.*

PROOF. Put $\sigma = \tau = r = s = 0$ in Theorem 1, and use the convenction $0 \cdot (+\infty) = 0$.

COROLLARY 2 (Reissig [5], Theorem 5). *The ordinary equation*

$$x'' + f(x') + Kx + \gamma(x) = e(t) ,$$

*where $f$, $\gamma$, $e$ are continuous and $e$ is $p$-periodic, has at least one $p$-periodic solution when $0 < K < \omega^2$, $|\gamma(x)| \leqslant P$.*

PROOF. Put $Kx = g(x)$, $e(t) - \gamma(x) = h(t, x)$, and use Corollary 1.

PROOF OF THEOREM 1. We use a result of coincidence degree theory. Let $X_i$ $(i = 0, 1, 2)$ be Banach spaces, $X_2 \subseteq X_1 \subseteq X_0$ with completely continuous embeddings. Let $L: X_2 \to X_0$ be a continuous linear Fredholm map of index zero. This means that $\operatorname{im} L$ is closed and $\dim \ker L = \dim \operatorname{coker} L < \infty$. As a consequence, we can find two continuous projections $P: X_1 \to \ker L$, $(I - Q): X_0 \to \operatorname{im} L$. The restriction $L: X_2 \cap \ker P \to \operatorname{im} L$ is bijective: we call $K$ its inverse. Let $N: X_1 \to X_0$ be an $L$-completely continuous map: this means that $QN: X_1 \to X_0$ is continuous and maps bounded sets into bounded sets, and that $K(I - Q)N: X_1 \to X_1$ is completely continuous. Actually the map $A: X_1 \to X_0$, $Ax = Px$, is $L$-completely continuous. In fact, $QA: X_1 \to X_0$ and $K(I - Q)A: X_1 \to X_2$ are linear bounded (and the embedding $X_2 \to X_1$ is completely continuous). Moreover,

$$\ker (L - A) = \{0\} .$$

Then it follows directly from a theorem by Mawhin (see [3]) that if there exists $\varrho > 0$ such that $|x|_{X_1} < \varrho$ whenever $(\lambda, x) \in ]0,1[ \times X_2$ satisfies

$$Lx = (1 - \lambda) Ax + \lambda Nx ,$$

then the equation $Lx = Nx$ has at least one solution $x \in X_2$.

We shall apply this result with $X_i = C^i(p, R)$ $(i = 0, 1, 2)$. We define $L: C^2(p, R) \to C^0(p, R)$, $(Lx)(t) = - x''(t)$. It is well known that $L$ is a continuous linear Fredholm map of index zero. Moreover the projections

$$P: C^1(p, R) \to \ker L = \{\text{constants maps } R \to R\}$$

and

$$Q\colon C^0(p, R) \to \{\text{constants maps } R \to R\}$$

can be choosen as follows:

$$(Px)(t) = (1/p)\int_0^p x(\xi)\, d\xi\,, \qquad (Qx)(t) = (1/p)\int_0^p x(\xi)\, d\xi\,.$$

We define $N\colon C^1(p, R) \to C^0(p, R)$

$$(Nx)(t) = f\big(x'(t + \sigma(t))\big) + g\big(x(t + \tau(t))\big) - h\big(t,\, x(t + r(t)),\, x'(t + s(t))\big)\,.$$

Since $f$, $g$, $h$ are continuous, and $Q$ is linear bounded, we have easily that the composite map $QN\colon C^1(p, R) \to C^0(p, R)$ is continuous and maps bounded sets into bounded sets. Moreover $K(I - Q)\colon C^0(p, R) \to$ $\to C^2(p, R)$ is linear bounded; hence $K(I - Q)N\colon C^1(p, R) \to C^1(p, R)$ is completely continuous. It follows that $N$ is $L$-completely continuous.

Now equation (1) has a $p$-periodic solution $x$ if and only if the coincidence equation $Lx = Nx$ has a solution $x \in C^2(p, R)$. So, to prove the existence of a $p$-periodic solution of (1), in virtue of the Mawhin's theorem, we need only to show that there exists a constant $\varrho > 0$ such that, if $\lambda \in ]0, 1[$ and $x \in C^2(p, R)$ verify

$$Lx = (1 - \lambda)\, Ax + \lambda Nx \tag{2}$$

(where $Ax = (1/p)\int_0^p x(\xi)\, d\xi$), then we have $|x'|_\infty + |x|_\infty < \varrho$.

First we prove the existence of a bound for $|x'|^\infty$. If we multiply (2) by $-x''$ and we integrate on $[0, p]$, we have easily

$$|x''|_2^2 = -\lambda\int_0^p (Nx)\, x''\, dt\,.$$

We shall use now the definition of $N$, the boundedness of $h$ (condition (i)), the upper bound of $g'$ (condition (ii)), and, possibly, the

Lipschitz constants of $f$ and $g$:

$$-\int_0^p (Nx)x''\,dt = -\int_0^p f(x'(t))x''(t)\,dt - \int_0^p \big(f(x'(t+\sigma(t))) - f(x'(t))\big)x''(t)\,dt -$$

$$-\int_0^p g(x(t))x''(t)\,dt - \int_0^p \big(g(x(t+\tau(t))) - g(x(t))\big)x''(t)\,dt +$$

$$+\int_0^p h(t, x(t+r(t)), x'(t+s(t)))x''(t)\,dt \leqslant$$

$$\leqslant 0 + L_f|x'(\cdot+\sigma) - x'|_2\,|x''|_2 + K|x'|_2^2 + L_g|x(\cdot+\tau) - x|_2\,|x''|_2 + Mp^{\frac{1}{2}}|x''|_2 .$$

It follows from Lemma 1 that

$$|x'(\cdot+\sigma) - x'|_2 = \Big|\int_0^{\sigma(t)} x''(t+\xi)\,d\xi\Big|_2 \leqslant \delta(\sigma)|x''|_2 ,$$

$$|x(\cdot+\tau) - x|_2 = \Big|\int_0^{\tau(t)} x'(t+\xi)\,d\xi\Big|_2 \leqslant \delta(\tau)|x'|_2 .$$

Using the Wirtinger inequality $\omega|x'|_2 \leqslant |x''|_2$ we obtain, since $0<\lambda<1$,

$$|x''|_2^2 \leqslant -\int_0^p (Nx)x''\,dt \leqslant \left(L_f\delta(\sigma) + \frac{1}{\omega}L_g\delta(\tau) + \frac{1}{\omega^2}K\right)|x''|_2^2 + Mp^{\frac{1}{2}}|x''|_2 .$$

It follows from condition (iii) that $|x''|_2 \leqslant$ const, and this implies, by an elementary argument, that there exists a constant $\alpha>0$ such that

$$|x'|_\infty \leqslant \alpha .$$

In order to show the existence of a bound for $|x|_\infty$, we shall use the condition (iv). There is no loss of generality if we assume that $g(x)\operatorname{sign} x \to +\infty$ $\big($as $|x|\to+\infty\big)$. In fact, if $g(x)\operatorname{sign} x \to -\infty$, we have only to define the map $A\colon X_1 \to X_0$ in the « abstract » part by $Ax = -Px$ instead of $Ax = Px$, that is, for the « concrete » case, $(Ax)(t) = -(1/p)\int_0^p x(\xi)\,d\xi$. It is easy to see that, with this sign modification, the a priori bound $|x'|_\infty \leqslant \alpha$ is still true, and that the a priori bound for $|x|_\infty$ we shall prove for the case $g(x)\operatorname{sign} x \to +\infty$ can be obtained, in the case $g(x)\operatorname{sign} x \to -\infty$, with the same argument.

We compute the average for both terms of (2): we have $-Qx'' = (1-\lambda)QAx + \lambda QNx$, that is

$$0 = (1-\lambda)Ax + \lambda QNx\,. \tag{3}$$

*Claim.* There exists $\beta > 0$ such that, for any $x \in C^2(p, R)$ which satisfies (3) with some $\lambda \in ]0, 1[$,

$$|Ax| \leqslant \beta\,.$$

This statement guarantees the existence of a bound for $|x|_\infty$. In fact, for each $x \in C^1(p, R)$, for every $t \in [0, p]$, there exist two points $\xi$, $\eta$ such that $x(t) = Ax + x'(\xi)(t-\eta)$. It follows that if $x$ is a solution of (2) then $|x - Ax|_\infty \leqslant \alpha p$, and so, if the claim is true, we obtain $|x|_\infty \leqslant \alpha p + \beta$.

Let us assume our claim is false. We can find a suitable sequence of pairs $(\lambda_n, x_n) \in ]0, 1[ \times C^2(p, R)$ such that

(j) for every $n$, $0 = (1-\lambda_n)Ax_n + \lambda_n QNx_n$,

(jj) the sequence $\lambda_n$ is convergent to some point of the closed interval $[0, 1]$,

(jjj) $Ax_n \to +\infty$ or $Ax_n \to -\infty$.

By definition, $QNx_n$ is equal to the sum of the sequence

$$a_n = (1/p)\int_0^p g\big(x_n(t+\tau(t))\big)\,dt$$

and of another sequence of the form

$$b_n = (1/p)\int_0^p \big(f(x'(\ldots)) - h(\ldots)\big)\,dt\,.$$

Clearly $b_n$ is bounded $\big($by $\sup\limits_{|x'|\leqslant\alpha} |f(x')| + M\big)$. Let us consider $a_n$. We assume that the function $g$ reaches its minimum, on the interval

$[Ax_n - \alpha p, Ax_n + \alpha p]$, at the point $u_n$, and its maximum on the same interval at the point $v_n$. Since

$$a_n = (1/p)\int_0^p g\big(x_n(t+\tau(t)) - Ax_n + Ax_n\big)\,dt\,,$$

and since

$$\sup_{t\in[0,p]} |x_n(t+\tau(t)) - Ax_n| \leqslant \sup_{t\in[0,p]} |x_n(t) - Ax_n| \leqslant \alpha p\,,$$

we obtain easily that $g(u_n) \leqslant a_n \leqslant g(v_n)$. Thus, if $Ax_n \to +\infty$, we must have $u_n \to +\infty$. It follows from condition (iv) that $g(u_n) \to +\infty$ and hence $a_n \to +\infty$. This is a contradiction with (j), since we have simultaneously $Ax_n \to +\infty$ and $QNx_n \to +\infty$. On the other hand, if $Ax_n \to -\infty$, we obtain $g(v_n) \to -\infty$ and $a_n \to -\infty$, which is again a contradiction with (j).

This proves our claim and completes the proof of the theorem.

As a consequence of Corollary 1 we obtain the result that the non-linear ordinary differential equation

$$x'' + f(x') + g(x) = h(t)\,, \tag{4}$$

where $f \in C^0(R, R)$, $g \in C^1(R, R)$, $h \in C^0(p, R)$ has at least one $p$-periodic solution if $g'(\cdot) \leqslant K < 0$. In fact this condition implies that (ii) and (iv) hold.

A natural question arises: do the monotonicity condition $g'(\cdot) \leqslant K < 0$ imply the uniqueness of the periodic solution of (4)? We are able to give an affermative answer provided that $f$ satisfies only a regularity condition: $f$ is of class $C^1$. For instance, all the viscous dampings $f(x') = \beta|x'|^\varrho \operatorname{sign}(x')$, with $\beta > 0$, $\varrho \geqslant 1$, can be considered.

THEOREM 2. *The ordinary differential equation*

$$x'' + f(x') + g(x) = h(t) \tag{5}$$

*where $h$ is continuous and $p$-periodic, $g \in C^1(R, R)$, and $g'(\cdot) \leqslant K < 0$, has exactly one $p$-periodic solution whatever $f \in C^1(R, R)$ may be.*

PROOF. The existence follows from Corollary 1. Let us assume that $x$, $y$ are $p$-periodic solutions of (5). Then the difference $z = x - y$ is a $p$-periodic function which satisfies the linear homogeneous equation

$$z''(t) + a(t)z'(t) + b(t)z(t) = 0 ,$$

where

$$a(t) = \int_0^1 f'(sx'(t) + (1-s)y'(t))\,ds , \quad b(t) = \int_0^1 g'(sx(t) + (1-s)y(t))\,ds ,$$

are continuous coefficients with $b(\cdot) < 0$. Let us define the auxiliary function $w = e^A(z^2)'$, where $A(t) = \int_0^t a(s)\,ds$. We have

$$w' = 2e^A(z'^2 + z(z'' + az')) = 2e^A(z'^2 - bz^2) \geqslant 0 ,$$

hence $w$ is increasing. We consider the set $N = \{t \in R: z'(t) = 0\}$. Since $z$ is periodic, $N$ is not empty, and inf $N = -\infty$, sup $N = +\infty$. But clearly $w(N) = \{0\}$, and thus the monotonicity of $w$ implies that $w(t) = 0$ for every $t$. From the definition of $w$, it follows that $z$ = a constant. Now the condition $b < 0$ implies that if $z$ is a constant solution of $z'' + az' + bz = 0$, then we must have $z = 0$.

REMARK 2. Theorem 2 can be proved using the Caccioppoli global inversion method (see [4]). In fact we can define a map

$$T: x \in C^2(p, R) \to x'' + f(x') + g(x) \in C^0(p, R)$$

and we need only to prove that $T$ is proper and that at each point $x$ the differential $DT(x)$ is bijective. The differential $DT(x)$ is a linear map defined by $DT(x)[v] = v'' + f(x')v' + g'(x)v$. Since $g'(x) < 0$, the argument of Theorem 2 shows that $DT(x)$ is one-to-one, hence it is onto by the Fredholm Alternative. To prove the properness of $T$, we take the $L^2$-inner product of $Tx = h$ with $x''$: we have

$$|x''|_2^2 - \int_0^p g'(x(t))x'^2(t)\,dt = \int_0^p h(t)x''(t)\,dt .$$

It follows $|x''|_2^2 \leqslant K|x'|_2^2 + |h|_2|x''| \leqslant p^{\frac{1}{2}}|h|_\infty |x''|_2$. The usual technique yelds that $|x'|_\infty$ and consequently $|f(x')|_\infty$ is bounded in terms of $|h|_\infty$. using $Tx = h$, we deduce that $|g(x)|_2$ is bounded. Now it is easy to see that $|x|_2$ is bounded: in fact, for $s \neq 0$, we have $(g(s) - g(0))/s \leqslant K < 0$, and so $(g(s) - g(0))^2/s^2 \geqslant K^2 > 0$, that is $s^2 \leqslant (1/K)^2 \cdot (g(s) - g(0))^2$, or $s^2 \leqslant c_1|g(s)|^2 + c_2|g(s)| + c_3$, with $c_1 > 0$, $c_2, c_3 \geqslant 0$. This last inequality holds for every $s$. In particular, for $s = x(t)$, we can deduce that $|x|_2$ is bounded. An elementary argument shows that $|x|_\infty$ is bounded in terms of $|h|_\infty$. This implies that $T$ is a proper map.

In this way we obtain the further result that *the unique $p$-periodic solution $x$ of the equation* (5) *$C^1$-depends upon the forcing term $h$.*

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