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## Abelian Groups with Anti-Isomorphic Endomorphism Rings.

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All groups considered in this paper are abelian. We say that a group  $G$  is *E-dual* if there exists a group  $H$  such that the endomorphism rings  $E(G)$  and  $E(H)$  are anti-isomorphic;  $G$  is said to be *E-self-dual* if  $E(G)$  has an anti-automorphism. In this note we investigate some properties of *E-dual* and *E-self-dual* groups. In section 1, we examine some closure properties of the classes of *E-dual* and *E-self-dual* groups. In fact, we prove that direct summands of *E-self-dual* groups are not necessarily *E-self-dual*, and direct sums of *E-self-dual* groups are not necessarily *E-dual*. In section 2, we show that a torsion group  $G$  is *E-dual* if and only if, for every prime  $p$ , its  $p$ -component  $t_p(G)$  is either a  $p$ -group of finite rank or a torsion-complete  $p$ -group with finite Ulm invariants. In section 3, we describe some classes of *E-dual* cotorsion groups. As we shall see, a reduced cotorsion group  $G$  is *E-dual* if and only if, for every prime  $p$ , the  $p$ -adic component of  $G$  is either a  $J_p$ -module of finite rank or the  $p$ -adic completion of an *E-dual* reduced  $p$ -group. We also prove that a divisible group  $G$  is *E-dual* if and only if  $G$  is either a torsion *E-dual* group or a torsion-free group of finite rank. In section 4, we show that plenty of reduced torsion-free groups are *E-dual*. In fact, every controlled group  $G$  such that  $E(G)$  is of cardinality  $< \aleph_i$ , the first strongly inaccessible cardinal, is an *E-dual* group. In the torsion-free case some pathologies of the class of *E-dual* groups appear. For instance, by Corner's realization

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theorems, completely different reduced torsion-free groups have anti-isomorphic endomorphism rings. Finally, we remark that there exist torsion, mixed and torsion-free  $E$ -dual groups which are not  $E$ -self-dual.

For all unexplained terminology and notation we refer to ([5]); in particular  $\mathbb{N}$  is the set of natural numbers,  $P$  the set of prime numbers;  $\mathbb{Z}, \mathbb{Q}, J_p$  are respectively the groups (or rings) of integers, rational numbers,  $p$ -adic integers;  $\mathbb{Z}(p)$  is the group (or field) with  $p$  elements. If  $G$  is a group and  $g \in G$ , then  $o(g)$  is the order of  $g$  and, if  $G$  is a  $p$ -group,  $o(g) = p^{e(g)}$ , where  $e(g)$  is the exponent of  $g$ . If  $G'$  is a pure subgroup of  $G$ , we write  $G' \leq G$ . If  $G$  is torsion-free and  $S$  is a subset of  $G$ , then  $\langle S \rangle_*$  is the pure subgroup of  $G$  generated by  $S$ . For every set  $X$ ,  $G^{(X)}$  is the group of all functions from  $X$  to  $G$  with finite support. If  $R$  is a ring, then  $R^o$  is its opposite ring and, for every  $n \in \mathbb{N}$ ,  $M_n(R)$  is the ring of all  $n \times n$  matrices with entries in  $R$ . For every  $p$ -group  $G$  and every ordinal  $\sigma$ ,  $f_\sigma(G)$  is the  $\sigma$ -th Ulm invariant of  $G$ . When we shall say that  $B = \bigoplus_{n \in \mathbb{N}} B_n$  is a basic subgroup of the  $p$ -group  $G$ , we always adopt the convention that  $B_n$  is a direct sum of cyclic groups of order  $p^n$ . If  $G$  is a reduced cotorsion group, then we write  $G = \prod_{p \in P} G_p$ , where each  $G_p$  is the  $p$ -adic component of  $G$ .

§ 1. Let  $G$  and  $H$  be groups and assume there is an anti-isomorphism between  $E(G)$  and  $E(H)$ . Since idempotents of  $E(G)$  are mapped onto idempotents of  $E(H)$ , the following lemma is obvious.

LEMMA 1.1. *Direct summands of  $E$ -dual groups are  $E$ -dual.*

The situation is different in the class of  $E$ -self-dual groups.

LEMMA 1.2. *Direct summands of  $E$ -self-dual groups are not necessarily  $E$ -self-dual.*

PROOF. We shall prove that if  $G = \bigoplus_{i=1}^4 G_i$ , where  $G_1 = G_2 = \prod_{p \in P} \mathbb{Z}(p)$ ;  $G_3 = G_4 = \bigoplus_{p \in P} \mathbb{Z}(p)$ , then  $G$  is  $E$ -self-dual, but there exists a direct summand of  $G$  which is not  $E$ -self-dual. In the following  $\Pi$  denotes the group (or ring)  $\prod_{p \in P} \mathbb{Z}(p)$  and  $\Sigma$  denotes the group  $\bigoplus_{p \in P} \mathbb{Z}(p)$ . Thus  $\text{Hom}(\Sigma, \Pi) \cong \Pi$ ;  $\text{Hom}(\Pi, \Sigma) \cong \Sigma$ ;  $E(G_i) \cong \Pi$  ( $1 \leq i \leq 4$ ). Let  $A$  and  $A^*$  be the following subrings of  $M_4(\Pi)$ :

$$A = \{a = [a_{ij}] \in M_4(\Pi) : a_{ij} \in \Sigma; 3 \leq i \leq 4; 1 \leq j \leq 2\}$$

$$A^* = \{a = [a_{ij}] \in M_4(\Pi) : a_{ij} \in \Sigma; 1 \leq i \leq 2; 3 \leq j \leq 4\}.$$

Then, by ([5] Theorem 106.1),  $A$  is isomorphic to  $E(G)$  and, using the decomposition  $G = G_3 \oplus G_4 \oplus G_1 \oplus G_2$ , the same applies to  $A^*$ . Since the transposition of  $M_4(\mathbf{\Pi})$  induces an anti-isomorphism between  $A$  and  $A^*$ , we conclude that  $G$  is  $E$ -self-dual. To complete the proof, we now show that  $G' = \bigoplus_{i=1}^3 G_i$  is not an  $E$ -self-dual group. Otherwise, suppose  $E(G')$  has an anti-automorphism which takes  $e_i$ , the projection of  $G'$  onto  $G_i$ , to a suitable  $\varepsilon_i \in E(G')$  ( $1 \leq i \leq 3$ ). Then  $G' = \bigoplus_{i=1}^3 H_i$ , where  $H_i = \varepsilon_i(G')$  for every  $i$ . Evidently  $E(H_i) \cong \mathbf{\Pi}$  and  $\bigcap_{p \in P} p^\infty H_i = 0$ ; hence, by ([15] Theorem 2),  $\mathbf{\Sigma} \leq H_i \leq \mathbf{\Pi}$  ( $1 \leq i \leq 3$ ). Also note that  $\text{Hom}(H_3, H_i) \cong \text{Hom}(G_i, G_3) \cong \mathbf{\Sigma}$  ( $1 \leq i \leq 2$ ). For every prime  $p$ , let  $1_p$  be the unit of  $\mathbf{Z}(p)$  and let  $x = (1_p)_{p \in P} \in G_1$ ;  $y = (1_p)_{p \in P} \in G_2$ . On the other hand  $x = \sum_{i=1}^3 x_i$ ;  $y = \sum_{i=1}^3 y_i$  for some  $x_i, y_i \in H_i$  ( $1 \leq i \leq 3$ ). To find a contradiction, we first prove that  $x_i, y_i \in \mathbf{\Sigma}$  ( $1 \leq i \leq 2$ ). For instance, we show that  $x_1 \in \mathbf{\Sigma}$ . Since  $\varepsilon_1(ax) = a\varepsilon_1(x) = ax_1 \in H_1$  ( $a \in G_1 = \mathbf{\Pi}$ ) and  $H_3 \leq \mathbf{\Pi}$ , there exists a homomorphism  $\varphi: H_3 \rightarrow H_1$  such that  $\varphi(z) = zx_1$  for every  $z \in H_3$ ; therefore  $n\varphi = 0$  for some  $n \in \mathbf{N}$ . Since  $nx_1\mathbf{\Sigma} = n\varphi(\mathbf{\Sigma}) \leq n\varphi(H_3) = 0$ , we must have  $x_1 \in \mathbf{\Sigma}$ , as claimed. Consequently  $P^* = \{p \in P / t_p(G_1) \oplus t_p(G_2) \leq \text{Ker}(\varepsilon_1 + \varepsilon_2)\}$  contains all but finitely many primes. Fix  $p \in P^*$  and let  $j: \bigoplus_{i=1}^3 G_i \rightarrow \bigoplus_{i=1}^3 H_i$  denote the identity map of  $G'$ . Then the choice of  $p$  implies  $j(t_p(G_1) \oplus t_p(G_2)) \leq H_3$ ; on the other hand  $t_p(G_1), t_p(G_2)$  and  $t_p(H_3)$  are all isomorphic to  $\mathbf{Z}(p)$ . This contradiction establishes that  $G'$  is not  $E$ -self-dual, and the lemma follows. Another application of ([5] Theorem 106.1) shows that  $E(G')$  is anti-isomorphic to  $E(G'')$ , where  $G'' = G_2 \oplus G_3 \oplus G_4$ . Hence  $G''$  is another direct summand of  $G$  which is not  $E$ -self-dual.  $\square$

**LEMMA 1.3.** *Finite direct sums of  $E$ -self-dual groups are not necessarily  $E$ -dual.*

**PROOF.** It is enough to observe that  $\mathbf{Q}, \mathbf{Z}(p^\infty), \mathbf{Z}$  are clearly  $E$ -self-dual; however, as we shall see in sections 3 and 4, the groups  $\mathbf{Q} \oplus \mathbf{Z}(p^\infty), \mathbf{Q} \oplus \mathbf{Z}, \mathbf{Z}(p^\infty) \oplus \mathbf{Z}$  are not  $E$ -dual.  $\square$

Before classifying all  $E$ -dual and  $E$ -self-dual torsion groups by means of a suitable realization of their endomorphism ring, we summarize the results previously obtained about this kind of problem.

Liebert has shown ([10] Lemma A) that the endomorphism ring of a finite  $p$ -group has an anti-automorphism. By a result of Faltings ([4] Lemma 2.10), the same property holds for every torsion-complete  $p$ -group with finite Ulm invariants. A new theorem of Liebert ([12] Theorem 8.1) states that if  $G$  is a torsion-complete  $p$ -group, then  $E(G)$  has an anti-automorphism if and only if  $G$  has finite Ulm invariants.

§ 2. In the first part of this section we prove that if  $G$  is an  $E$ -dual reduced  $p$ -group, then  $G$  must be a torsion-complete  $E$ -self-dual  $p$ -group. We begin with two lemmas.

LEMMA 2.1. *Let  $G$  be a reduced  $p$ -group. If  $G$  is  $E$ -dual, then  $f_\sigma(G)$  is finite, for every  $\sigma < \omega$ .*

PROOF. Let  $G$  be as in the hypotheses and assume  $B = \bigoplus_{n \in \mathbb{N}} B_n$  is basic in  $G$ . We now prove that  $B_1$  is finite. By 1.1, there exists a group  $H$  and an anti-isomorphism  $f: E(B_1) \rightarrow E(H)$ . Since  $pH = 0$ , an application of ([1] General Existence Theorem, p. 193) shows that  $B_1$  is finite. An elementary proof of this fact is the following. Assume  $|B_1| \geq \aleph_0$ . To find a contradiction, it is enough to prove that  $B_1$  cannot be of cardinality  $\aleph_0$ . Suppose the contrary. Then  $H$  is not finite and  $E(B_1)$  has only one proper two-sided ideal consisting of all endomorphisms of finite rank ([8], Chapter 4; also see [1], p. 198). Since the endomorphism ring of an uncountable vector space has at least two proper two-sided ideals, i.e. the ideals of all endomorphisms of finite or countable rank, we conclude that  $|H| = \aleph_0$ . Let  $\pi$  be a minimal idempotent of  $E(B_1)$  and let  $\pi' = f(\pi)$ . Then  $|E(B_1)\pi| = |\text{Hom}(\pi(B_1), B_1)| = \aleph_0$ , while  $|\pi'E(H)| = |\text{Hom}(H, \pi'(H))| = 2^{\aleph_0}$ . This contradiction establishes that  $B_1$  is finite. To complete the proof, it is enough to check that  $B_{n+1}$  is finite ( $n \in \mathbb{N}$ ). Fix  $n \in \mathbb{N}$ , and let  $f: E(B_{n+1}) \rightarrow E(H)$  be an anti-isomorphism. Remark that  $H$  is a direct sum of cyclic groups of order  $p^{n+1}$ , because  $p^{n+1}H = 0$  and  $E(H)$  has no idempotent of order  $< p^{n+1}$ . Let  $\sigma_1: E(B_{n+1}) \rightarrow E(B_{n+1})/p^n E(B_{n+1})$  and  $\sigma_2: E(H) \rightarrow E(H)/p^n E(H)$  be the natural homomorphisms. Since  $\text{Ker } \sigma_1 = \text{Ker } \sigma_2 f$  and  $f$  is an anti-isomorphism, there exists an anti-isomorphism  $\tilde{f}: \sigma_1(E(B_{n+1})) \rightarrow \sigma_2(E(H))$  such that  $\tilde{f}\sigma_1 = \sigma_2 f$ . Evidently  $\sigma_1(E(B_{n+1})) \cong E(p^n B_{n+1})$  and  $\sigma_2(E(H)) \cong E(p^n H)$ . Therefore  $B_{n+1}$  is finite, and the lemma follows.  $\square$

LEMMA 2.2. *Let  $G$  be a reduced  $p$ -group. If  $G$  is  $E$ -dual, then  $p^\omega G = 0$ .*

PROOF. As before, let  $B = \bigoplus_{n \in \mathbb{N}} B_n$  be a basic subgroup of  $G$ . Let  $\pi_n$  denote the projection of  $G$  onto  $B_n$  with  $\text{Ker } \pi_n = \bigoplus_{m \neq n} B_m + p^n G$  ( $n \in \mathbb{N}$ ). Assume  $G$  is  $E$ -dual; then, there is an anti-isomorphism  $f: E(G) \rightarrow E(H)$  for some  $H$ . To see that  $G$  is separable, we shall use the following properties of  $H$ :

(1)  $t(H) = t_p(H)$ . Let  $q \in P$ ;  $q \neq p$ . Since  $G$  has no summand whose endomorphism ring is isomorphic to  $J_q$  or to  $\mathbb{Z}(q^n)$  for some  $n \in \mathbb{N}$ , the same applies to  $H$ . Therefore  $t_q(H) = 0$ .

(2)  $H$  is reduced. By the previous remark, it is enough to observe that  $\mathbb{Q}$  and  $\mathbb{Z}(p^\infty)$  cannot be subgroups of  $H$ .

(3)  $H$  is a  $J_p$ -module. This follows from the fact that the center of  $E(H)$  is isomorphic to the center of  $E(G)$ , and the center of  $E(G)$  is isomorphic to  $J_p$  or to  $\mathbb{Z}(p^n)$  for some  $n \in \mathbb{N}$  ([5] Theorem 108.3).

(4) If  $\pi'_n = f(\pi_n)$  and  $\pi'_n(H) = B'_n$  ( $n \in \mathbb{N}$ ), then  $B' = \bigoplus_{n \in \mathbb{N}} B'_n$  is a basic subgroup of  $t_p(H)$ . Since  $B'$  is a direct sum of cyclic groups and  $B' \leq^* t_p(H)$ , there is a basic subgroup  $B'' = \bigoplus_{n \in \mathbb{N}} B''_n$  such that  $B' \leq B''$ .

Our claim is that  $B' = B''$ . Assume this is not true. Choose  $m \in \mathbb{N}$  such that  $B'_m < B''_m$ . Let  $\eta'$  be a projection of  $H$  onto  $B''_m$ , and let  $\eta' = f(\eta)$ . Therefore  $\eta(G)$  is a direct sum of cyclic groups of order  $p^m$  and clearly  $B_m < \eta(G)$ . This contradiction proves that  $B' = B''$ .

(5)  $p^\omega t_p(H) = 0$ . To see this, suppose the contrary. Then, there is  $\varphi \in E(G)$  such that  $0 \neq f(\varphi) = \varphi' \in E(H)[p]$  and  $\varphi'(H) \leq p^\omega t_p(H)$ . Since  $\pi'_n \varphi' = 0$ , we get  $\varphi \pi_n = 0$  ( $n \in \mathbb{N}$ ). It follows that  $\varphi = 0$ , and this contradicts the hypothesis that  $\varphi' \neq 0$ ; consequently  $p^\omega t_p(H) = 0$ .

The last remark tells us that, if  $H$  is a  $p$ -group,  $G$  is separable. To end the proof, assume that  $p^\omega G \neq 0$ . Then there is a suitable  $\varphi \in E(G)[p]$  such that  $\varphi \neq 0$ ;  $\pi_n \varphi = 0$  ( $n \in \mathbb{N}$ ). Let  $\varphi' = f(\varphi)$ ; since  $\varphi' \pi'_n = 0$  ( $n \in \mathbb{N}$ ),  $\varphi'$  is 0 on  $B'$ . Using (1), (2) and (4), we conclude that  $\varphi'(t(H)) = 0$ , and therefore  $\text{Hom}(H/t(H), H[p]) \neq 0$ . This implies that  $H/t(H)$  is not  $p$ -divisible; hence, by (3), there exists  $x \in H$ ,  $x \notin t(H)$  such that  $J_p x$  is a direct summand of  $H$ . But this is impossible, because  $G$  has no summand isomorphic to  $J_p$  or  $\mathbb{Z}(p^\infty)$ . This contradiction proves that  $p^\omega G = 0$ .  $\square$

REMARK. In  $G$  is an infinite reduced  $E$ -dual  $p$ -group,  $|G| = 2^{\aleph_0}$ . In fact, with the notations of 2.2,  $|G| \leq \left| \prod_{n \in \mathbb{N}} B_n \right| = 2^{\aleph_0}$ . To prove the

reverse inequality, take  $m \in \mathbb{N}$  such that  $B_m \neq 0$ . Since  $|H/p^m H| \geq |B'/p_{\mathbb{1}}^m B'| = \aleph_0$ , we clearly have

$$\begin{aligned} |G| \geq |G[p^m]| &= |\text{Hom}(B_m, G)| = |E(G)\pi_m| = |\pi'_m E(H)| = \\ &= |\text{Hom}(H, B'_m)| = |\text{Hom}(H/p^m H, B'_m)| \geq 2^{\aleph_0}. \end{aligned}$$

Following ([12], p. 350), we say that a  $p$ -group  $G$  is torsion-compact if  $G$  is torsion-complete and every Ulm invariant of  $G$  is finite. We now give a realization of the endomorphism ring of a torsion-compact  $p$ -group  $G$ . If  $G$  is finite, an application of ([5] Theorem 106.1) shows that there exist  $r, n \in \mathbb{N}$  such that  $E(G)$  is isomorphic to a subring of  $M_r(\mathbb{Z}(p^n))$  fully invariant under the transposition of  $M_r(\mathbb{Z}(p^n))$ . This is an elementary proof of a result ([10] Lemma A) mentioned in section 1. Suppose now that  $G$  is not finite. Fix a basic subgroup  $B = \bigoplus_{n \in \mathbb{N}} \langle x_n \rangle$  of  $G$  such that  $o(x_r) \leq o(x_s)$  if  $r \leq s$ . Regarding  $G$  as embedded in  $\prod_{n \in \mathbb{N}} \langle x_n \rangle$ , let  $\pi_n$  denote the projection of  $G$  onto  $\langle x_n \rangle$  ( $n \in \mathbb{N}$ );

for every  $x \in G$  we may write  $x = (\alpha_n x_n)_{n \in \mathbb{N}}$ , where  $\alpha_n x_n = \pi_n(x)$  and  $\alpha_n$  is a suitable  $p$ -adic integer ( $n \in \mathbb{N}$ ). Let  $h$  be the  $p$ -adic valuation of  $J_p$  and let  $A$  be additive group of all  $\aleph_0 \times \aleph_0$  matrices over  $J_p$  of the form  $a = [\alpha_{rs}]$ , where  $h(\alpha_{rs}) \geq \lambda_{rs} = \max(0, e(x_r) - e(x_s))$  for every  $r, s \in \mathbb{N}$ . Then  $A$  is a ring with the usual rows by columns product and the subset  $I$  of all  $a = [\alpha_{rs}] \in A$  such that  $h(\alpha_{rs}) \geq e(x_r)$  ( $r, s \in \mathbb{N}$ ) is a two-sided ideal of  $A$ . Remark that every  $\varphi \in E(G)$  is completely determined by the elements  $\{\varphi(x_s) = (\alpha_{rs} x_r)_{r \in \mathbb{N}} (s \in \mathbb{N})\}$ . With these notations, let  $\varrho: E(G) \rightarrow A/I$  be the map defined by  $\varrho(\varphi) = [\alpha_{rs}] + I$  for all  $\varphi \in E(G)$ . Evidently  $\varrho$  is a group isomorphism. We claim that  $\varrho$  is a ring isomorphism. To see this, choose  $\varphi, \psi \in E(G)$ ; then there are suitable  $\alpha_{rs}, \beta_{rs}, \delta_{rs} \in J_p$  ( $r, s \in \mathbb{N}$ ) such that  $\varrho(\varphi) = [\alpha_{rs}] + I$ ;  $\varrho(\psi) = [\beta_{rs}] + I$ ;  $\varrho(\psi\varphi) = [\delta_{rs}] + I$ . Fix  $r, s \in \mathbb{N}$ ; then there exists some  $k \in \mathbb{N}$  such that  $e(x_k) \geq e(x_r) + e(x_s)$ . Consequently  $\varphi(x_s) \equiv \sum_{i=1}^k \alpha_{is} x_i \pmod{p^{e(x_r)} G}$ . Since

$$\delta_{rs} x_r = \pi_r(\psi\varphi(x_s)) = \pi_r\left(\sum_{i=1}^k \alpha_{is} \psi(x_i)\right) = \left(\sum_{i=1}^k \beta_{ri} \alpha_{is}\right) x_r,$$

we obtain  $\delta_{rs} \equiv \sum_{i=1}^k \beta_{ri} \alpha_{is} \pmod{p^{e(x_r)} J_p}$ . Using the hypothesis that  $h(\alpha_{ns}) \geq e(x_r)$  for all  $n > k$ , we conclude that  $\delta_{rs} \equiv \sum_{n \in \mathbb{N}} \beta_{rn} \alpha_{ns} \pmod{p^{e(x_r)} J_p}$ .

This proves that  $\varrho$  is a ring isomorphism, because  $\varphi, \psi \in E(G)$  and  $r, s \in \mathbb{N}$  are arbitrary elements.

The following theorem characterizes all  $E$ -dual reduced  $p$ -groups.

**THEOREM 2.3.** *Let  $G$  be a reduced  $p$ -group. The following are equivalent:*

- (1)  $G$  is torsion-compact.
- (2)  $G$  is  $E$ -self-dual.
- (3)  $G$  is  $E$ -dual.

**PROOF** (1)  $\Rightarrow$  (2). As already observed, finite  $p$ -groups are  $E$ -self-dual. Assume  $G$  is not finite and fix a basic subgroup  $B = \bigoplus_{n \in \mathbb{N}} \langle x_n \rangle$  of  $G$  such that  $e(x_r) \leq e(x_s)$  if  $r \leq s$  ( $r, s \in \mathbb{N}$ ). Let  $A$  be the ring of all matrices  $a = [\alpha_{rs}]$ , where  $\alpha_{rs} = p^{\lambda_{rs}} \gamma_{rs}$  for some  $\gamma_{rs} \in J_p$  and  $\lambda_{rs} = \max(0, e(x_r) - e(x_s))$  ( $r, s \in \mathbb{N}$ ). Evidently the map  $t: A \rightarrow A$  such that  ${}^t a = {}^t [p^{\lambda_{rs}} \gamma_{rs}] = [p^{\lambda_{rs}} \gamma_{sr}]$  ( $a \in A$ ) is a group isomorphism. To show that  $t$  is a ring anti-automorphism, pick  $a, a' \in A$ . Let  $a = [p^{\lambda_{rs}} \gamma_{rs}]$ ;  $a' = [p^{\lambda'_{rs}} \gamma'_{rs}]$ ;  $a'a = [\beta_{rs}]$  and  ${}^t a {}^t a' = [\delta_{rs}]$ . For every  $r, s \in \mathbb{N}$  we may write  $\beta_{rs} = \sum_{n \in \mathbb{N}} p^{\sigma_n} \gamma'_{rn} \gamma_{ns}$ ;  $\delta_{sr} = \sum_{n \in \mathbb{N}} p^{\tau_n} \gamma'_{rn} \gamma_{ns}$  where  $\sigma_n = \lambda_{rn} + \lambda_{ns}$  and  $\tau_n = \lambda_{sn} + \lambda_{nr}$  ( $n \in \mathbb{N}$ ). Let  $r \leq s$ ; it is easy to check that

$$\sigma_n = \begin{cases} e(x_r) - e(x_n) \\ 0 \\ e(x_n) - e(x_s) \end{cases} \quad \tau_n = \begin{cases} e(x_s) - e(x_n) & e(x_n) \leq e(x_r) \\ e(x_s) - e(x_r) & e(x_r) \leq e(x_n) \leq e(x_s) \\ e(x_n) - e(x_r) & e(x_s) \leq e(x_n) \end{cases}$$

Since  $\lambda_{sr} = e(x_s) - e(x_r)$ , we clearly have  $\tau_n = \sigma_n + \lambda_{sr}$  ( $n \in \mathbb{N}$ ), and therefore  $\delta_{sr} = p^{\lambda_{sr}} \beta_{rs}$ . Let  $r \geq s$ ; this hypothesis implies

$$\sigma_n = \begin{cases} e(x_r) - e(x_n) \\ e(x_r) - e(x_s) \\ e(x_n) - e(x_s) \end{cases} \quad \tau_n = \begin{cases} e(x_s) - e(x_n) & e(x_n) \leq e(x_s) \\ 0 & e(x_s) \leq e(x_n) \leq e(x_r) \\ e(x_n) - e(x_r) & e(x_n) \leq e(x_r) \end{cases}$$

Since  $\lambda_{rs} = e(x_r) - e(x_s)$ , we get  $\tau_n = \sigma_n - \lambda_{rs}$  ( $n \in \mathbb{N}$ ); thus  $\delta_{sr} = p^{-\lambda_{rs}} \beta_{rs}$ . Consequently  ${}^t [\beta_{rs}] = [\delta_{rs}]$ , and  $t$  is a ring anti-automorphism, as required. Let  $r, s \in \mathbb{N}$  and  $\gamma_{rs} \in J_p$ ; then  $h(p^{\lambda_{rs}} \gamma_{rs}) \geq e(x_r)$  if and only

if  $h(p^{\lambda_{rs}}\gamma_{rs}) \geq e(x_s)$ . Hence  $t$  induces an anti-automorphism of  $A/I$ , that we still call  $t$ . Since  $A/I$  is isomorphic to  $E(G)$ ,  $G$  is  $E$ -self-dual.

(2)  $\Rightarrow$  (3): This is obvious.

(3)  $\Rightarrow$  (1): Let  $G$  be an  $E$ -dual  $p$ -group. If  $G$  is finite,  $G$  is clearly torsion-compact. If  $G$  is not finite, Lemmas 2.1 and 2.2. enable us to assume that  $G < \bar{B} = t\left(\prod_{n \in \mathbb{N}} \langle x_n \rangle\right)$ , where  $B = \bigoplus_{n \in \mathbb{N}} \langle x_n \rangle$  is a basic subgroup of  $G$  and  $e(x_r) \leq e(x_s)$  if  $r \leq s$  ( $r, s \in \mathbb{N}$ ). To prove that  $\bar{B} \leq G$ , we introduce some endomorphisms of  $G$  similar to those used in ([9] Theorem 28). Let  $\pi_n$  be the projection of  $G$  onto  $\langle x_n \rangle$  ( $n \in \mathbb{N}$ ). Then, for all  $r, s \in \mathbb{N}$  we define  $e_{rs} \in E(G)$  as follows:  $e_{rs}(1 - \pi_s) = 0$  and  $e_{rs}(x_s) = p^{\lambda_{rs}}x_r$ , where  $\lambda_{rs} = \max(0, e(x_r) - e(x_s))$ . Since  $G$  is  $E$ -dual, there exists an anti-isomorphism  $f: E(G) \rightarrow E(H)$  for some  $H$ . Write  $f(e_{rs}) = e'_{sr}$  ( $r, s \in \mathbb{N}$ ) and choose  $y_n \in H$  such that  $e'_{nn}(H) = \langle y_n \rangle$  ( $n \in \mathbb{N}$ ). For every  $r, s \in \mathbb{N}$ , let  $\varepsilon_{rs}$  be the endomorphism of  $H$  uniquely determined by the following conditions  $\varepsilon_{rs}(y_s) = p^{\lambda_{rs}}y_r$ ,  $\varepsilon_{rs}(1 - e'_{ss}) = 0$ . Remark that  $e_{rs} = e_{rr}e_{rs}e_{ss}$  implies  $e'_{sr} = e'_{ss}e'_{sr}e'_{rr}$ ; therefore  $e'_{sr} = u_{sr}\varepsilon_{sr}$  for some  $u_{sr} \in J_p \setminus pJ_p$  ( $r, s \in \mathbb{N}$ ). Assume  $G < \bar{B}$  and choose  $x \in \bar{B} \setminus G$ . Then there exist  $\alpha_n \in J_p$  ( $n \in \mathbb{N}$ ) and  $m \in \mathbb{N}$  such that  $x = (\alpha_n x_n)_{n \in \mathbb{N}}$  and  $o(x_n) \geq o(x)$  for all  $n \geq m$ . Let  $\alpha_n^* = 0$  if  $n < m$ , and let  $\alpha_n^* = \alpha_n$  if  $n \geq m$ . We claim that  $x^* = (\alpha_n^* x_n)_{n \in \mathbb{N}} \in G$ . To see this, let  $\varphi'$  denote an endomorphism of  $H$  with the following properties:  $\varphi' = e'_{mm}\varphi'$ ,  $\varphi'(y_n) = \alpha_n^* u_{mn} y_m$  ( $n \in \mathbb{N}$ ). Now consider the endomorphism  $\varphi$  of  $G$  such that  $f(\varphi) = \varphi'$ . By hypothesis, we clearly have  $e'_{mm}\varphi' e'_{nn} = \alpha_n^* u_{mn} \varepsilon_{mn} = (\alpha_n^* u_{mn})(u_{mn}^{-1} e'_{mn}) = \alpha_n^* e'_{mn}$  ( $n \in \mathbb{N}$ ). It follows that  $\varphi(x_m) = (\alpha_n^* x_n)_{n \in \mathbb{N}} = x^* \in G$ . Since  $x \in x^* + B$ , we obtain  $x \in G$ . This contradiction shows that  $G = \bar{B}$  and the proof is complete.  $\square$

REMARK. Let  $G$  be an infinite torsion-compact  $p$ -group. With the previous notations, the anti-automorphism  $t$  of  $E(G)$  defined in the first part of the proof has the property that  $e'_{rs} = \varepsilon_{rs}$ ; thus we may assume  $u_{rs} = 1$  for every  $r, s \in \mathbb{N}$ . This follows from the fact that  $t$  is induced by the most obvious transposition of the matrix ring  $A$ .

A remark of ([12], p. 352) states that if  $G$  is a  $p$ -group, then  $G$  is  $E$ -self-dual if and only if  $G$  is either a torsion-compact group or a divisible group of finite rank. As we shall see, if a divisible  $p$ -group is  $E$ -dual then it is  $E$ -self-dual, but there exist  $E$ -dual  $p$ -groups which are not  $E$ -self-dual.

**THEOREM 2.4.** *Let  $G$  be a divisible  $p$ -group. Then  $G$  is  $E$ -dual if and only if it is of finite rank.*

**PROOF.** If  $G$  is a divisible  $p$ -group of rank  $n$ , then  $E(G) \cong M_n(J_p)$  clearly has an anti-automorphism. By 1.1, to prove the theorem, it is enough to show that a divisible  $p$ -group of rank  $\aleph_0$  cannot be  $E$ -dual. Assume this is not true. Write  $G = \bigoplus_{n \in \mathbb{N}} G_n$ , where  $G_n \cong \mathbb{Z}(p^\infty)$  for every  $n \in \mathbb{N}$ , and choose an anti-isomorphism  $f: E(G) \rightarrow E(H)$  for some  $H$ . Let  $\pi_n$  be the projection of  $G$  onto  $G_n$  ( $n \in \mathbb{N}$ ), and let  $\varphi' = f(\varphi)$  ( $\varphi \in E(G)$ ). First note that the groups  $\pi'_n(H)$  ( $n \in \mathbb{N}$ ) are all isomorphic. Otherwise, there exist  $m, n \in \mathbb{N}$  such that  $\pi'_m(H) \cong \mathbb{Z}(p^\infty)$  and  $\pi'_n(H) \cong J_p$ . But this is impossible, because  $\pi'_n E(H) \pi'_m = 0$ , while  $\pi_m E(G) \pi_n \neq 0$  (compare with ([13] Lemma 1.2)). Remark now the following properties of  $H$ :

(1)  $H$  is not a divisible  $p$ -group. Suppose (1) does not hold. Let  $\sigma_1: E(G) \rightarrow E(G)/pE(G)$ ;  $\sigma_2: E(H) \rightarrow E(H)/pE(H)$  be the natural homomorphisms. Then  $\sigma_1(E(G))$ ,  $\sigma_2(E(H))$  are isomorphic to the endomorphism rings of two infinite vector spaces over  $\mathbb{Z}(p)$ , and this is a contradiction. In fact, the existence of  $f$  implies that  $\sigma_1(E(G))$  is anti-isomorphic to  $\sigma_2(E(H))$ . Hence (1) is true.

(2)  $H$  is torsion-free. Since  $G$  has no finite summand, the same holds for  $H$ . Consequently  $t_q(H) = 0$  for every prime  $q \neq p$ , because  $G$  cannot have  $J_q$  or  $\mathbb{Z}(q^\infty)$  for a summand. It remains to check that  $t_p(H) = 0$ . Assume the contrary. Then  $H$  is a mixed group and  $t(H)$  is a divisible  $p$ -group. Thus  $t(H)$  is a proper fully invariant direct summand of  $H$ . On the other hand  $G$  has no proper fully invariant direct summand. This contradiction proves that  $H$  is torsion-free.

(3)  $H$  is a reduced  $J_p$ -module. Since  $H$  is torsion-free and  $G$  is a  $p$ -group, it suffices to repeat the proof of (2) and (3) of Lemma 2.2.

It is now clear that we may assume  $\pi'_1(H) = J_p$ . As an immediate consequence  $J_p^{\mathbb{N}} \cong \text{Hom}(G, G_1) \cong \text{Hom}(\pi'_1(H), H) \cong \text{Hom}_{J_p}(\pi'_1(H), H)$ . By (3), the map which takes  $\varphi'$  to  $\varphi'(1)$  for all  $\varphi' \in \text{Hom}(\pi'_1(H), H)$  is an isomorphism between  $\text{Hom}(\pi'_1(H), H)$  and  $H$ ; therefore  $H \cong J_p^{\mathbb{N}}$ . Since  $H$  properly contains the  $p$ -adic completion of  $\bigoplus_{n \in \mathbb{N}} \pi'_n(H)$ , the group  $\bar{H} = H \left| \bigoplus_{n \in \mathbb{N}} \pi'_n(H) \right.$  is not  $p$ -divisible. This implies that  $\bar{H}$  has a pure subgroup, hence a direct summand, isomorphic to  $J_p$ . Since  $\text{Hom}(\bar{H}, H) \neq 0$ , there exists a non-zero endomorphism  $\varphi'$  of  $H$  such

that  $\varphi' \left( \bigoplus_{n \in \mathbb{N}} \pi'_n(H) \right) = 0$ . But this means that if  $\varphi \in E(G)$  and  $f(\varphi) = \varphi'$ , then  $\varphi \neq 0$ , while  $\pi_n \varphi = 0$  ( $n \in \mathbb{N}$ ). This contradiction establishes that divisible  $p$ -groups of infinite rank are not  $E$ -dual.  $\square$

REMARK. Let  $G$  be a divisible  $p$ -group of infinite rank  $\mathfrak{m}$ . Then, as observed in ([5] vol. II, p. 220)  $E(G)$  is isomorphic to the ring of all column-convergent  $\mathfrak{m} \times \mathfrak{m}$  matrices with entries in  $J_p$  (i.e. in every column almost all entries are divisible by  $p^n$  for any  $n \in \mathbb{N}$ ). In this case the asymmetry between rows and columns cannot be removed by means of a suitable transposition, as in the case of infinite torsion-compact  $p$ -groups.

THEOREM 2.5. *Let  $G$  be a  $p$ -group. Then  $G$  is  $E$ -dual if and only if either  $G$  is torsion-compact or  $G$  is of finite rank.*

PROOF. By the previous results, we assume  $G$  is neither reduced nor divisible. Suppose first that  $G$  has finite rank. Then  $G$  has a decomposition  $G = \bigoplus_{i=1}^r G_i$  with the following properties:  $G_i$  is a cyclic group, if  $1 \leq i \leq n$ , and  $G_i \cong \mathbb{Z}(p^\infty)$ , if  $n + 1 \leq i \leq r$ . Using ([5] Theorem 106.1), we identify  $E(G)$  with the ring of all  $r \times r$  matrices  $[\alpha_{ij}]$ , where  $\alpha_{ij} \in \text{Hom}(G_j, G_i)$ . Define  $H$  to be the group  $H = \bigoplus_{i=1}^r H_i$ , where  $H_i = G_i$  if  $1 \leq i \leq n$ , and  $H_i = J_p$  if  $n + 1 \leq i \leq r$ . Another application of ([5] Theorem 106.1) shows that  $E(H)$  is isomorphic to the ring of all  $r \times r$  matrices  $[\alpha'_{ij}]$ , where  $\alpha'_{ij} \in \text{Hom}(H_j, H_i)$ . Identifying the groups  $\text{Hom}(G_j, G_i)$  and  $\text{Hom}(H_j, H_i)$  ( $1 \leq i, j \leq n$ ), let  $t: E(G) \rightarrow E(H)$  be the map that sends  $a = [\alpha_{ij}]$  to  $t a = [\alpha'_{ij}]$  for all  $a \in E(G)$ . Since  $t$  is a ring anti-isomorphism,  $G$  is  $E$ -dual. Conversely, let  $G = D \oplus R$  be an  $E$ -dual  $p$ -group, where  $D \cong (\mathbb{Z}(p^\infty))^r$  for some  $r \in \mathbb{N}$  and  $R$  is reduced. Our claim is that  $R$  is finite. Suppose  $R$  is not finite and fix an anti-isomorphism  $f: E(G) \rightarrow E(H)$ . Let  $\pi_1, \pi_2$  be the projections of  $G$  onto  $D$  and  $R$  respectively, and let  $\pi'_i = f(\pi_i)$ ,  $H_i = \pi'_i(H)$  ( $1 \leq i \leq 2$ ). Then  $H = H_1 \oplus H_2$ . Since  $\pi'_1 E(H) \pi'_2 = 0$ ,  $H_1$  is isomorphic to  $J_p^{\aleph_0}$ . The proof of Lemma 2.2 enables us to regard  $H_2$  as embedded in a group of the form  $\prod_{n \in \mathbb{N}} B'_n$ , where  $B' = \bigoplus_{n \in \mathbb{N}} B'_n$  is isomorphic to a basic subgroup  $B = \bigoplus_{n \in \mathbb{N}} B_n$  of  $R$ . Since  $|R/B| = 2^{\aleph_0}$ , we get  $|\text{Hom}(R, D)| = \left| \text{Hom}(B, \mathbb{Z}(p^\infty)) \oplus J_p^{2^{\aleph_0}} \right| > 2^{\aleph_0}$  ([5] Theorem 47.1). This contradicts the fact that  $|\text{Hom}(H_1, H_2)| \leq \left| \prod_{n \in \mathbb{N}} \text{Hom}(J_p^n, B'_n) \right| \leq 2^{\aleph_0}$ . Therefore  $R$  is finite, and the proof is complete.  $\square$

**COROLLARY 2.6.** *Let  $G$  be a torsion group. Then  $G$  is  $E$ -dual if and only if  $t_p(G)$  is  $E$ -dual for every prime  $p$ .*

**PROOF.** Necessity follows from 1.1. Assume that  $t_p(G)$  is  $E$ -dual ( $p \in P$ ). Let  $H_p = t_p(G)$  if  $t_p(G)$  is reduced, and let  $H_p = J_p' \oplus R$  if  $t_p(G) \cong (\mathbf{Z}(p^\infty))^r \oplus R$ , where  $r \in \mathbf{N}$  and  $R$  is reduced. If  $H = \bigoplus_{p \in P} H_p$ , then  $E(H) \cong \prod_{p \in P} E(H_p)$ . Since  $E(t_p(G))$  is anti-isomorphic to  $E(H_p)$  for all  $p \in P$ ,  $E(G)$  is anti-isomorphic to  $E(H)$ . Thus  $G$  is  $E$ -dual.  $\square$

**COROLLARY 2.7.** *If  $G$  and  $H$  are torsion groups with anti-isomorphic endomorphism rings, the following conditions hold for every prime  $p$ :*

(i)  $t_p(G)$  is either reduced or divisible.

(ii)  $t_p(G)$  is isomorphic to  $t_p(H)$ .

**PROOF.** (i) Assume  $t_p(G)$  is neither reduced nor divisible. Then our hypotheses imply that  $t_p(H)$  is a group of the form  $t_p(H) = D \oplus R$ , where  $D$  is divisible,  $R$  is reduced and  $\text{Hom}(D, R) \neq 0$ . Since this is clearly impossible, (i) holds.

(ii) If  $t_p(G)$  is reduced, then the proof of 2.2 indicates that  $t_p(G)$  and  $t_p(H)$  must be torsion-compact  $p$ -groups with the same Ulm invariants. Consequently  $t_p(G)$  is isomorphic to  $t_p(H)$ . If  $t_p(G)$  is divisible, the statement is obvious, because  $t_p(G)$  and  $t_p(H)$  must have the same rank.  $\square$

**REMARK.** By 2.6 and 2.7, if  $G$  and  $H$  are torsion groups and there exists an anti-isomorphism  $f: E(G) \rightarrow E(H)$ , then  $G$  belongs to a restricted class of torsion groups and  $H$  is isomorphic to  $G$ . In particular, let  $G, H, f$  be as above; then the following conditions are equivalent:

(1)  $f$  is induced by a group isomorphism  $\tau: G \rightarrow H$  (i.e.  $f(\varphi) = \tau\varphi\tau^{-1}(\varphi \in E(G))$ ).

(2)  $E(G)$  is commutative.

In fact, assume first that (1) is true. Since  $\tau\varphi\tau^{-1} = \tau\varphi\tau^{-1}$  ( $\varphi, \psi \in E(G)$ ), (2) clearly holds. This completes the proof, because the implication (2)  $\Rightarrow$  (1) follows from the Baer-Kaplansky theorem. Hence, by ([15] Theorem 1), condition (1) is not generally satisfied.

§ 3. The characterization of all  $E$ -dual torsion groups enables us to prove the following

**THEOREM 3.1.** *Let  $G = \prod_{p \in P} G_p$  be a reduced cotorsion group. Then  $G$  is  $E$ -dual if and only if, for every  $p \in P$ , its  $p$ -adic component  $G_p$  is either a  $J_p$ -module of finite rank or the  $p$ -adic completion of a torsion-compact  $p$ -group.*

**PROOF.** Necessity. Suppose  $G$  is  $E$ -dual and fix a prime  $p$ . To prove that  $G_p$  has the required properties, we distinguish three cases.

(i)  $G_p$  adjusted. Since  $E(G_p)$  is isomorphic to  $E(t_p(G_p))$  ([13] Theorem 3.3), from Theorem 2.3 we deduce that  $G_p$  is the  $p$ -adic completion of a torsion compact  $p$ -group.

(ii)  $G_p$  torsion-free. An application of ([11] Theorem 5.5) shows that  $E(G_p) \cong E(\mathbf{Q}_p/J_p \otimes G_p)$ , where  $\mathbf{Q}_p$  is the field of  $p$ -adic numbers. Using 2.4, we conclude that  $G_p$  is isomorphic to  $J_p^r$  for some  $r \in \mathbf{N}$ .

(iii)  $G_p$  neither adjusted nor torsion-free. It is not restrictive to assume  $G_p = J_p^r \oplus G'_p$ , where  $r \in \mathbf{N}$  and  $G'_p$  is adjusted. We claim that  $G'_p$  is finite. Suppose the contrary. Let  $B = \bigoplus_{n \in \mathbf{N}} B_n$  be a basic subgroup of  $t_p(G'_p)$  and assume  $G'_p \leq \prod_{n \in \mathbf{N}} B_n$ . Fix a group  $H$  and an anti-isomorphism between  $E(G_p)$  and  $E(H)$ , which takes  $\varphi$  to  $\varphi'$  ( $\varphi \in E(G_p)$ ). Let  $\pi_1, \pi_2$  be the projections of  $G_p$  onto  $J_p^r$  and  $G'_p$  respectively, and let  $H_i = \pi'_i(H)$  ( $1 \leq i \leq 2$ ). Evidently  $H = H_1 \oplus H_2$  and  $H_1 \cong (\mathbf{Z}(p^\infty))^r$  because  $H_1$  is fully invariant in  $H$ . Since  $E(G'_p)$  is isomorphic to  $E(t_p(G'_p))$ , the proof of Lemma 2.2 shows that  $H_2$  is a reduced  $J_p$ -module. The hypothesis that  $G'_p$  is not finite and the fact that  $H_1 \cong (\mathbf{Z}(p^\infty))^r$  guarantee that  $H_2$  is not a  $p$ -group. Let  $x$  be a torsion-free element of  $H_2$ , and let  $y = ux$ , where  $u$  is a  $p$ -adic integer algebraically independent over  $\mathbf{Z}_p$ . Choose  $\bar{x}, \bar{y} \in H_1$  such that  $\bar{y} \neq u\bar{x}$ . Then  $H$  has an endomorphism  $\varphi'$  that maps  $x$  and  $y$  onto  $\bar{x}$  and  $\bar{y}$  respectively. Since  $\text{Hom}(J_p^r, G'_p) \leq \text{Hom}_{J_p}(J_p^r, \prod_{n \in \mathbf{N}} B_n)$ , the center of  $E(G)$  is isomorphic to  $J_p$  and the same applies to  $E(H)$ . But this is a contradiction, because  $u\varphi' \neq \varphi'u$ . Consequently  $G'_p$  is finite.

Sufficiency. Assume  $G = \prod_{p \in P} G_p$ , where each  $G_p$  is as in (i), (ii) or (iii). Then 2.5, 2.6 and the result used in (i) tell us that, for every  $p \in P$ , there exists a group  $H_p$  such that  $E(G_p)$  is anti-isomorphic to  $E(H_p)$

and  $H_p$  is fully invariant in  $H = \bigoplus_{p \in P} H_p$ . Since  $E(H) \cong \prod_{p \in P} E(H_p)$ ,  $G$  is  $E$ -dual.  $\square$

**COROLLARY 3.2.** *Let  $G = \prod_{p \in P} G_p$  be a reduced cotorsion group. The following are equivalent:*

(1) *For every prime  $p$ ,  $G_p$  is either a torsion-free  $J_p$ -module of finite rank or the  $p$ -adic completion of a torsion-compact  $p$ -group.*

(2)  *$G$  is  $E$ -self-dual.*

**PROOF.** Since  $G$  is  $E$ -self-dual if and only if the  $p$ -adic component  $G_p$  of  $G$  is  $E$ -self-dual ( $p \in P$ ), the result is an immediate consequence of the previous theorem. In fact, the first part of the proof of 3.1 shows that if  $G_p$  is  $E$ -dual, then  $G_p$  is  $E$ -self-dual if and only if it is either adjusted or torsion-free.  $\square$

**REMARK.** In ([9], p. 73) Kaplansky asserts that there are reasons for believing that two modules with isomorphic (or anti-isomorphic) endomorphism rings are isomorphic or «dual». This suggests that we translate 3.1 and 3.2 as follows: The correspondence given by Harrison ([7]) between torsion groups and reduced cotorsion groups induces a correspondence between  $E$ -dual ( $E$ -self-dual) torsion groups and  $E$ -dual ( $E$ -self-dual) reduced cotorsion groups. It is natural to compare this statement with a result of May and Tubassi ([13] Main Theorem) about groups with isomorphic endomorphism rings, i.e. the characterization of all groups  $G$  and  $H$  such that  $E(G) \cong E(H)$  and  $t(G) \cong t(H)$ . Even in this case, a theory of duality, more precisely Harrison's duality, clarifies the situation.

**THEOREM 3.3.** *Let  $G$  be a divisible group. Then  $G$  is  $E$ -dual if and only if either  $G = \bigoplus_{p \in P} D_p$  with  $D_p$  a divisible  $p$ -group of finite rank or  $G$  is a torsion-free group of finite rank.*

**PROOF.** Sufficiency immediately follows from 2.5, because if  $G$  is torsion-free of finite rank  $r$ , then  $E(G) \cong M_r(\mathbb{Q})$ . Since arguments very similar to those used in the first part of 2.1 show that infinite dimensional vector spaces over  $\mathbb{Q}$  cannot be  $E$ -dual, it remains to prove that the group  $G = \mathbb{Q} \oplus \mathbb{Z}(p^\infty)$  is not  $E$ -dual ( $p \in P$ ). Suppose this does not hold. Let  $\pi_1, \pi_2$  be the projections of  $G$  onto  $\mathbb{Q}$  and  $\mathbb{Z}(p^\infty)$  respectively, and let  $\pi'_1, \pi'_2$  be the corresponding elements under an anti-isomorphism between  $E(G)$  and  $E(H)$  for some  $H$ . Write

$H = H_1 \oplus H_2$ , where  $H_i = \pi'_i(H)$  ( $1 \leq i \leq 2$ ). Since  $H_1 \cong \mathbb{Q}$  and  $\pi'_2 E(H) \pi'_1 = 0$ , we must have  $H_2 \cong J_p$ . Therefore  $|E(H)| > 2^{\aleph_0} = |E(G)|$ , and this contradiction proves that divisible mixed groups are not  $E$ -dual.  $\square$

The following result is an obvious consequence of Theorem 3.3.

**COROLLARY 3.4.** *Let  $G$  be a divisible group. Then  $G$  is  $E$ -dual if and only if  $G$  is  $E$ -self-dual.*

**COROLLARY 3.5.** *Let  $G = D \oplus R$  and let  $D$ , the divisible part of  $G$ , be non-zero and torsion-free. Then  $G$  is  $E$ -dual if and only if  $D$  and  $R$  are  $E$ -dual and  $R$  is a torsion group.*

**PROOF.** Let  $G$  be an  $E$ -dual group as in the hypotheses. Fix a group  $H$  such that  $E(G)$  and  $E(H)$  are anti-isomorphic. Write  $H = H_1 \oplus H_2$ , where  $\text{Hom}(H_1, H_2) \cong \text{Hom}(R, D)$ ;  $\text{Hom}(H_2, H_1) = 0$  and  $E(H_1)$ ,  $E(H_2)$  are anti-isomorphic to  $E(D)$  and  $E(R)$  respectively. Then  $H_1$  is isomorphic to  $D$ , while  $H_2$  is a reduced torsion group. By symmetry, we conclude that  $R$  is a torsion group. The other assertions follow from Lemma 1.1 and the fact that  $D$  and  $R$  are fully invariant in  $G$ .  $\square$

**COROLLARY 3.6.** *Let  $G = D \oplus R$ ; let  $D$  be a non-zero divisible torsion group and  $R = \prod_{p \in P} R_p$  an adjusted cotorsion group. Then  $G$  is  $E$ -dual if and only if  $D$  is  $E$ -dual and  $R$  is finite.*

**PROOF.** By 1.1 and 2.6, we need only prove that if  $G$  is an  $E$ -dual group as in the hypotheses, then  $R$  is finite. To see this, fix a group  $H$  and an anti-isomorphism between  $E(G)$  and  $E(H)$  mapping  $\varphi$  onto  $\varphi'$  for every  $\varphi \in E(G)$ . Let  $\pi_1, \pi_2$  be the projections of  $G$  onto  $D$  and  $R$  respectively, and let  $H = H_1 \oplus H_2$ , where  $H_i = \pi'_i(H)$  ( $1 \leq i \leq 2$ ). Assume first that  $R = R_p$  for some prime  $p$ . Our claim is that  $R$  is finite. Suppose the contrary. Then there exists  $\varphi \in E(G)$  such that  $\varphi \neq 0$ ,  $\varphi(t(G)) = 0$ . Since  $G/t(G)$  is divisible and torsion-free,  $\varphi \in p^\omega E(G)$  ([5] vol. I, p. 182) and, obviously,  $\varphi = \varphi \pi_2$ . Hence  $\varphi'(H)$  is a non-zero subgroup of  $p^\omega H_2$ . On the other hand, by 3.1,  $t(R)$  is a torsion-compact  $p$ -group. Since  $E(R)$  is isomorphic to  $E(t(R))$ , the proof of Lemma 2.2 assures us that  $p^\omega H_2 = 0$ . This contradiction establishes that  $R$  is finite. To complete the proof, it remains to show that the hypothesis that  $G$  is  $E$ -dual always implies that  $R$  is finite. Assume this is not

true. Then, as before, there exists  $\varphi \in E(G)$  such that  $\varphi \neq 0$  and  $\varphi(t(G)) = 0$ . For every prime  $p$ , let  $e_p$  denote the projection of  $G$  onto  $R_p$ . Remark that  $\varphi'(H) \cap t_p(H_2) = 0$ , because  $t_p(H_2) = e'_p(H)$  and  $\varphi'(H) \cap e'_p(H) = 0$  ( $p \in P$ ). Since  $G/t(G)$  is divisible and torsion-free, it follows that  $\varphi \in \bigcap_{p \in P} p^\omega E(G)$ . Therefore  $\varphi'(H)$  must be a torsion-free divisible subgroup of  $H_2$ , and this is clearly impossible. In fact,  $R$  has no subgroup isomorphic to  $\mathbf{Q}$  and the same applies to  $H_2$ . This contradiction proves that  $R$  is finite, and the proof is complete.  $\square$

REMARK 1. Let  $G$  be as in 3.6. Then a necessary and sufficient condition for  $G$  to be  $E$ -dual is that  $D$  and  $R$  are  $E$ -dual with  $R$  a torsion group. In fact, by ([5] Corollary 54.4), reduced cotorsion torsion groups are bounded. The result now follows from 2.6.

REMARK 2. The hypotheses of 3.6 cannot be weakened, because there exist reduced  $E$ -dual groups  $G$  of the form  $G = T \oplus R$ , where  $T$  is a non-zero torsion group and  $R$  is an infinite adjusted cotorsion group. For instance, Lemma 1.2 tells us that the group  $G = \bigoplus_{p \in P} \mathbf{Z}(p) \oplus \prod_{p \in P} \mathbf{Z}(p)$  is  $E$ -dual.

PROPOSITION 3.7. *If  $G$  is a mixed  $E$ -dual group, the following facts hold:*

- (i)  $G/t(G)$  is not necessarily  $E$ -dual.
- (ii)  $t_p(G)/p^\omega t_p(G)$  is  $E$ -dual, for every prime  $p$ .

PROOF. (i) Since  $G = \prod_{p \in P} \mathbf{Z}(p)$  is  $E$ -dual and  $G/t(G)$  is a divisible torsion-free group of rank  $2^{\aleph_0}$ , (i) follows from 3.3.

(ii) Let  $B = \bigoplus_{n \in \mathbf{N}} B_n$  be a basic subgroup of  $t_p(G)$ . Since  $B_n$  is a summand of  $G$ ,  $B_n$  is finite ( $n \in \mathbf{N}$ ). If  $B$  is finite, then the statement clearly holds. Assume  $B$  is not finite. Then there exist suitable  $x_n \in B$  ( $n \in \mathbf{N}$ ) such that  $B = \bigoplus_{n \in \mathbf{N}} \langle x_n \rangle$  and  $o(x_r) \leq o(x_s)$  ( $r, s \in \mathbf{N}; r < s$ ). Fix pairwise orthogonal projections  $\pi_n: G \rightarrow \langle x_n \rangle$  ( $n \in \mathbf{N}$ ) so that if  $\eta: t_p(G) \rightarrow \prod_{n \in \mathbf{N}} \langle x_n \rangle$  is the product map, i.e.  $\eta(x) = (\pi_n(x))_{n \in \mathbf{N}}$  ( $x \in t_p(G)$ ), then  $\text{Ker } \eta = p^\omega t_p(G)$ . It remains to show that  $t\left(\prod_{n \in \mathbf{N}} \langle x_n \rangle\right) \leq \eta(t_p(G))$ . Let  $e_{rs}$  denote the endomorphism of  $G$  uniquely defined by the following conditions:  $e_{rs}(1 - \pi_s) = 0$  and  $e_{rs}(x_s) = p^{\lambda_{rs}} x_r$ , where  $\lambda_{rs} =$

$= \max(0, e(x_r) - e(x_s))$  ( $r, s \in \mathbb{N}$ ). As in the proof of 2.3, the existence of these elements implies  $t\left(\prod_{n \in \mathbb{N}} \langle x_n \rangle\right) \leq \eta(t_p(G))$ .  $\square$

REMARK. Condition (ii) indicates that only very particular torsion groups may be the torsion part of an  $E$ -dual group. We don't know examples of  $E$ -dual groups  $G$  such that, for some prime  $p$ ,  $p^\omega t_p(G)$  is not divisible. However, we can give a sufficient condition in order that  $p^\omega t_p(G)$  is divisible. In fact, if  $G$  and  $H$  have anti-isomorphic endomorphism rings and  $H/t(H)$  is  $p$ -divisible, then  $t_p(G)$  is  $E$ -dual. To see this, assume the anti-isomorphism between  $E(G)$  and  $E(H)$  takes  $\varphi$  to  $\varphi'$  ( $\varphi \in E(G)$ ). Write  $t_p(G) = D \oplus R$ ;  $t_p(H) = D' \oplus R'$  where  $D, D'$  are divisible and  $R, R'$  are reduced. We claim that  $p^\omega R = 0$ . Suppose this does not hold. Then there is an endomorphism  $\varphi$  of  $G$  such that  $0 \neq \varphi(G) \leq p^\omega R[p]$ . Let  $\pi_n$  ( $n \in \mathbb{N}$ ) be as before. An argument similar to that used in 2.2 shows that  $\bigoplus_{n \in \mathbb{N}} \pi'_n(H)$  is a basic subgroup of  $R'$ . Since  $\pi_n \varphi = 0$  ( $n \in \mathbb{N}$ ),  $\varphi'$  is 0 on  $t(H)$  and clearly  $0 \neq \varphi'(H) \leq H[p]$ . But this is impossible, because  $H/t(H)$  is  $p$ -divisible. This contradiction establishes that  $p^\omega R = 0$ . Consequently  $p^\omega t_p(G)$  is divisible.

§ 4. In this section we investigate some properties of torsion-free  $E$ -dual groups. Since Corollary 3.5 gives the structure of an  $E$ -dual group containing  $\mathbb{Q}$ , we can confine ourselves to the reduced case. First we recall some definitions.

If  $G$  is any group, the finite topology of  $E(G)$  has the family of all  $U_X = \{\varphi \in E(G) : \varphi(X) = 0\}$ , with  $X$  a finite subset of  $G$ , as a basis of neighborhoods of 0. It is well known ([5] Theorem 107.1) that  $E(G)$ , with respect to the finite topology, is a complete Hausdorff topological ring. According to ([2], p. 63), reduced torsion-free groups of cardinality  $< 2^{\aleph_0}$  are called control groups. If  $G$  is a group and, for some control group  $C$ , every subgroup of  $G$  of finite rank is isomorphic to a subgroup of  $C$ , then  $G$  is a controlled group. In the following,  $\aleph_i$  denotes the first strongly inaccessible cardinal ([5] vol. II, p. 129).

THEOREM 4.1. *If  $G$  is a controlled group and  $E(G)$  is of cardinality  $< \aleph_i$ , then  $G$  is  $E$ -dual.*

PROOF. It is enough to show that the ring  $A = (E(G))^0$ , equipped with the discrete topology, satisfies the hypotheses of ([2] Theorem 2.2). This clearly holds, if we only show that the group  $E(G)$  is controlled. To this purpose, regard  $E(G)$  as embedded in  $\prod_X E(G)/U_X$ , the product

being extended over all finite subsets  $X$  of  $G$ , and let  $K$  be a subgroup of  $E(G)$  of finite rank. Take linearly independent elements  $\varphi_1, \dots, \varphi_r \in E(G)$  such that  $K \leq \langle \varphi_1, \dots, \varphi_r \rangle_* \leq E(G)$ . Then there exist a finite subset  $X'$  of  $G$  such that the natural projection  $\pi: \prod_X E(G)/U_X \rightarrow E(G)/U_{X'}$  maps  $\varphi_1, \dots, \varphi_r$  onto linearly independent elements. Since  $K \cap \text{Ker } \pi = 0$ ,  $K$  is isomorphic to a subgroup of  $E(G)/U_{X'}$ . The choice of  $K$  assures us that every subgroup of  $E(G)$  of finite rank is isomorphic to a subgroup of  $\bigoplus_X E(G)/U_X$ . Using ([2] Proposition 2.1), we conclude that  $E(G)$  is controlled. This completes the proof.  $\square$

**COROLLARY 4.2.** *If  $G$  is a reduced torsion-free separable group and  $E(G)$  is of cardinality  $< \aleph_i$ , then  $G$  is  $E$ -dual.*

**PROOF.** Let  $C = \bigoplus_{p \in P} \mathbb{Z}_p^{(\mathbb{N})}$ . Since  $C$  is a control group and every subgroup of  $G$  of finite rank may be embedded in  $C$ , the result follows from the previous theorem.  $\square$

**REMARK.** There exists an  $E$ -dual group  $G$  such that  $G^{(\mathbb{N})}$  is  $E$ -dual. In fact, the group  $\mathbb{Z}^{(\mathbb{N})}$  satisfies the hypotheses of 4.1. Observe that, by 2.5 and 3.3, this possibility cannot occur if  $G$  either a torsion or a divisible group.

Comparing 3.1 and 3.3 with 4.1, we see that the behaviour of torsion-free cotorsion groups is completely different from that of torsion-free non cotorsion groups. Also note that, by Corner's theorems, very complicated torsion-free groups have uncomplicated, even commutative, endomorphism rings ([14], p. 180; [15], p. 62). On the other hand, if  $G$  and  $H$  are arbitrary reduced torsion-free groups with anti-isomorphic endomorphism rings, then  $H$  does not generally inherit many properties of  $G$ . For instance, it has been proved ([6] Theorem 1.2) that if  $G = \mathbb{Z}^{(\mathbb{N})}$ , then there is no reduced torsion-free group  $H$  of the same type as  $\mathbb{Z}$  such that  $E(G)$  and  $E(H)$  are anti-isomorphic. More generally, we have the following

**PROPOSITION 4.3.** *There exist a free group  $G$  and a non controlled group  $H$  such that  $E(G)$  and  $E(H)$  are anti-isomorphic.*

**PROOF.** Let  $G = \mathbb{Z}^{(\mathbb{N})}$  and  $A = (E(G))^0$ . We shall show first that  $A$ , endowed with the discrete topology, satisfies the hypotheses of ([3] Theorem 1). Fix a prime  $p$ . Since  $|A| = 2^{\aleph_0}$  and  $p^{\omega}A = 0$ , it suffices to prove that  $J_p$  is linearly disjoint from the group  $A$ , that

is from the group  $E(G)$ , over  $\mathbf{Z}_p$  (i.e. if  $\sum_{i=1}^n \alpha_i \varphi_i = 0$  in  $\widehat{E(G)}$ , the  $p$ -adic completion of  $E(G)$ , with  $\varphi_1, \dots, \varphi_n \in E(G)$ ;  $\alpha_1, \dots, \alpha_n \in J_p$  and linearly independent over  $\mathbf{Z}_p$ , then  $\varphi_1 = \dots = \varphi_n = 0$ ). Assume this is not true. Then we may write  $\sum_{i=1}^n \alpha_i \varphi_i = 0$ , where  $\varphi_i \in E(G)$  ( $1 \leq i \leq n$ ),  $\varphi_1 \neq 0$  and the  $\alpha_i$ 's are as before. Let  $G = \bigoplus_{n \in \mathbf{N}} G_n$ , where  $G_n = \mathbf{Z}$ ,  $x_n = 1 \in G_n$  and  $\pi_n$  is the projection of  $G$  onto  $G_n$  ( $n \in \mathbf{N}$ ). By hypothesis, there exist  $r, s \in \mathbf{N}$  such that  $\pi_r \left( \sum_{i=1}^n \alpha_i \varphi_i(x_s) \right) = 0$  is a linear combination of the  $\alpha_i$ 's with coefficients in  $\mathbf{Z}$  not all equal to 0. This contradiction establishes that  $A$  has the required property. We claim that there exists a non controlled group  $H$  whose endomorphism ring, with the finite topology, is the discrete ring  $A$ . In fact, for every  $a \in A$  we can choose a  $p$ -adic integer  $\alpha(a)$  with the following properties:

(i) The set  $\{\alpha(a) : a \in A\}$  is algebraically independent over  $\mathbf{Z}_p$ .

(ii)  $J_p$  has transcendence degree  $2^{\aleph_0}$  over the subring generated by the  $\alpha(a)$ 's. Let  $H$  be the following pure subgroup of the  $p$ -adic completion  $\widehat{A}$  of  $A$

$$H = \langle A, A\alpha(a)(a \in A) \rangle_* \leq \widehat{A}.$$

Since ([3] Theorem 1) assures that  $E(H)$  is isomorphic to  $A$ , it remains to check that  $H$  is not controlled. To see this, let  $S$  denote the subset of all  $\varphi \in E(G)$  such that  $\pi_1 \varphi \pi_1 = \pi_1$ ;  $\pi_r \varphi \pi_s = 0$  ( $r, s \in \mathbf{N}$ ;  $r \neq s$ ). Now consider the pure subgroups  $S'$  and  $S''$ , where

$$S' = \langle 1, \alpha(\varphi) (\varphi \in S) \rangle_* \leq J_p; \quad S'' = \langle \pi_1, \pi_1 \varphi \alpha(\varphi) (\varphi \in S) \rangle_* \leq \widehat{E(G)}.$$

By ([3] Proposition 1),  $S'$  is not controlled. Since  $S''$  is isomorphic to  $S'$  and  $S'' \leq H$ , we conclude that  $H$  is not controlled.  $\square$

**PROPOSITION 4.4.** *There exists a countable reduced torsion-free group  $G$  such that  $E(G)$  is not anti-isomorphic to the endomorphism ring of a countable reduced torsion-free group.*

**PROOF.** Let  $G = \mathbf{Z}^{(\mathbf{N})}$ . With the same notations of 4.3, let  $e_{rs}$  be the endomorphism of  $G$  defined by  $e_{rs}(x_s) = x_r$ ;  $e_{rs}(1 - \pi_s) = 0$  ( $r, s \in \mathbf{N}$ ). Let  $f: E(G) \rightarrow E(H)$  be any anti-isomorphism. To end the proof, it is enough to show that  $H$  is not countable. Assume the contrary.

Then  $H$  is a countable reduced torsion-free group and  $f$  is continuous with respect to the finite topologies of  $E(G)$  and  $E(H)$ . This is an immediate consequence of ([14] Lemma 4.3), because if  $U$  is a subgroup of  $E(H)$ , then

$$U \text{ open} \Leftrightarrow E(H)/U \cong E(G)/f^{-1}(U) \text{ countable} \\ \text{reduced torsion-free} \Leftrightarrow f^{-1}(U) \text{ open} .$$

Since  $E(G)$  is not discrete, the same applies to  $E(H)$ . Therefore  $E(H)$  has a proper open left ideal  $U$  and  $f^{-1}(U)$  is a proper open right ideal of  $E(G)$ . It is now clear that there exists  $m \in \mathbf{N}$  such that  $\{\varphi \in E(G) : \varphi\pi_r = 0 \ (1 \leq r \leq m)\} \leq V$ , where  $V$  is an open two-sided ideal of  $E(G)$  and  $V \leq f^{-1}(U)$ . Choose  $\varphi \in E(G) \setminus V$  and define  $\varphi', \varphi'' \in E(G)$  as follows:  $\varphi'\pi_r = \varphi\pi_r, \varphi''\pi_r = 0 \ (1 \leq r \leq m); \varphi'\pi_r = 0, \varphi''\pi_r = \varphi\pi_r \ (r > m)$ .

Evidently  $\varphi = \varphi' + \varphi''$  and  $\varphi'' \in V$ . Write  $\varphi' = \sum_{r=1}^m \varphi_r$ , where  $\varphi_r\pi_r = \varphi'\pi_r, p_r(1 - \pi_x) = 0 \ (1 \leq r \leq m)$ . Then there are suitable  $n_{kr} \in \mathbf{Z} \ (k \in \mathbf{N}; 1 \leq r \leq m)$  almost all 0 such that  $\varphi_r = \sum_{k \in \mathbf{N}} n_{kr} e_{kr}$ . Since

$$e_{kr} = e_{k,m+1} e_{m+1,r} \in V \quad (k \in \mathbf{N}; 1 \leq r \leq m) ,$$

we conclude that  $\varphi \in V$ . This contradiction proves that  $H$  is not countable, and the proof is complete.  $\square$

**REMARK 1.** There exists a non commutative topological ring  $A$  such that  $A$  is the endomorphism ring of a countable reduced torsion-free group and the same applies to its opposite ring. In fact, let  $G = \mathbf{Z}^{(\mathbf{N})}$  and, using the notations of 4.4, let  $A$  be the subring of  $E(G)$  consisting of all  $\varphi$  such that  $\pi_r \varphi \pi_s = 0 \ (r, s \in \mathbf{N}; r > s)$ , i.e.  $A$  is isomorphic to the subring of all upper triangular  $\aleph_0 \times \aleph_0$  matrices with entries in  $\mathbf{Z}$  ([5] Theorem 106.1). It is easy to see that  $A$ , with the topology induced by the finite topology of  $E(G)$ , has a family of two-sided ideals as a basis of neighborhoods of 0 and satisfies the hypotheses of ([2] Theorem 1.1).

**REMARK 2.** The direct sum of two reduced torsion-free  $E$ -self-dual groups is not necessarily  $E$ -dual. To prove this, fix a prime  $p$  and let  $G = \mathbf{Z} \oplus J_p$ . We claim that  $G$  is not  $E$ -dual. Otherwise,  $E(G)$  is anti-isomorphic to  $E(H)$  for some group  $H$  of the form  $H = H' \oplus H''$ , where  $E(H') \cong \mathbf{Z}; E(H'') \cong J_p; \text{Hom}(H', H'') = 0$  and

$\text{Hom}(H'', H') \cong J_p$ . Since  $H'$  is reduced and torsion-free, we may assume  $H'' = J_p$ . Consequently  $\text{Hom}(H'', H') = \text{Hom}_{J_p}(H'', H')$ . Choose a non-zero homomorphism  $\varphi: H'' \rightarrow H'$  and regard  $H'$  as a pure subgroup of its  $\mathbb{Z}$ -adic completion  $\hat{H}' = \prod_{q \in P} \hat{H}'_q$ . Since  $\varphi(H'') = J_p \varphi(1) \leq \leq H' \cap \hat{H}'_p$ , there exists  $x \in H' \setminus pH'$  such that  $\varphi(1) = p^n x$  for some  $n \in \mathbb{N}$ . These conditions imply that  $J_p x$  is a pure subgroup, hence a direct summand, of  $H'$ . But this is clearly impossible, because  $E(H')$  is isomorphic to  $\mathbb{Z}$ . This contradiction establishes that  $G'$  is not  $E$ -dual.

The previous example suggests that we determine some properties of all  $E$ -dual groups admitting a free summand.

**PROPOSITION 4.5.** *If  $G = \mathbb{Z} \oplus G'$  is  $E$ -dual, then the following conditions hold:*

- (i)  $G'$  is reduced and torsion-free.
- (ii)  $G'$  is not cotorsion.
- (iii)  $G'$  is not necessarily a controlled group.

**PROOF.** (i) We first prove that  $G'$  is torsion-free. Suppose  $E(G)$  is anti-isomorphic to  $E(H)$ . Then  $H$  has a decomposition  $H = H' \oplus H''$ , where  $E(H') \cong \mathbb{Z}$  and  $\text{Hom}(H'', H') \cong \text{Hom}(\mathbb{Z}, G') \cong G'$ . Since  $H'$  is torsion-free, the same applies to  $G'$ . Using Corollary 3.5, we conclude that  $G'$  is reduced.

(ii) This immediately follows from (i) and Remark 2.

(iii) Fix a prime  $p$ . Let  $G'$  denote a pure subgroup of  $J_p$  with the following properties:  $1 \in G'$ ;  $|G'| = 2^{\aleph_0}$  and the transcendence degree of  $J_p$  over the subring generated by  $G'$  is  $2^{\aleph_0}$ . Let  $G = \mathbb{Z} \oplus G'$ ; then, as in 4.3, one can show that the ring  $(E(G))^0$ , with the discrete topology, satisfies the hypotheses of ([3] Theorem 1). Thus  $G$  is  $E$ -dual and, by ([3] Proposition 1),  $G'$  is not controlled.  $\square$

**REMARK.** More generally, if  $R$  is a rational group,  $p$  is a prime and  $G = R \oplus G'$  is  $E$ -dual, then  $pR \neq R$  implies  $t_p(G') = 0$ . In fact, we can find a group  $H = H' \oplus H''$  such that  $E(G)$  is anti-isomorphic to  $E(H)$ ,  $E(H') \cong E(R)$  and  $\text{Hom}(H'', H') \cong \text{Hom}(R, G')$ . Since  $H'$  is torsion-free and  $R \neq pR$ ,  $G'$  has no element of order  $p$ . Finally note that if  $R = \mathbb{Q}$ , then the structure of  $G'$  is completely determined by 3.5.

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