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Extremal Units in an Archimedean Riesz Space.

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Let $A$ be an archimedean Riesz space (= vector lattice) with distinguished weak unit $e_A$, and for any $e \in A$, let $X(e)$ be the compact space of $e$-maximal ideals. A natural map $\sigma^e: X(e) \to X(e_A)$ is a continuous extension of the inclusion $X(e) \cap X(e_A) \hookrightarrow X(e_A)$; natural $\sigma_e: X(e_A) \to X(e)$ is defined dually, only for weak units $e$.

This paper concerns when natural $\sigma^e$ (or $\sigma_e$) exists, and those $(A, e_A)$ such that for every $e \in A$, $\sigma^e$ (or $\sigma_e$) exists. We then call $e_A$ $X$-strong (or $X$-costrong). These conditions are treated in terms of the Yosida representation $\hat{A}$ in $D(X(e_A))$.

Some of the results: (2.5 and 3.1) $\sigma^e$ exists iff $p \neq q$ in $X(e_A)$ implies $a \in A$ with $a \in \mathcal{O}_p$ and $e - ae \mathcal{O}_q$. (§ 6) $\sigma_e$ exists iff whenever $U_1$ and $U_2$ are $\hat{A}$-cozeros in $X(e_A)$ for which there is $\hat{a} \in \hat{A}$ which is $\hat{e}$ on $U_1$ and $0$ on $U_2$, then $\overline{U_1} \cap \overline{U_2} = \emptyset$. (§ 4) $e_A$ is $X$-strong iff each prime ideal of $A$ contains a unique $\mathcal{O}_p$ ($p \in X(e_A)$ iff to each open cover of $X(e_A)$ are subordinate finite $\hat{A}$-partitions of every $e \in A$. (§ 5) $e_A$ will be $X$-strong if $e_A$ is a strong unit, or if $A$ is an $l$-algebra with identity $e_A$, or if $A$ has the principal projection property. $e_A$ will be $X$-costrong if $A$ is Cantor complete or has the principal projection property.

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1. Representation.

We sketch those aspects of the Yoshida representation [Y] which are needed for the sequel. More detail appears in [HR]; see also [LZ].

Let $A$ be an archimedean Riesz space (= vector lattice over the reals $R$), and let $0 < e \in A$. A Riesz ideal $M$ which is maximal with respect to the property of not containing $e$ will be called $e$-maximal, and the set of these will be denoted $X(e)$. Any such $M$ is prime, hence $A/M$ is totally ordered (see also I 3, here). Regarding the following, see also 4.5, below.

1.1. Let $P$ be a prime ideal in $A$, let $A \rightarrow A/P$ denote the quotient, and let $e \in A^+$. These are equivalent: $P \in X(e)$; there is no nonzero $q(e)$-infinitesimal in $A/P$; the principal ideal $I(q(e))$ is the smallest nonzero ideal in $A/P$.

When this occurs, $I(q(e)) = \{ tq(e) | t \in R \}$.

Consider the extended reals $\bar{R} = R \cup \{ \pm \infty \}$, with the obvious order and topology and partly defined addition and scaler multiplication extending these operations from $R$.

1.2. (a) Let $M \in X(e)$, with $A \rightarrow A/M$ the quotient. Define $\gamma^*_M : A \rightarrow \bar{R}$ by:

\[
\gamma^*_M(a) = t \quad \text{if } q(a) = tq(e) \quad (t \in R);
\]

\[
\gamma^*_M(a) = + \infty \quad \text{if } 0 < q(a) \notin I(q(e)), \text{ and}
\]

\[
= - \infty \quad \text{if } 0 > q(a) \notin I(q(e)).
\]

(b) Define $\gamma^* : A \rightarrow \bar{R}^{X(e)}$ by:

\[
\gamma^*(a)(M) = \gamma^*_M(a).
\]

Now, when $X$ is a topological space, let $D(X)$ denote those continuous $f : X \rightarrow \bar{R}$ with $\mathcal{R}(f)$ dense, where $\mathcal{R}(f) = f^{-1}(\bar{R})$. $D(X)$ is a lattice admitting scaler multiplication. For $f, g, h \in D(X)$, $f + g = h$ means that $f(x) + g(x) = h(x)$ for $x \in \mathcal{R}(f) \cap \mathcal{R}(g)$. A « Riesz space in $D(X)$ » is a sublattice $A$ with $ra \in A$ when $a \in A$ and $r \in R$, and « closed under addition ».

Let $0 < e \in A$. 
1.3. **Theorem.** (a) In the hull-kernel topology, \(X(e)\) is a nonvoid compact Hausdorff space.

(b) \(\gamma^*\) is a homomorphism of \(A\) onto a Riesz space in \(D(X(e))\), with kernel \(\gamma^* = e^\perp\) and with \(\gamma^*(e)\) the constant function 1. So \(\gamma^*\) is an isomorphism iff \(e^\perp = \{0\}\), i.e., \(e\) is a weak unit.

(c) If \(K_1\) and \(K_2\) are disjoint closed sets in \(X(e)\), then there is \(a \in A\) with \(0 < a < e\), hence \(0 < \gamma^*(a) < 1\), and \(\gamma^*(a) = 1\) on \(K_1\) and 0 on \(K_2\).

(d) Let \(e\) be a weak unit and let \(\gamma: A \to D(X)\) be an isomorphism, with \(X\) compact, \(\gamma(e) = 1\), and with \(\gamma(A)\) separating the points of \(X\). Then there is a homeomorphism \(h: X(e) \to X\) with \(\gamma(a) = \gamma^*(a) \circ h\) for each \(a \in A\).

1.4. **Notation.** Throughout the paper, we use the following abbreviations: \(A \in \mathcal{L}\) means that \(A\) has a distinguished positive weak unit \(e_A\). For \(A \in \mathcal{L}\) the isomorphic representation \(\gamma^* : A \to D(X(e_A))\) is denoted \(\mathcal{A}\). For another \(e \in A^+\), we always write \(\gamma^*\).

2. **Natural mappings: topology and functions.**

We begin the comparison of representations and maximal ideal spaces. Throughout the section, \(A \in \mathcal{L}\) (which presumes \(e_A\)), and \(0 < e \in A^+\). We state the results and sketch the development, then proceed to the proofs.

2.1. **Definition.** A natural mapping \(\sigma^* : X(e) \to X(e_A)\) is a continuous extension of the inclusion \(X(e) \cap X(e_A) \hookrightarrow X(e_A)\). (Such a mapping is unique.)

2.2. **Main Lemma.** Let \(Y_* = \text{coz } \ell \cap K(e) \subset X(e_A)\), and for \(p \in Y_*\), let \(\tau(p) = M_p = \{a|\hat{a}(p) = 0\}\). Then

(a) Each \(M_p \in X(e)\).

(b) \(\tau\) is a homeomorphism of \(Y_*\) onto \(\text{coz } \gamma^*(e_A) \cap K(\gamma^*(e_A))\).

(c) \(\gamma^*(a) \cdot \tau = (1/e) \hat{a}|Y_*\).

2.3. **Corollary.** A natural map \(\sigma^*\) is exactly a continuous function \(\sigma^* : X(e) \to X(e_A)\) with \(\sigma^* \circ \tau\) the identity on \(Y_*\).
2.4. **THEOREM.** $\sigma^e$ exists iff whenever $K_1$ and $K_2$ are disjoint closed sets in $X(e_a)$, then there is $a \in A$ with $\hat{a} = \hat{b}$ on $K_1$, and $\hat{a} = 0$ on $K_2$.

The condition in 2.4 is reminiscent of the theory of normal topological spaces. The analogy goes quite far:

2.5. **PROPOSITION.** The following are equivalent (to $\sigma^e$ exists):

(a) If $K_1$ and $K_2$ are disjoint closed sets in $X(e_a)$, then there is $a \in A$ with $\hat{a} = \hat{b}$ on $K_1$ and $\hat{a} = 0$ on $K_2$.

(b) If $G_1, G_2 \in \text{coz } A$ and $\overline{G_1} \cap G_2 = \emptyset$, then there is $a \in A$ with $\hat{a} = \hat{b}$ on $G_1$ and $\hat{a} = 0$ on $G_2$.

(c) If $p \neq q$ in $X(e_a)$, then there is $a \in A$ with $\hat{a} = \hat{b}$ on a neighborhood of $p$ and $\hat{a} = 0$ on a neighborhood of $q$.

(d) If $G$ an $H$ are open in $X(e_a)$ with $\overline{G} \subset H$, then there is $a \in A$ with $\hat{a} = \hat{b}$ on $G$ and $a = 0$ off $H$.

In each case, we may take $0 < a < e$.

2.6. **DEFINITION.** A partition of $e$ is a family $\psi \subset A$ such that $\bigvee \psi = e$ (supremum in $A$).

Let $\mathcal{G}$ be an open cover of $X(e_a)$. A family $\psi \subset A$ is subordinate to $\mathcal{G}$ if $\text{coz } \psi$ refines $\mathcal{G}$. (I.e., for each $f \in \psi$, $\text{coz } f$ is contained in some $g \in \mathcal{G}$; we do not assume $\mathcal{G}$ to be a cover).

2.7. **PROPOSITION.** $\sigma^e$ exists iff to each (finite) open cover of $X(e_a)$ is subordinate a finite partition of $e$.

(2.6 and 2.7 are suggested by the proof of 5.5 (a)).

We turn to the proofs.

**Proof of 2.2.** (a) Obviously, $M_\pi$ is prime. To show $e$-maximality consider the quotient $q: A \to A/M_\pi$. Using §1, we show that $Z(q(e)) = (0)$.

Suppose $a \in A^+$, and $tq(a) < q(e)$. Then $0 < q(e - ta)$, and by definition of the order in $A/M_\pi$, there is $m \in M_\pi$ with $m < e - ta$.

Then $0 = \tilde{m}(p) < \ell(p) - t\hat{a}(p)$. If this holds for every $t \in R$, then $\tilde{m}(p) = 0$ since $\ell(p) \in R$. Thus, $a \in M_\pi$ and $q(a) = 0$ as desired.

(b) We use the facts that $\text{coz } \hat{A}$ is an open base in $X(e_a)$ and $\text{coz } \gamma^*(A)$ is an base in $X(e)$: $\tau$ is continuous from the equations $\tau(\text{coz } \gamma^*(a)) = \text{coz } \hat{a} \cap Y_\pi (a \in A)$. $\tau$ has dense image and is open onto its range from the equations $\tau(\text{coz } \hat{a} \cap Y_\pi) = \text{coz } \gamma^*(a) \tau(Y_\pi) (a \in A)$. 

\(\tau\) is one-to-one (and thus a homeomorphism) because \(\hat{A}\) separates the points in \(X(e_A)\).

We finish the proof of (b) below.

(c) Let \(a \in A\), and \(p \in Y_e\). Consider the definition of \(\gamma^x(a)(\tau(p)) = \gamma^x(a)(M_p)\) from \(A \xrightarrow{\gamma} A/M_p \xrightarrow{\tau} \hat{R}\) described in \(\S\ 1\).

Suppose first that \(\partial(p) = t \in R\). Then \(\partial(p) \partial(p) = t \partial(p)\), or \(a - (t/\partial(p)) \in M_p\): Thus,

\[0 = q \left(a - \frac{t}{\partial(p)} e\right),\]

whence

\[0 = \gamma^x(a)(M_p) - \frac{t}{\partial(p)} \gamma^x(e)(M_p), \quad \text{or} \quad \gamma^x(a)(M_p) = \frac{1}{\partial(p)} \partial(p),\]

as desired.

Thus the equation in (c) holds on the dense set \(R(\partial) \cap Y_e\), and by continuity, it holds on \(Y_e\).

(b) continued. \(\tau(Y_e) \subseteq \coz(\gamma^x(e_A)) \cap R(\gamma^x(e_A))\) follows from the equation in (c), with \(a = e_A\). For the reverse inclusions let \(x \in \coz(\gamma^x(e_A)) \cap R(\gamma^x(e_A))\) and consider \(M = \{a \mid \gamma^x(a)(x) = 0\}\). This is a prime ideal \(e_A \notin M\) because \(x \in \coz(\gamma^x(e_A))\) and an argument as in (a) shows \(M\) is \(e_A\)-maximal, because \(x \in R(\gamma^x(e_A))\). Thus \(M = M_p\) for unique \(p \in X(e_A)\), by \(\S\ 1\). It follows that \(x = \tau(p)\).

Proof of 2.1 (uniqueness) and 2.3. By 2.2, we may identify \(Y_e\) and \(\tau(Y_e)\) as subspaces of, say, the prime ideal space with the hull kernel topology and this space is \(X(e) \cap X(e_A)\), which is dense in \(X(e)\) by 2.2 (b).

Thus \(\sigma^e\) is unique when it exists. Upon the identification, the inclusion \(X(e) \cap X(e_A) \hookrightarrow x(e_A)\) restricted to \(\tau(Y_e)\) is \(\tau^{-1}\). Thus 2.3.

2.8. Lemma ([E], p. 110). Let \(Y\) be dense in \(X'\) and let \(i: Y \to X\) be continuous, with \(X\) compact. There is a continuous extension \(\sigma: X' \to X\) iff whenever \(K_1\) and \(K_2\) are disjoint closed sets in \(X\), then \(i^{-1}(K_1)\) and \(i^{-1}(K_2)\) have disjoint closures in \(X'\).

2.9. Lemma. Let \(A \in L, 0 < e \in A^+\), and let \(K_1, K_2 \subseteq X(e_A)\). These are equivalent:

(a) The closures in \(X(e)\) of \(\tau(K_1)\) and \(\tau(K_2)\) are disjoint.
(b) There is \(a \in A\) with \(\gamma^x(a) = 1\) on \(\tau(K_1)\) and \(\gamma^x(a) = 0\) on \(\tau(K_2)\).
(c) There is \(a \in A\) with \(\partial = \partial\) on \(K_1\) and \(\partial = 0\) on \(K_2\).
PROOF. Any Yosida representation separates closed sets (§ 1), so (a) \Leftrightarrow (b). The equation (c) in 2.1 shows (b) \Leftrightarrow (e).

PROOF OF 2.4. By 2.3, 2.8, and 2.9 (using \( \tau^{-1} \) as the \( \iota \) of 2.8).

PROOF OF (2.5). (a) \Rightarrow (b). Obvious.

(b) \Rightarrow (c). Choose coz \( \mathcal{A} \)-neighborhoods with disjoint closures.

(c) \Rightarrow (a). Let \( K_1 \) and \( K_2 \) be given. For each \( p \in K_1 \) and \( q \in K_2 \), choose \( a^p_\iota \in \mathcal{A} \) with

\[
a^p_\iota = \begin{cases} 
\delta & \text{on a neighborhood } G^p_\iota \text{ of } p, \\
0 & \text{on a neighborhood } H^p_\iota \text{ of } q.
\end{cases}
\]

Fix \( p \). \( \{H^p_\iota | q \in K_2\} \) covers \( K_2 \), and so does a finite subset \( \{H^p_\iota\} \). Let \( a^p = \bigcap_i a^p_\iota \). Then

\[
a^p = \begin{cases} 
\delta & \text{on } G^p = \bigcap_i G^p_\iota, \\
0 & \text{on } K_2.
\end{cases}
\]

Then \( \{G^p | p \in K_1\} \) covers \( K_1 \), and so does a finite subset \( \{G^p\} \). Let \( a = \bigvee_i a^p_\iota \), so that

\[
a = \begin{cases} 
\delta & \text{on } \bigcup_i G^p \supset K_1, \\
0 & \text{on } K_2.
\end{cases}
\]

(d) \Leftrightarrow (a) is routine.

Finally, in any of the conditions we may replace \( a \) by \( a_1 = |a| \wedge e \). Then \( a_1 \) also works and \( 0 < a_1 < e \).

PROOF OF 2.7. Suppose \( \sigma^e \) exists, let \( \mathcal{G} \) be an open cover and \( \{G_i\} \) a finite subcover. Let \( \{W_i\} \) be a «shrinkage» of \( \{G_i\} \): an open cover with \( \overline{W_i} \supset G_i \) for each \( i \). (Such exists by [E], p. 266). For each \( i \), choose \( a_i \in \mathcal{A} \) with \( a_i = \delta \) on \( W_i \) and \( a_i = 0 \) off of \( G_i \) and \( 0 < a_i < e \) (by 2.5 (d)). Then \( \{a_i\} \) is a finite partition of \( a \), subordinate to \( \{G_i\} \) and to \( \mathcal{G} \). To show \( \bigvee_i a_i = e; \ \delta \geq \bigvee_i a_i \) because \( \delta \geq e \) each \( a_i \). And, if \( eX(eA), \) then \( x \in \text{some } W_i \supset G_i, \) and \( \delta(x) = a_i(x), \) hence \( e(x) < \bigvee_i a_i(x) \).

Conversely, let \( G, H \) be open with \( \overline{G} \subset H \). Then \( \{X - \overline{G}, H\} \) is an open cover, and we assume there is a finite partition \( \psi \) of \( e \) subordinate. Let \( a = \bigvee \{f \in \psi | \text{coz } f \subset H\} \). Then \( a \) satisfies 2.5 (d).
3. Natural mappings: ideals.

We find some what more algebraic conditions that $\sigma^*$ exists in terms of certain ideals of $A$. We state the results, then proceed to the proofs.

3.1. Let $A \in \mathcal{F}$, let $p \in X(e_A)$, and define

$$O_p = \{ a \in A | \forall \delta = 0 \text{ on a neighborhood of } p \}.$$ 

Then $O_p$ is an ideal, $e_A \notin O_p$, and $M_p$ is the unique $e_A$-maximal ideal containing $O_p$.

3.2. Theorem. Let $A \in \mathcal{F}$, and let $0 < e \in A^+$.

(a) There is natural $\sigma^*: X(e) \to X(e_A)$ iff each $M \in X(e)$ contains a unique $O_p$ ($p \in X(e_A)$).

(b) Assuming this, $\sigma^*(M) = p$ iff $M \supset O_p$.

The point in (a) is the uniqueness of $O_p$ as the following shows.

3.3. Proposition. Let $A \in \mathcal{F}$. Each prime ideal of $A$ contains on $O_p$.

3.2 (b) says that the condition in (a) provides a canonical description of $\sigma^*$. Another such comes from the following, applied to $M \in X(e)$.

(3.4 was obtained jointly with Giuseppe De Marco).

3.4. Theorem. Let $A \in \mathcal{F}$, let $p \in X(e_A)$, and let $P$ be a prime ideal of $A$. Then $P \supset O_p$ iff $P$ is comparable with $M_p$.

3.5. Corollary. These are equivalent.

(a) $\sigma^*$ exists.

(b) Each prime $P$ with $e \notin P$ contains unique $O_p$ (or is comparable with unique $M_p$) for $p \in X(e_A)$.

(c) Each $M \in X(e)$ is comparable with unique $M' \in X(e_A)$.

(d) Each $M \in X(e)$ with $e_A \in M$ contains unique $M' \in X(e_A)$;

3.4 and 3.5 can be used to derive conditions on mappings of prime ideal spaces. We postpone such a discussion to a later paper, as it would curry us too far afield.

We turn to the proofs.
PROOF OF 3.1. $O_p \subseteq M_a \Rightarrow p = q$.

3.6. Let $p \in X(e_A)$. Define

$$X^p(e) = \{ M \in X(e) | M \supseteq O_p \}.$$ 

Then, clearly,

$$p \in Y_e \Rightarrow e \notin O_p \iff X^p(e) \neq \emptyset.$$ 

3.7. LEMMA. Let $p, q \in Y_e$:

(a) $X^p(e) = \cap \{ \tau(G) | G \text{ a neighborhood of } p \}$.

(b) $X^p(e) \cap X^q(e) = \emptyset$ iff there is $a \in O_p$ with $e - a \in O_q$.

PROOF. Recall from 2.1 that

$$\gamma^r(a) \tau = \frac{1}{e} \delta | Y_e.$$ 

We shall use this several times.

(a) Let $M \in X(e)$, and let $G$ always be a neighborhood of $p$.

If $M \in X^p(e) - \tau(G)$, then there is $a \in A$ with $0 < a \leq e$, $\gamma^r(a) = 0$ on $\tau(G)$ and $\gamma^r(a)(M) = 1$. \((*)\) shows that $\delta = 0$ on $G \cap Y_e$. Since $0 < a \leq e$, we have $\delta = 0$ on $G$. Thus, $a \in O_p \subseteq M$, contradiction.

Suppose $M \supseteq O_p$, so there is a $\epsilon O_p - M$. Then there is $G$ with $\delta = 0$ on $G$, and \((*)\) shows that $\gamma^r(a) = 0$ on $\tau(G \cap Y_e)$, hence on $\tau(G)$. Since $a \notin M$, $\gamma^r(a)(M) \neq 0$ and $M \notin \tau(G)$.

(b) Given such $a$, there are neighborhoods $G, H$ of $p, q$ with $\delta = 0$ on $G$, $\delta = \epsilon$ on $H$. Using \((*)\) as before, it follows that $\gamma^r(a) = 0$ on $\tau(G)$ and $\gamma^r(a) = 0$ on $\tau(H)$. So $\tau(G)$ and $\tau(H)$ are disjoint, hence so are $X^p(e)$ and $X^q(e)$, by (a).

Let $X^p(e) \cap X^q(e) = \emptyset$. We claim there are neighborhoods $G, H$ of $p, q$ respectively, with $\tau(G) \cap \tau(H) = \emptyset$. Then choose $a \in A$ with $0 < a < e$, $\gamma^r(a) = 0$ on $\tau(G)$ and $\gamma^q(a) = 1$ on $\tau(H)$. Using \((*)\) as usual, we get $a \in O_p$, $e - a \in O_q$. To obtain such $G, H$: If for all such $G, H$, $\tau(G) \cap \tau(H) \neq \emptyset$, then $J = \{ \tau(G) \cap \tau(H) | G, H \}$ has the finite intersection property and there is $M \in \cap J$ by compactness. Such $M \in X^p(e) \cap X^q(e)$ by (a), a contradiction.

This completes the proof of 3.3.
proof of 3.2. (a) follows immediately from 3.7 and 2.6.

(b) Let \( \sigma^* (M) = p \). If \( G \) is a neighborhood of \( p \), then \( M \in (\sigma^*)^{-1}(G) \).
Now, \( (\sigma^*)^{-1}(G) \cap \mathcal{Y}_\bullet = \tau (G) \), and this set is dense in \( (\sigma^*)^{-1}(G) \). Thus \( M \in \tau (G) \). Since this is true for every \( G \), \( M \in X^p (e) \) by 3.7.

If \( M \in X^p (e) \), let \( \sigma (M) = q \). The preceding shows that \( M \in X^p (e) \).
By 3.2 (a), \( p = q \).

3.8. Lemma [LZ]. Let \( A \) be a Riesz space and \( P \) an ideal. These are equivalent.

(a) \( a \wedge b \in P \Rightarrow a \in P \) or \( b \in P \) (\( P \) is prime).

(b) \( |a| \wedge |b| = 0 \Rightarrow a \in P \) or \( b \in P \):

(c) \( A / P \) is totally ordered.

(d) The set of ideals containing \( P \) is totally ordered by set-inclusion.

Proof of 3.3. Suppose \( O_q \not\subseteq P \) for each \( q \). Then, for each \( q \), there is \( a_q \not\in P \) with \( a_q = 0 \) on a neighborhood \( G_q \) of \( q \). From \( \{ G_q | q \in X(e_A) \} \), we extract the finite subcover \( \{ G_q \} \). Then \( \bigwedge |a_q| = 0 \in P \), and by 3.8, \( P \) is not prime.

The following interesting lemma was contributed by Giuseppe De Marco, considerably simplifying our proofs and essentially extending part of 3.5 to 3.4.

3.9. Lemma (De Marco). Let \( q \in X(e_A) \). If \( P \) is a prime ideal of \( A \) with \( O_q \subseteq P \), then there is a prime ideal \( Q \) with \( Q \subseteq P \) and \( Q \subseteq M_q \).

Proof. First, let \( S \) be any subset of positive elements (of any Riesz space) such that \( 0 \not\in S \) and \( u, v \in S \Rightarrow u \wedge v \in S \). Then (with an argument by Zorn's lemma), there is an ideal \( Q \) which is maximal with respect to the property \( Q \cap S = 0 \). And \( Q \) is prime: if \( u \wedge v \in Q \), then one of \( u, v \) is not in \( S \); say \( u \not\in S \). Then \( u \in Q \), for if not, the ideal generated by \( Q \) and \( u \) still misses \( S \) and contradicts maximality of \( Q \).

Now let \( P \) be prime, \( O_q \subseteq P \). Let \( S_1 = (A - M_q)^+ \), \( S_2 = (A - P)^+ \). These are « meet-closed » because \( M_q \) and \( P \) are prime ideals. Then \( S = S_1 \cup S_2 \cup \{ u \wedge v | u \in S_1, v \in S_2 \} \) is meet-closed too. Also \( 0 \not\in S \): Certainly \( 0 \not\in S_1 \cup S_2 \). Suppose \( 0 = u \wedge v \) for \( u \in S_1 \). Then \( 0 = \hat{u} \wedge \hat{v} \) (identically in \( D(X(e_A)) \)). Since \( u \not\in M_q \), \( \hat{u}(q) \neq 0 \), and it follows that \( \hat{v} \) is \( 0 \) on a neighborhood of \( q \), i.e., \( v \in O_q \), so \( v \not\in S_2 \).

Applying the first paragraph to this \( S \) products the desired prime \( Q \).

Proof of 3.4. Let \( P \supset O_q \). By 3.9, choose prime \( Q \) with \( Q \subseteq P, M_q \).
By 3.8 then, \( P \) and \( M_q \) are comparable.
Conversely, let $P$ and $M_p$ be comparable. If $P \supseteq M_p$, certainly $P \subseteq O_p$. When $P \subseteq M_p$, choose $q$ with $P \subseteq O_q$ by 3.3. Then $O_q \subseteq M_p$, and $q = p$ follows.

3.10. For any subset $M$ of $A$: $M$ is comparable with $M_p$ iff either $e_A \notin M$ and $M \subseteq M_p$, or $e_A \in M$ and $M \supseteq M_p$ properly.

**Proof of 3.5.** Each $M \in X(e)$ is prime, of course: (a) $\iff$ (e) by 3.2 and 3.4. (c) $\iff$ (d) by 3.10. The two parts of (b) are equivalent by 3.4. (b) $\implies$ (e), clearly.

(c) $\implies$ (b). Let $e \notin P$. By 3.3, $P \supseteq$ some $O_p$. Suppose also that $O_q \subseteq P$. Now, $P$ is contained in unique $M \in X(e)$ (by Zorn’s lemma and 3.8). So $O_p, O_q \subseteq M$. By (e) (and 3.4 and 3.2), $p = q$.

4. **$X$-strong units.**

This section is essentially a summary of conditions on $A$ and $e_A$ such that $\sigma^*$'s always exists.

4.1. **Definition.** Let $A \in \mathfrak{A}$. If for each $e \in A^+$, natural $\sigma^*: X(e) \to X(e_A)$ exists, we call $e_A$ an $X$-strong unit.

We are not convinced that the terminology is the best. The motivation is that such an $e_A$ behaves like a strong unit with respect to the spaces $X(e)$:

4.2. **Proposition.** (a) If $te_A \geq e$ for some $t \in R^+$, then $\sigma^*$ exists.

(b) A strong unit is $X$-strong.

**Proof.** (a) Given $G \subset H$, choose $u \in A$ with $0 < \hat{u} < 1, \hat{u} = 1$ on $G$ and $\hat{u} = 0$ off $H$ (from §1). Then $a = tu \wedge e$ (when $te_A \geq e$) satisfies $\hat{a} = \hat{e}$ on $G$, $\hat{a} = 0$ off $H$.

(b) follows from (a).

4.3. **Proposition.** These conditions on $A \in \mathfrak{A}$ are equivalent.

(a) $e_A$ is $X$-strong.

(b) $\sigma^*$ exists $\forall$ weak unit $e \in A^+$.

(c) $\sigma^*$ exists $\forall e \geq e_A$.

(d) $\forall e \geq e_A$, the natural map $\sigma^*: X(e_A) \to X(e)$ (existing by 4.2) is a homeomorphism.
PROOF. \((a) \Rightarrow (b) \Rightarrow (c)\) are clear.

\((c) \Rightarrow (d)\). Assuming \((c)\), \(\sigma^\alpha \circ \sigma^\delta A\) and \(\sigma^\delta A \circ \sigma^\varepsilon\) are identities on dense sets, hence identities because the spaces are Hausdorff. So each is a homeomorphism.

\((d) \Rightarrow (e)\). \(\sigma^\varepsilon = (\sigma^\delta A)^{-1}\).

\((e) \Rightarrow (a)\). Let \(0 < e \in A^+\), and let \(G \subset H\). Since \(e_1 = e \vee e_\delta \geq e_\delta\), there is \(\sigma^{\alpha_i}\) and hence there is \(a_i \in A\) with \(a_i = \delta_i\) on \(G\) and \(a_i = 0\) off \(H\). Since \(e_i \geq e\), we have \(e_i \wedge e = e\) and \(a_i \wedge e\) works.

4.3 will be useful later. The following just restates part of § 3.

4.4. THEOREM. These conditions on \(A \in \mathcal{C}\) are equivalent.

\(a)\) \(e_\delta A\) is \(X\)-strong.

\((b)\) Whenever \(e \in A\) and \(G_1, G_2 \subset X(e_\delta A)\) (which may be assumed arbitrary, open, or in \(\text{coz} \hat{A}\), with \(G_1 \cap G_2 = 0\), then there is \(a \in A\) with \(a = \delta\) on \(G_1\) and \(a = 0\) on \(G_2\).

\((c)\) Whenever \(e \in A\) and \(G, H\) are open (or in \(\text{coz} \hat{A}\)) with \(G \subset H\), then there is \(a \in A\) with \(a = \delta\) on \(G\) and \(a = 0\) off \(H\).

\((d)\) Whenever \(a \in A\) and \(G\) is an open cover of \(X(e_\delta A)\), then there is a finite partition of \(e\) subordinate to \(G\).

\((e)\) Whenever \(e \in A\) and \(p \neq q\) in \(X(e_\delta A)\), then there is \(a \in A\) with \(a \in O_p\) and \(e - a \in O_q\).

Finally, there are the more algebraic conditions from § 3. For a better statement of the results, we insert a preliminary

4.5. LEMMA (cf. 1.1). Let \(A\) be a Riesz space and \(M\) an ideal. These are equivalent.

\(a)\) There is \(e \in A\) with \(M \in X(e)\).

\((b)\) \(A/M\) is totally ordered, and there is \(x \in A/M\) such that \(0\) is the only \(x\)-infinitesimal.

\((c)\) \(A/M\) is a subdirectly irreducible Riesz space.

\((d)\) In \(A/M\) there is a smallest nonzero ideal.

\((e)\) If \(\mathcal{G}\) is a family of ideals in \(A\) each properly containing \(M\), then \(\mathcal{G}\) properly contains \(M\).

We call such an \(M\) completely meet-irreducible (emm).
PROOF (sketch). $(a) \iff (b)$. Use $x = e + M$ (and 3.8; see §1).

$(b) \implies (d)$. The principal ideal generated by $x$.

$(c) \iff (e)$. See the treatment in [B] of subdirectly irreducible
abstract algebras.

$(d) \iff (e)$. From the correspondence between ideals in $A/M$
and ideals in $A$ containing $M$.

From 4.5 and § 3, we have immediately.

4.6. THEOREM. These conditions on $A$ are equivalent.

$(a)$ $e_A$ is $X$-strong.

$(b)$ Each proper prime ideal contains unique $O_p$ (or, is comparable
with unique $M_p$) for $p \in X(e_A)$.

$(c)$ Each cmm ideal contains unique $O_p$ (or, is comparable with
unique $M_p$) for $p \in X(e_A)$.

5. $X$-strong units versus other properties.

We recall some definitions and relevant facts:

5.1. Let $A \in \mathcal{L}$. $A$ is called a $\Phi$-algebra [HJ] if $e_A$
is the identity
for an $f$-ring multiplication on $A$. It is shown in [HR] that when $A$
is a $\Phi$-algebra, the Riesz isomorphism $A \rightarrow \hat{A} \subseteq D(X(e_A))$
preserves the multiplication.

Let $A \in \mathcal{L}$. $A$ is called convex [AH] if $\hat{A}$ is a convex subset of
$D(X(e_A))$, that is, if $f \in D(X(e_A))$ and $|f| < \hat{a}$ for some $a \in A$
imples $f \in A$.

If $A$ is convex and $a \in A$, then $\mathcal{R}(\hat{a})$ is $C^*$-embedded in $X(e_A)$ and $A$
is $e_A$-uniformly complete, whence [HR] $\hat{A}^* = C(X(e_A))$.

Let $A$ be a Riesz space. $A$ has the principal projection property,
or $ppp$, if for each $a \in A$, $A = a^+ \oplus a^-$. Then, given $f \in A$, $f =
= p_a f + b$ with $p_a f \in a^+$ and $b \in a^-$, uniquely. See Chapt. 4 of [LZ].
Such $A$ is archimedean. For $A \in \mathcal{L}$, if follows easily from 24.9 of [LZ]
that $A$ has the $ppp$ iff for each $a \in A$, $\text{coz } \hat{a}$ is open. Then, $(p_a f)^\wedge = \hat{f}$
on $\text{coz } \hat{a}$ and 0 off $\text{coz } \hat{a}$.

5.2. THEOREM. Let $A \in \mathcal{L}$. Any of the following imply that $e_A$
is $X$-strong: $A$ is a $\Phi$-algebra; $A$ is convex; $e_A$ is a strong unit;
$A$ has the $ppp$. 
PROOF. Let \( a \in A^+ \), and using 2.6, let \( G, H \) be open with \( \overline{G} \subset H \).
Choose \( u \in A \) with \( \hat{u} = 1 \) on \( G \), 0 off \( H \), and \( 0 < u \leq e_A \).

If \( A \) is a \( \Phi \)-algebra, then \( ua \in A \), and \( (ua)^\Lambda = \hat{u}\hat{a} \) is the desired function.

If \( A \) is convex, \( \hat{u}\hat{a} \) is continuous on \( \Re(\hat{a}) \), hence extends to \( f \in D(X(e_A)) \). Clearly, \( 0 < f < \hat{a} \), so by convexity \( f \in \hat{A} \). And \( f \) is the desired function.

If \( e_A \) is strong, then \( a \leq te_A \) for some real \( t \). Then \( tu\hat{a} \setminus \hat{a} \) is the desired function.

If \( A \) has the ppp, then we resort to assuming that \( G \in \text{coz} \ A \) (per 4.4 (c)). Thus \( \overline{G} \) is open. Then the function \( f = \hat{a} \) on \( G \) and \( f = 0 \) off \( G \) is in \( \hat{A} \) (per 5.1; \( f \) is a certain \( (p, a)^\Lambda \)), and serves the purpose.

REMARK. If \( A \) is either a \( \Phi \)-algebra or convex, then \( \hat{A} \cdot \hat{A} \subseteq \hat{A} \) (see \([AH]\)) and as the proof above shows, this property implies that \( e_A \) is \( X \)-strong.

5.3. DEFINITION. Let \( A \in \mathfrak{L} \). Let \( \text{loc} \ A \) be the set of functions which are locally in \( A \), that is, \( f \in \text{loc} \ A \) iff \( f : X(e_A) \to \overline{\Re} \) is a function such that for each \( p \in X(e_A) \) there are a neighborhood \( G \) of \( p \) and \( a \in A \) such that \( f = \hat{a} \) on \( G \).

If \( \text{loc} \ A = A \), we call \( A \) local.

5.4. REMARKS. Note that \( \text{loc} \ A \) is a Riesz space in \( D(X(e_A)) \).

By compactness, if \( f \in \text{loc} \ A \), then there is a finite open cover \( \{G_i\} \)
of \( X(e_A) \) and \( \{a_i\} \subseteq A \), with \( f = \hat{a}_i \) on \( G_i \) for each \( i \).

Each \( \mathfrak{L} \)-morphism (see §1) \( \psi : A \to L \) with \( L \) local extends to an \( \mathfrak{L} \)-morphism \( \overline{\psi} : \text{loc} \ A \to L \) (using \([HR]\)). Thus \( A \to \text{loc} \ A \) is what is called a reflection in category theory.

5.5. THEOREM. Let \( A \in \mathfrak{L} \).

(a) If \( e_A \) is \( X \)-strong, then \( A \) is local.

(b) If \( A \) is local and \( X(e_A) \) is totally disconnected, then \( e_A \) is \( X \)-strong.

PROOF. (a) Let \( f \in (\text{loc} \ A)^+ \). For each \( p \in X(e_A) \), choose a neighborhood \( G_p \) and \( a_p \in A \) with \( f = \hat{a}_p \) on \( G_p \). Let \( \{G_p\} \) be a finite sub-cover of \( \{G_p|p \in X(e_A)\} \), with \( \{a_i\} \) the associated elements of \( A \). We may take \( \{a_i\} \subseteq A^+ \).

The finite cover \( \{G_i\} \) has a « shrinkage » by \([E]\), p. 266: A finite open cover \( \{W_i\} \) with \( \overline{W}_i \subset G_i \) for each \( i \).
For each $i$, choose $b_i \in A$ with $0 < b_i < a_i$, $b_i = a_i$ on $W_i$ and $b_i = 0$ off of $G_i$. This is possible because $e_i$ is $X$-strong. Note that $0 < b_i < f$: On $G_i$, $b_i(x) = 0 < f(x)$, and on $G_i$, $b_i(x) < a_i(x) = f(x)$.

Then $f = \bigvee b_i$ as in the proof of 2.7.

(b) Let $a \in A$ and let $G$ and $H$ be open with $\overline{G} \subset H$. A compactness argument products clopen $C$ with $\overline{G} \subset C \subset H$. Let $f = \hat{a}$ on $C$, $0$ off of $C$. Since $C$ is clopen, $f \in \text{loc } A = A$.

For $X$ a Hausdorff uniform space, let $U(X)$ be the Riesz space of all uniformly continuous functions to the reals $R$ ($R$ having the usual uniformity), with weak unit $1$. (See [I]).

5.6. PROPOSITION. (a) Any $U(X)$ is local.

(b) In $U(R)$, $1$ is not $X$-strong.

5.7. LEMMA. The Yosida representation of $U(X)$ is extension over the Samuel compactification $sX$.

PROOF. Essentially by definition, $sX$ is the « compact reflection » of $X$ in Hausdorff uniform spaces: there is a uniformly continues dense homeomorphism $s_x : X \to sX$ such that whenever $f : X \to K$ is uniform with $K$ compact, there is unique $sf : sX \to K$ with $(sf) \circ s_x = f$. See [I].

Let $\hat{A} = \{sf | f \in U(X)\}$. It follows that $\hat{A}^* = C(sX)$, hence $A$ separates the points of $sX$. Since $1 \in \hat{A}$, from 1.3 we see that $sX = X(1)$ and $\hat{A}$ is the Yosida representation.

PROOF OF 5.6. (a) Let $f \in \text{loc } U(X)$. We are to show that given $\varepsilon > 0$, $f^{-1}S(\varepsilon)$ is a uniform cover, where $S(\varepsilon)$ is the cover of $R$ consisting of $\varepsilon$-balls. (We are using the covering description of uniform spaces per [I]).

There is a finite cover $\{G_i\}$ of $sX$ and $\{a_i\} \subseteq U(X)$ such that $f = \hat{a}_i$ on $G_i$. Thus each $f|G_i \cap X$ is uniformly continuous, and so there is a uniform cover $\mathcal{U}_i$ such that $f^{-1}S(\varepsilon)(G_i) > \bigcup\mathcal{U}_i\cap G_i$ (the notation meaning the cover traced on the subset; $>$ means « is refined by »). Thus, $f^{-1}S(\varepsilon) > \{G_i\} \wedge \bigcap\mathcal{U}_i$ (where $\wedge$ means « least common refinement »).

Now $\{G_i\}$ is an open cover of compact $sX$, hence uniform, and its trace on $X$ is uniform. Since $\bigwedge\mathcal{U}_i$ is uniform, so is $f^{-1}S(\varepsilon)$.

(b) We use 2.5. Let $\tilde{f}$ be the extension of $|\sin x|$ over $sR = X(1)$. Let $K_1 = \{x | \tilde{f}(x) = 1\}, K_2 = \{x | \tilde{f}(x) = 0\}$. Let $\hat{a}$ be the extension of
\( a(x) = x \). There is no \( g \in A \) with \( g = a \) on \( K_1 \) and \( g = 0 \) on \( K_2 \). Because, for such \( g \), \( g = g|X \) would be \( x \) on \( \{(2n + 1)\pi/2|n \text{ integral}\} \) and \( 0 \) on \( \{n\pi|n \text{ integral}\} \), and therefore not uniformly continuous.

6. \( X \)-costrong units.

We discuss those \( A \in \mathcal{L} \) for which \( e_A \) has the property «dual» to being \( X \)-strong.

6.1. Definition. \( e_A \) is \( X \)-costrong if there is a natural mapping \( \sigma_\varepsilon: X(e_A) \to X(e) \) whenever \( \varepsilon \) is a positive weak unit.

The condition is «dual» to 4.3 (b). It doesn’t make sense to postulate such \( \sigma_\varepsilon \) when \( \varepsilon \) is not a weak unit: the existence of \( \sigma_\varepsilon \) implies \( e_\varepsilon = e_A \) (using 2.2).

6.2. A natural \( \sigma_\varepsilon: X(e_A) \to X(e) \) is exactly an extension of the \( \tau: Y \to X(\varepsilon) \) of 2.2.

Proof. Such \( \sigma_\varepsilon \) is (by 2.3) a function with \( \sigma_\varepsilon \circ \tau' \) the identity on a certain subset of \( X(\varepsilon) \), where \( \tau' \) is as in 2.2 with \( \varepsilon \) and \( e_A \) interchanged. By 2.2, \( \tau \) itself is \( (\tau')^{-1} \).

One can get a lot of properties equivalent to the existence of \( \sigma_\varepsilon \) by interchanging \( \varepsilon \) and \( e_A \) in the results of §’s 2 and 3. This interchanging can get confusing, and we shall be content with essentially one condition anyhow (the converse of 2.5 (b)); so we proceed directly from 2.8 and 2.9.

6.2. Proposition. Let \( A \in \mathcal{L} \) and let \( \varepsilon \in A^+ \) be a weak unit. Then these are equivalent.

(a) Natural \( \sigma_\varepsilon: X(e_A) \to X(\varepsilon) \) exists.

(b) If \( U_1, U_2 \in \text{coz } \hat{A} \) and there is \( a \in A \) with \( \hat{a} = \hat{\varepsilon} \) on \( U_1 \) and \( \hat{a} = 0 \) on \( U_2 \), then \( \overline{U_1} \setminus \overline{U_2} = \emptyset \).

Proof. By 2.4, there is \( a \in A \) with \( \hat{a} = \hat{\varepsilon} \) on \( U_1 \) and \( \hat{a} = 0 \) on \( U_2 \) iff \( \tau(U_1) \) and \( \tau(U_2) \) have disjoint closures in \( X(\varepsilon) \).

(a) \( \Rightarrow \) (b). Let \( U_1, U_2 \), and \( a \) be as in (b). If \( \sigma_\varepsilon \) exists, then by 2.8, \( \tau^{-1}\tau(U_1) \) and \( \tau^{-1}\tau(U_2) \) have disjoint closures in \( X(e_A) \). But \( \overline{U_i} = \tau^{-1}\tau(\overline{U_i}) \) since \( U_i \cap Y = \tau^{-1}\tau(U_i) \) and \( Y = \tau^{-1}\tau(U_i) \) is dense.
(b) $\Rightarrow$ (a). Applying 2.8, let $K_1$ and $K_2$ be closed and disjoint in $X(\varepsilon)$. Choose $a_i \in A$ with $K_i \subseteq \text{coz } \gamma^*(a_i)$ and with the closures in $X(\varepsilon)$ of $\text{coz } \gamma^*(a_i)$ disjoint. Let $U_i = \text{coz } a_i$. Using (c) 2.2, $\tau(U_i) = \tau(Y_\varepsilon) \cap \text{coz } \gamma^*(a_i)$; and $\tau(U_i) = \text{coz } \gamma^*(a_i)$. Thus, $\tau(U_1) \cap \tau(U_2) = \emptyset$ and by 2.9 there is $a \in A$ with $\hat{a} = \hat{\varepsilon}$ on $U_1$ and $\hat{a} = 0$ on $U_2$. By (b), $U_1 \cap U_2 = \emptyset$. Now $U_1 \subseteq \tau(K_1)$; 2.8 yields $\sigma_\varepsilon$.

6.2 immediately gives a workable condition that $\varepsilon_\lambda$ be $X$-costrong. The following makes the statement of the result more concise.

6.3. TERMINOLOGY. Let $U_1, U_2 \in \text{coz } A$. $U_1$ and $U_2$ are adjacent if $U_1 \cap \overline{U_2}$ is nonempty with empty interior.

Let $a_1, a_2 \in A$. We say $a_1$ is adjacent to $a_2$ if there are adjacent $U_1, U_2$, and $a \in A$, with $\hat{a} = a_1$ on $U_1$ and $\hat{a} = a_2$ on $U_2$.

Thus, immediately from 6.2:

6.4. THEOREM. Let $A \in \mathcal{L}$. These conditions are equivalent.

(a) $\varepsilon_\lambda$ is $X$-costrong.

(b) No weak unit is adjacent to 0.

(c) If $a$ is adjacent to 0, then $\hat{a} = 0$ on some nonvoid open set in $X(\varepsilon_\lambda)$.

6.5. COROLLARY. Let $A \in \mathcal{L}$. The following are equivalent, and implied by $\varepsilon_\lambda$ is $X$-costrong $\Rightarrow$

(a) If $\varepsilon$ is a weak unit, then $\text{pos } \varepsilon \cap \text{neg } \varepsilon = \emptyset$.

(b) If $U_1, U_2 \in \text{coz } A$, $U_1 \cap U_2 = \emptyset$, and $U_1 \cup U_2$ is dense, then $U_1 \cap \overline{U_2} = \emptyset$.

(c) If $U_1 \in \text{coz } A$ is complemented (meaning: there is $U_2 \in \text{coz } A$ with $U_1 \cap U_2 = \emptyset$, $U_1 \cup U_2$ dense), then $U_1$ is open.

(d) If $a, b \in A$, then either $\{ x | \hat{a}(x) = b(x) \}$ has interior, or $\{ x | \hat{a}(x) \not\hat{a} b(x) \}$ has open closure.

PROOF. Let $e_\lambda$ be $X$-costrong, and let $e$ be a weak unit. Then $|e|$ is a positive weak unit. Let $a = e^+$. Then clearly, $\hat{a} = |\varepsilon|$ on $\text{pos } \varepsilon$ and $\hat{a} = 0$ on $\text{neg } \varepsilon$. Of course, $\text{pos } \varepsilon \cap \text{neg } \varepsilon = \emptyset$ by 6.2 (b) (or 6.4 (b)).

(a) $\Rightarrow$ (b). $U_i = \text{coz } \hat{a}_i$ for $a_i \geq 0$. Then apply (a) to $\varepsilon = a_1 - a_2$.

(b) $\Rightarrow$ (c). By (b), $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. Since $\overline{U}_1 = X(\varepsilon_\lambda) - \overline{U}_2$, $\overline{U}_1$ is open.
(c) $\Rightarrow$ (d). If $\text{int}\{x|\hat{a}(x) = \hat{b}(x)\} = \emptyset$, then $\{x|\hat{a}(x) > \hat{b}(x)\}$ is complemented by $\{x|\hat{a}(x) < \hat{b}(x)\}$.

(d) $\Rightarrow$ (a). Apply (d) to $a = e\vee 0$ and $b = (-e)\vee 0$. Then $Z(\hat{e}) = \{x|\hat{a}(x) = \hat{b}(x)\}$ has no interior, so $\text{pos }\hat{e} = \{x|\hat{a}(x) > \hat{b}(x)\}$ has open closure. Since $\text{pos }\hat{e} \subset \text{pos }\hat{e} \cup Z(\hat{e})$, we have $\text{pos }\hat{e} \cap \text{neg }\hat{e} = \emptyset$. Since $\text{pos }\hat{e}$ is open, (a) follows.

The converse to 6.5 fails. We postpone the examples to a later paper treating the ideas of this section with more care.

6.6. A topological space is called quasi-$F$ [DHH] if each dense cozero set is $C^*$-embedded. This is equivalent to each of: [HJ] $D(X)$ is a Riesz space; $C(X)$ (or $D(X)$) is Cantor complete (Dashiell), where a Riesz space $A$ is called Cantor complete (Everett, Papangelou) if each order-Cauchy sequence order-converges, where $\{a_n\}$ is called order-Cauchy if there is $\{u_n\}$ with $u_n \gg u_{n+1} \gg \ldots \gg 0$ with $\bigwedge u_n = 0$ such that for each $n$, $|a_n - a_{n+p}| \leq u_n$ for all $p \geq 0$; and order-convergence is similarly defined.

Every Riesz space (archimedean or not) has a Cantor completion. It was shown in [AH], and independently by Dashiell, that for $A \in \mathcal{L}$, $A$ is Cantor complete iff $X(e_A)$ is quasi-$F$ and $A$ is convex (= an ideal in the Riesz space $D(X(e_A))$).

6.7. COROLLARY. Let $A \in \mathcal{L}$. If $A$ is Cantor complete, or if only $X(e_A)$ is quasi-$F$, then $e_A$ is $X$-costrong.

PROOF. If $e$ is a weak unit, then $Y_e$ is a dense cozero set, hence $C^*$-embedded and hence $\tau: Y_e \to X(e)$ has the extension $\sigma_e: X(e_A) \to (e)$.

6.8. There is $A \in \mathcal{L}$ with $e_A$ $X$-costrong, but $X(e_A)$ not quasi-$F$.

A class of examples is as follows: Let $Y$ be a compact totally disconnected space, and let $A$ consist of all locally constant functions on $Y$, with $e_A = 1$. (Otherwise put, $A$ is the linear span of the continuous characteristic functions). Since $Y$ is totally disconnected, $A$ separates the points. Hence the given presentation of $A$ is the Yosida representation by §1. By compactness, each coz $a$ is clopen. Thus 6.4 (b) holds vacuously.

These examples can be classified in two ways: First, whenever $e$ is a weak unit, $e$ is never 0 and $e_A < te$ for some $t \in \mathbb{R}$; thus $\sigma_e: X(e_A) \to X(e)$ exists by 4.2. Second, the examples above have the ppp (because any coz $a$ is clopen; see 5.1), and
6.9. **Corollary.** If $A$ has the ppp, then any weak unit is $X$-costrong.

**Proof.** Given weak units $e$ and $e'$, $e'$ is $X$-strong by §5, so $\sigma : X(e) \to X(e')$ exists and thus $e$ is $X$-costrong.

6.10. **Remarks.** As in the proof of 4.11, it follows that these conditions on $A$ are equivalent:

(a) Each weak unit is $X$-strong.

(b) Each weak unit is $X$-costrong.

(c) Each weak unit is both $X$-strong and $X$-costrong.

(d) There is a weak unit which is $X$-strong and $X$-costrong (if there is any weak unit).

(e) All spaces $X(e)$ ($e$ a weak unit) are naturally homeomorphic.

(For (e), the existence of natural maps $X(e) \cong X(e')$ implies the maps are mutually inverse, hence each is a "natural" homeomorphism.

Hence, if $A$ has the ppp or is Cantor complete, then the above conditions hold.

We shall return to the general subject of "$X$-equivalence" and "$X$-uniqueness" in another paper.

*Added in proof.* The topic of this paper is explored further in "Retracting the prime spectrum of a Riesz space", by Giuseppe De Marco and the first author (to appear).

**References**


Extremal units in an archimedean Riesz space


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