

RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

G. DA PRATO

**Some remarks on an operational time
dependent equation**

Rendiconti del Seminario Matematico della Università di Padova,
tome 59 (1978), p. 247-262

[<http://www.numdam.org/item?id=RSMUP_1978__59__247_0>](http://www.numdam.org/item?id=RSMUP_1978__59__247_0)

© Rendiconti del Seminario Matematico della Università di Padova, 1978, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

Some Remarks on an Operational Time Dependent Equation.

G. DA PRATO (*)

Introduction.

Let E be a complex Hilbert space and $\{A(t)\}_{t \in [0, T]}$; $\{B(t)\}_{t \in [0, T]}$ two families of linear operators (generally not bounded) in E .

Consider the Cauchy problem:

$$\begin{cases} U'(t) = A(t) U(t) + U(t) B^*(t) + f(t, U(t)) , \\ U(0) = U_0 , \end{cases}$$

where f is a mapping $[0, T] \times Q \rightarrow \mathcal{L}(E)$ and $Q \subset \mathcal{L}(E)$.

Problems of this kind arise in several fields as Optimal Control theory ([2], [3], [7], [8], [9]) and the Hartree-Fock time dependent problem in the case of finite Fermi system ([1]).

In this paper we generalize the results contained in [3] and we give some new regularity result for the case where $A(t)$ and $B(t)$ generates « hyperbolic » semi-groups.

1. The semi-group $T \rightarrow e^{tA} T e^{tB}$.

Let E be a complex Hilbert space (norm $|| \cdot ||$, inner product $(\cdot, \cdot)$). We note by $\mathcal{L}(E)$ (resp. $H(E)$) the complex (resp. real) Banach space of linear bounded (resp. hermitian) operators $E \rightarrow E$ and by $H_+(E)$ the cone of positive operators.

(*) Indirizzo dell'A.: Dipartimento di Matematica - Università di Trento - 38050 Povo (Trento).

Let A and B be the infinitesimal generators of two semi-groups e^{tA} and e^{tB} ; we assume that:

$$(1.1) \quad |e^{tA}| \leq M_A \exp(w_A t), \quad |e^{tB}| \leq M_B \exp(w_B t).$$

We note finally by $\mathfrak{L}_s(E)$ (resp. $H_s(E)$) the set $\mathfrak{L}(E)$ (resp. $H(E)$) endowed by the strong topology; $\mathfrak{L}_s(E)$ is a locally convex space.

Consider the following semi-group in $\mathfrak{L}_s(E)$:

$$(1.2) \quad G_t(T) = e^{tA} T e^{tB}, \quad \forall T \in \mathfrak{L}(E), \quad t \geq 0,$$

G_t is not strongly continuous in $\mathfrak{L}(E)$, but it is sequentially strongly continuous in $\mathfrak{L}_s(E)$, that is:

$$T_n \rightarrow T \text{ in } \mathfrak{L}_s(E) \Rightarrow G_t(T_n) \rightarrow G_t(T) \text{ in } \mathfrak{L}_s(E)$$

and the mapping:

$$\overline{\mathbf{R}}_+ \rightarrow \mathfrak{L}_s(E), \quad t \rightarrow G_t(T)$$

is continuous $\forall T \in \mathfrak{L}_s(E)$.

If $B = A^*$ ⁽¹⁾ it is:

$$(1.3) \quad G_t(T) \in H(E), \quad \forall T \in H(E).$$

Put:

$$(1.4) \quad D(L) = \left\{ T \in \mathfrak{L}(E); \exists \lim_{h \rightarrow 0} \frac{1}{h} (G_h(T)x - Tx), \forall x \in E \right\},$$

$$(1.5) \quad L(T)x = \lim_{h \rightarrow 0^+} \frac{1}{h} (G_h(T)x - Tx), \quad \forall T \in D(L), \quad \forall x \in E.$$

LEMMA 1.1. If $T \in D(L)$ and $x \in D(B)$ then $Tx \in D(A)$ and it is:

$$(1.6) \quad L(T)x = ATx + TBx.$$

PROOF. Let $T \in D(L)$, $x \in D(B)$, $y \in D(A^*)$; it is:

$$(L(T)x, y) = \frac{d}{dh} (Te^{hB}x, e^{hA^*}y)|_{h=0} = (TBx, y) + (Tx, A^*y).$$

⁽¹⁾ A^* is the adjoint of A .

It follows that the mapping:

$$D(A^*) \rightarrow \mathbf{C}, \quad y \rightarrow (Tx, A^*y) = (L(T)x, y) - (TBx, y)$$

is continuous, $Tx \in D(A)$ and:

$$(ATx, y) = (L(T)x, y) - (TBx, y) \neq$$

The following proposition is clear:

PROPOSITION 1.2. *If $T \in D(L)$ then $G_t(T) \in D(L)$ and it is:*

$$(1.7) \quad L(G_t(T)) = e^{tA}L(T)e^{tB},$$

$$(1.8) \quad \frac{d}{dt}(G_t(T)x) = e^{tA}L(T)e^{tB}x.$$

PROPOSITION 1.3. *L is closed in $\mathfrak{L}_s(E)$ and in $\mathfrak{L}(E)$.*

PROOF. Let $T_n \in D(L)$, $T_n \rightarrow T$, $S_n = L(T_n) \rightarrow S$ in $\mathfrak{L}_s(E)$; due to (1.8) it is:

$$e^{tA}T_ne^{tB}x - T_nx = \int_0^t e^{sA}S_ne^{sB}x ds$$

recalling the dominate convergence theorem we obtain:

$$\frac{1}{t}(G_t(T)x - Tx) = \frac{1}{t} \int_0^t G_s(S)x ds$$

it follows $T \in D(L)$ and $L(T) = S$. Therefore L is closed in $\mathfrak{L}_s(E)$ and consequently in $\mathfrak{L}(E)$. $\neq$

PROPOSITION 1.4. *$D(L)$ is dense in $\mathfrak{L}_s(E)$.*

PROOF. Put:

$$Q_t x = \frac{1}{t} \int_0^t G_s(T)x ds, \quad \forall T \in \mathfrak{L}(E), \quad \forall x \in E,$$

it is:

$$\lim_{t \rightarrow 0^+} Q_t = I \quad \text{in } \mathfrak{L}_s(E),$$

moreover

$$\frac{1}{h} (G_h(Q_t) - Q_t)x = \frac{1}{th} \left[\int_t^{t+h} - \int_0^h G_s(T)x ds \right]$$

it follows $D_t \in D(L)$ and therefore $D(L)$ is dense in $\mathfrak{L}_s(E)$. $\neq$

PROPOSITION 1.5. $\varrho(L) \supset]w_A + w_B, \infty[$ and it is ⁽²⁾:

$$(1.9) \quad R(\lambda, L)(T)x = \int_0^\infty e^{-\lambda t} e^{tA} T e^{tB} x dt, \quad \forall x \in E, \quad \forall \lambda > w_A + w_B,$$

$$(1.10) \quad \|R(\lambda, L)\|_{\mathfrak{L}(\mathfrak{L}(E))} \leq M_A M_B (\lambda - w_A - w_B)^{-1}, \quad w_A + w_B < \lambda \text{ } ^{(3)}.$$

PROOF. Put

$$F(T)x = \int_0^\infty e^{-\lambda t} e^{tA} T e^{tB} x dt, \quad \forall T \in \mathfrak{L}(E).$$

For every $T \in D(L)$ it is:

$$F(L(T))x = \int_0^\infty e^{-\lambda t} G'_t(T)x dt = (\lambda F(T) - T)x$$

moreover if $T \in \mathfrak{L}(E)$ it is:

$$\frac{1}{h} \{G_h(F(T)) - F(T)\}x = \frac{e^{\lambda h} - 1}{h} \int_h^\infty e^{-\lambda t} G_t(T)x dt - \frac{1}{h} \int_0^h e^{-\lambda t} G_t(T)x dt$$

it follows

$$L(F(T))x = (\lambda F(T) - T)x. \quad \neq$$

⁽²⁾ If L is a linear operator, $\varrho(L)$ is the resolvent set and $R(\lambda, L)$ the resolvent of L .

⁽³⁾ $\mathfrak{L}(\mathfrak{L}(E))$ is the Banach space of the linear bounded operators $\mathfrak{L}(E) \rightarrow \mathfrak{L}(E)$. We note $\| \cdot \|$ the norm in $\mathfrak{L}(\mathfrak{L}(E))$.

PROPOSITION 1.6. *If $T_n \rightarrow T$ in $C([0, T]; \mathfrak{L}_s(E))$ ⁽⁴⁾ then*

$$G_i(T_n(t)) \rightarrow G_i(T(t)) \quad \text{in } C([0, T]; \mathfrak{L}_s(E)) .$$

PROOF. Let $x \in E$; for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$, $f_1, \dots, f_{n_\varepsilon}$ in $C([0, T])$ and $x_1, \dots, x_{n_\varepsilon} \in E$ such that:

$$\left| e^{tB}x - \sum_{i=1}^{n_\varepsilon} f_i(t)x_i \right| < \varepsilon, \quad \forall t \in [0, T],$$

it follows:

$$\begin{aligned} |G_i(T(t) - T_n(t))x| &\leq M_B \exp(|w_B|T) |(T(t) - T_n(t))e^{tB}x| \leq \\ &\leq M_B \exp(|w_B|T) \varepsilon (|T(t)| + |T_n(t)|) + \\ &\quad + M_B \exp(|w_B|T) \sum_{i=1}^{n_\varepsilon} |f_i(t)| |T(x)x_i - T_n(t)x_i|. \end{aligned}$$

Choose N such that $|T_n(t)| \leq N$, then:

$$\begin{aligned} |G_i(T(t) - T_n(t))x| &\leq 2NM_B \exp(|w_B|T) \varepsilon + \\ &\quad + M_B \exp(|w_B|T) \sum_{i=1}^{n_\varepsilon} |\varphi_i(t)| |T(x)x_i - T_n(x)x_i|. \end{aligned}$$

Choose n'_ε such that:

$$|T(t)x_i - T_n(t)x_i| \leq \varepsilon / (n_\varepsilon \max\{|\varphi_i|; i = 1, 2, \dots, n_\varepsilon\}), \quad \forall n > n_\varepsilon,$$

then

$$n > n'_\varepsilon \Rightarrow |G_i(T(t) - T_n(t))x| \leq (2N + 1)M_B \exp(|w_B|T) \varepsilon. \quad \neq$$

2. The linear problem.

Let $\mathcal{A} = \{A(t)\}_{t \in [0, T]}$, $\mathcal{B} = \{B(t)\}_{t \in [0, T]}$ be two families of linear operators in E .

Let F be a Hilbert space (norm $\| \cdot \|$, inner product $(\cdot, \cdot)$) continuously and densely embedded in E .

⁽⁴⁾ $C([0, T]; \mathfrak{L}_s(E))$ is the set of the mappings $[0, T] \rightarrow \mathfrak{L}_s(E)$ continuous; due to the Banach-Steinhaus theorem every $u \in C([0, T]; \mathfrak{L}_s(E))$ is bounded.

Let finally Z be an isometric isomorphism in $\mathfrak{L}(F, E)$.

We assume:

- (2.1) $\left\{ \begin{array}{l} a) \mathcal{A} \text{ (resp. } \mathcal{B}) \text{ is } (M_A, w_A)\text{-stable and } w_A\text{-measurable (resp.} \\ \quad (M_B, w_B)\text{-stable and } w_B\text{-measurable) in } E \text{ }^{(5)}. \\ b) \text{ It is } F \subset D(A(t)) \text{ (resp. } D(B(t))), A(t) \text{ (resp. } B(t)) \in \mathfrak{L}(F, E) \\ \quad \text{and } |A(t)| \text{ (resp. } |B(t)|) \text{ is bounded in } [0, T] \text{ }^{(6)}. \\ c) \text{ The mapping } A(\cdot)x \text{ (resp. } B(\cdot)x) \text{ is continuous } \forall x \in F. \\ d) \text{ There exists a mapping } H \text{ (resp. } K): [0, T] \rightarrow \mathfrak{L}(E) \text{ such} \\ \quad \text{that:} \\ d_1) H \text{ (resp. } K) \text{ is bounded in } [0, T] \text{ and strongly measur-} \\ \quad \text{able in } E. \\ d_2) \text{ It is:} \\ \quad ZA(t)Z^{-1}x = A(t)x + H(t)x, \quad \forall x \in D(A(t)), \\ \quad ZB(t)Z^{-1}x = B(t)x + K(t)x, \quad \forall x \in D(B(t)). \end{array} \right.$

If [2.1] is fulfilled it is known ([4], [6]) that there exists an evolution operator $G_A(t, s)$ (resp. $G_B(t, s)$) for the problem:

$$u' = A(t)u, \quad u(0) = x \quad (\text{resp. } u' = B(t)u, \quad u(0) = x).$$

Moreover G_A (resp. G_B): $\Delta = \{(t, s) \in [0, T]^2; t \geq s\} \rightarrow \mathfrak{L}(E)$ is strongly continuous and $G(r, s) \in \mathfrak{L}(F)$.

Finally it is:

$$(2.2) \quad \left\{ \begin{array}{l} \lim_{n \rightarrow \infty} G_{A,n} = G_A, \\ \lim_{n \rightarrow \infty} G_{B,n} = G_B, \end{array} \right. \quad \text{in } C(\Delta; \mathfrak{L}_s(E)),$$

⁽⁵⁾ $\mathcal{A}$ is w_A -measurable in E if $\varrho(A(t)) \supset]w_A, +\infty[$ and $R(\lambda, A(\cdot))$ is strongly measurable $\forall \lambda > w_A$.

$\mathcal{A}$ is (M_A, w_A) -stable in E if $\varrho(A(t)) \supset]w_A, +\infty[$ and it is:

$$\left| \prod_{i=1}^k R(\lambda, A(t_i)) \right| < M_A / (\lambda - w_A)^k$$

$\forall k \in \mathbf{N}, t_1 > t_2 > \dots > t_k, t_i \in [0, T], i = 1, \dots, n.$

⁽⁶⁾ With the topology of $\mathfrak{L}(F, E)$.

where $G_{A,n}$ (resp. $G_{B,n}$) is the evolution operator associated to the problem:

$$u'_n = A_n(t)u_n, \quad u_n(0) = x \quad (\text{resp. } u'_n = B_n(t)u_n, \quad u_n(0) = x)$$

where $A_n(t) = n^2 R(n, A(t)) - n$ and $B_n(t) = n^2 R(n, B(t)) - n$.

Consider now the problem:

$$(2.3) \quad \begin{cases} T'(t) = A(t)T(t) + T(t)B^*(t) + F(t), & F \in C([0, T]; \mathfrak{L}_s(E)) , \\ T(0) = T_0 \in \mathfrak{L}(E) . \end{cases}$$

We define $L(t)$ as in (1.4), (1.5) and write (2.3) in the following form:

$$(2.4) \quad \begin{cases} T'(t) = L(t)(T(t)) + F(t) , \\ T(0) = T_0 . \end{cases}$$

We consider also the approximate problem:

$$(2.5) \quad \begin{cases} T'_n(t) = L_n(t)(T(t)) + F(t) , \\ T_n(0) = T_0 , \end{cases}$$

where $L_n(t)(T) = A_n(t)T + TB_n^*(t)$.

We say that T is a *strong solution* of (2.4) if there exists:

$$(2.6) \quad \{T_k\} \subset D(L(t)) \cap C^1([0, T]; \mathfrak{L}_s(E)) \quad (7)$$

such that:

$$\begin{cases} T'_k - L(T_k) \rightarrow F & \text{in } C([0, T]; \mathfrak{L}_s(E)) , \\ T_k(0) \rightarrow T_0 & \text{in } \mathfrak{L}(E) . \end{cases}$$

If $T \in D(L(t)) \cap C^1([0, T]; \mathfrak{L}_s(E))$ and (2.4) is fulfilled we say that T is a *classical solution* of (2.4).

THEOREM 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two family of linear operators in E verifying (2.1). Then for every $T_0 \in \mathfrak{L}(E)$ and $F \in C([0, T]; \mathfrak{L}_s(E))$ the*

(7) $C^1([0, T]; \mathfrak{L}_s(E))$ is the set of the mappings $[0, T] \rightarrow \mathfrak{L}_s(E)$ strongly continuously differentiable.

problem (2.4) has a unique strong solution given by:

$$(2.7) \quad T(t)x = G_A(t, 0)T_0G_B^*(t, 0)x + \int_0^t G_A(t, s)F(s)G_B^*(t, s)x \, ds.$$

If $T_0 \in \mathfrak{L}(F)$ and $F \in C([0, Y]; \mathfrak{L}_s(F))$ then the solution T is classical.

PROOF. Let first $T_0 \in \mathfrak{L}(F)$ and $F \in C([0, T]; \mathfrak{L}_s(F))$; in this case we can easily verify that T is a classical solution.

In the general case by approximating T_0 and F we can show that $T(t)$, given by (2.7) is a strong solution.

Assume finally that T is a strong solution of (2.4) and take $\{T_k\}$ as in (2.6). Put $F_k = T'_k - L(T_k)$; it is:

$$\frac{d}{ds}(G_A(t, s)T_k(s)G_B(t, s)x) = G_A(t, s)F_k(s)G_B(t, s)x, \quad \forall x \in E,$$

by integration in $[0, t]$ it follows:

$$T_k(t)x = G_A(t, 0)T_k(0)G_B(t, 0)x + \int_0^t G_A(t, s)F_k(s)G_B(t, s)x \, ds$$

and, taking the limit for $k \rightarrow \infty$, the conclusion follows.

3. The quasi-linear problem.

Let Q a closed convex set in $\mathfrak{L}(E)$ and f a strongly continuous mapping

$$f: [0, T] \times Q \rightarrow \mathfrak{L}(E), \quad (t, S) \rightarrow f(t, S).$$

Consider the problems:

$$(3.1) \quad \begin{cases} U'(t) - L(t)(U(t)) + f(t, U(t)) = 0, \\ U(0) = U_0, \end{cases}$$

$$(3.2) \quad \begin{cases} U'_n(t) - L_n(t)(U(t)) + f(t, U_n(t)) = 0, \\ U_n(0) = U_0. \end{cases}$$

We say that U is a *strong solution* of (3.1) if there exists $\{U_k\} \subset D(L(t)) \cap C^1([0, T]; \mathfrak{L}_s(E))$ such that:

$$\begin{cases} U'_k - L(U_k) + f(t, U_k) \rightarrow 0 & \text{in } C([0, T]; \mathfrak{L}_s(E)) , \\ U_k(0) \rightarrow U_0 & \mathfrak{L}_s(E) . \end{cases}$$

If U belongs to $D(L(t)) \cap C^1([0, T]; \mathfrak{L}_s(E))$ and fulfils (3.1) we say that U is a *classical solution* of (3.1).

The following proposition is an immediate consequence of the Theorem 2.1.

PROPOSITION 3.1. U is a strong solution of (3.1) if and only if it is:

$$(3.3) \quad U(t)x = G_A(t, 0) U_0 B_B^*(t, 0)x - \int_0^t G_A(t, s) f(s, U(s)) G_B^*(t, s)x ds .$$

We remark now that $C([0, T]; \mathfrak{L}_s(E))$ is not a metric space, but we can define in it the following norm:

$$(3.4) \quad \|U\| = \text{Sup} \{|U(t)|, t \in [0, T]\} , \quad \forall U \in C([0, T]; \mathfrak{L}_s(E)) ,$$

by virtue of the Banach-Steinhaus theorem.

$C([0, T]; \mathfrak{L}_s(E))$ endowed by the norm (3.4) is a Banach space which we note by $B([0, T]; \mathfrak{L}_s(E))$.

LEMMA 3.2. Let K be a closed subset of $B([0, T]; \mathfrak{L}_s(E))$ and γ_n, γ mappings $K \rightarrow K$. Assume that:

$$(3.5) \quad \|\gamma_n(U) - \gamma_n(V)\| \leq \alpha \|U - V\| , \quad \alpha \in]0, 1[, \quad U, V \in K ,$$

$$(3.6) \quad \gamma_n(U) \rightarrow \gamma(U) \quad \text{in } C([0, T]; \mathfrak{L}_s(E)) , \quad \forall U \in K .$$

Then there exists $\{U_n\}$ and U unic in K such that:

$$(3.7) \quad \gamma_n(U_n) = U_n , \quad \gamma(U) = U ,$$

$$(3.8) \quad U_n \rightarrow U \text{ in } C([0, T]; \mathfrak{L}_s(E)) .$$

PROOF. By virtue of the contractions principle there exists U_n and U such that (3.7) is fulfilled.

To prove (3.8) fix Z in K ; it is:

$$U_n = \lim_{m \rightarrow \infty} \gamma_n^m(Z), \quad U = \lim_{m \rightarrow \infty} \gamma^m(Z) \quad \text{in } B([0, T]; \mathfrak{L}_s(E))$$

and

$$\|U_n - \gamma_n^m(Z)\| \leq \frac{\alpha^m}{1 - \alpha} (\|\gamma_n(Z)\| + \|Z\|)$$

therefore there exists $M > 0$ such that:

$$(3.9) \quad \|U_n - \gamma_n^m(Z)\| \leq M\alpha^m.$$

It is easy to show that:

$$(3.10) \quad \lim_{n \rightarrow \infty} \gamma_n^m(U) = \gamma^m(U) \quad \text{in } C([0, T]; \mathfrak{L}_s(E)), \quad \forall U \in K, \quad m \in \mathbb{N}$$

if $x \in E$ and $t \in [0, T]$ it follows:

$$\begin{aligned} |U(t)x - U_n(t)x| &\leq |U(t)x - \gamma^m(Z)(t)x| + \\ &\quad + |\gamma^m(Z)(t) - \gamma_n^m(Z)(t)x| + |U_n(t)x - \gamma_n^m(Z)(t)x| \end{aligned}$$

due to (3.9) it follows:

$$|U(t)x - U_n(t)x| \leq 2M\alpha^m |x| + |\gamma^m(Z)(t)x - \gamma_n^m(Z)(t)x|$$

and the conclusion follows from (3.10). $\neq$

We prove now the existence of the maximal solution for the problem (3.1).

We assume:

$$(3.11) \quad \left\{ \begin{array}{l} a) f \in C([0, T] \times Q_s^{(*)}; \mathfrak{L}_s(E)) \cap C([0, T] \times Q; \mathfrak{L}(E)), \\ b) \exists \mu: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ such that:} \\ \quad |f(t, T) - f(t, S)| \leq \mu(r) |T - S| \text{ if } |T| \leq r, |S| \leq r, \\ c) \exists \alpha: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ such that:} \\ \quad r > 0, |T| \leq r, T \in Q, \beta \in]0, \alpha(r)[\Rightarrow T - \beta f(t, T) \in Q. \end{array} \right.$$

We remark that $c)$ is trivial if $Q = \mathfrak{L}_s(E)$ or $H(E)$.

$(*) Q_s$ is endowed by the topology of $\mathfrak{L}_s(E)$.

LEMMA 3.3. Assume that:

- i) $\mathcal{A}$ and $\mathcal{B}$ verify (2.1), $U_0 \in Q$,
- ii) $T \in Q \Rightarrow \exp(sA(t))T \exp(sB(t)) \in Q, \forall t \in [0, T]$,
- iii) f verifies (3.11).

Take α, β such that:

$$\begin{cases} \alpha \geq M_A M_B \exp(|w_A| + |w_B|)T |U_0|, \\ \beta \leq \alpha(2a). \end{cases}$$

Then there exists $\tau > 0$ such that the problem (3.1) has a unique strong solution in $[0, \tau]$.

PROOF. Put:

$$\varphi(t, T) = T - \beta f(t, T)$$

then φ maps $[0, T] \times (Q \cap P(0, 2a))$ in $Q^{(*)}$ and it is:

$$(3.13) \quad |\varphi(t, T) - \varphi(t, S)| \leq (1 + \beta\mu(2a))|T - S|, \quad \forall T, S \in Q \cap P(0, 2a).$$

Problem (3.1) is equivalent to:

$$(3.14) \quad \begin{cases} U' - L(t)U + \frac{1}{\beta}\varphi(t, U) = 0, \\ U(0) = U_0, \end{cases}$$

put $U = \exp(-t/\beta)V$, then it is:

$$(3.15) \quad \begin{aligned} V(t)x &= G_A(t, 0)U_0 G_B^*(t, 0)x + \\ &+ \frac{1}{\beta} \int_0^t e^{s/\beta} G_A(t, s) \varphi(s, U(s)) G_B^*(t, s) x ds \end{aligned}$$

which is equivalent to the equation:

$$(3.16) \quad \begin{aligned} U(t) &= G_A(t, 0)U_0 G_B^*(t, 0)e^{-t/\beta} + \frac{1}{\beta} \int_0^t e^{-(t-s)/\beta} G_A(t, s) \varphi(s, U(s)) , \\ &G_B^*(t, s) ds = \gamma(U)(t). \end{aligned}$$

$$(*) P(0, r) = \{T \in \mathfrak{L}(E); |T| \leq r\}.$$

It is:

$$\|\gamma(U) - \gamma(V)\| \leq M_A M_B \exp\left((|w_A| + |w_B|)T\right)(1 - e^{-t/\beta})\|U - V\|, \\ \forall U, V \in C([0, \tau]; (Q \cap P(0, 2a))_s)$$

and

$$\|\gamma(U)\| \leq a + M_A M_B \exp\left((|w_A| + |w_B|)T\right)(1 + \mu(2a))2a + \\ + \sup\{|\varphi(t, 0)|, t \in [0, T]\}(1 - e^{-t/\beta}).$$

Therefore there exists $\tau > 0$ such that γ is a contraction in

$$C([0, \tau]; (Q \cap P(0, 2a))_s). \quad \neq$$

The following theorem is an immediate consequence of Lemma 3.3, Proposition 1.6, Lemma 3.2 and standard arguments.

THEOREM 3.4. *Assume that $\mathcal{A}$, $\mathcal{B}$, f verify the hypotheses of Lemma 3.3. Then there exists the maximal solution U of the problem (3.1). If I is the interval where U is defined it is:*

$$U_n \rightarrow U \quad \text{in} \quad C(I, \mathcal{L}_s(E))$$

U_n being the solution of (3.2). Finally if $\|U\|$ is bounded it is $I = [0, T]$.

PROPOSITION 3.5. *Assume that the hypotheses of Theorem 3.4 are fulfilled. Assume moreover:*

$$(3.18) \quad \left\{ \begin{array}{l} \text{i) } M_A = M_B = 1, \\ \text{ii) } \exists \omega_1 \in \mathbb{R} \text{ such that} \\ |T| \leq |T + \alpha(f(t, T) - f(t, 0) + \omega_1 T)|, \\ \forall \alpha \geq 0, t \in [0, T], T \in Q. \end{array} \right.$$

Then the maximal solution of (3.1) verifies the following inequality:

$$(3.19) \quad |U(t)| \leq \exp((w_A + w_B + \omega_1)t)|U_0| + \\ + \int_0^t \exp((w_A + w_B + \omega_1)(t-s))|f(s, 0)| ds.$$

PROOF. We remember (Kato [5]) that (3.18)-ii) is equivalent to:

$$(3.20) \quad \langle f(t, T) - f(t, 0), I \rangle \geq -\omega_1 |T|, \quad \forall I \in \partial |T|,$$

$\partial |T|$ being the sub-differential of the norm in $\mathfrak{L}(E)$.

Due to (3.18) for every $T \in D(L(s))$ there exists $I \in \partial |T|$ such that

$$(3.21) \quad \langle L(s)(T), I \rangle \leq (w_A + w_B) |T|.$$

Suppose first that U is a classical solution of (3.1); then

$$(3.22) \quad \begin{aligned} \frac{d^-}{dt} |U(t)| &= \inf \{ \langle U(t), I \rangle, I \in \partial |U(t)| \} \leq \\ &\leq \langle L(t)(U(t)), I \rangle - \langle f(t, U(t)) - f(t, 0), I \rangle + \langle f(t, 0), I \rangle \end{aligned}$$

if we take I such that

$$\langle L(t)(U(t)), I \rangle \leq (w_A + w_B) |U(t)|$$

it is

$$(3.13) \quad \frac{d^-}{dt} |U(t)| \leq (w_A + w_B + \omega_1) |U(t)| + |f(t, 0)|$$

which implies (3.19). If U is a strong solution the conclusion follows by approximation. $\neq$

4. Regularity.

If for every $V \in \mathfrak{L}(F)$ it is $f(t, V) \in \mathfrak{L}(F)$ we put

$$f_z(t, V) = Z f(t, Z^{-1} V Z) Z^{-1}.$$

THEOREM 4.1. *Assume that the hypotheses of Theorem 3.4 are fulfilled. Moreover assume that f maps $[0, T] \times \mathfrak{L}(F)$ in $\mathfrak{L}(F)$ and that f_z verifies (3.11); then if $U_0 \in \mathfrak{L}(E) \cap \mathfrak{L}(F)$ the maximal solution of (3.1) is classical and $U(t) \in \mathfrak{L}(F)$, $\forall t \in [0, T]$.*

PROOF. Consider the problems:

$$(4.1) \quad \begin{cases} V'(t) = (A(t) + H(t)) V(t) + V(t)(B(t) + K(t)) + f_{\sharp}(t, V), \\ B(0) = ZU_0Z^{-1}, \end{cases}$$

$$(4.2) \quad \begin{cases} V'_n(t) = (A_n(t) + H_n(t)) V_n(t) + \\ \quad + V_n(t)(B_n(t) + K_n(t)) + Zf_n(t, U_n)Z^{-1}, \\ V_n(0) = ZU_0Z^{-1}, \end{cases}$$

where

$$(4.3) \quad \begin{cases} H_n(t) = n^2 R(n, A(t)) H(t) R(n, A(t)) + H(t), \\ K_n(t) = n^2 R(n, B(t)) K(t) R(n, B(t)) + K(t). \end{cases}$$

By virtue of Theorem 3.4 the problems (4.1) and (4.2) have maximal solutions in $[0, \tau[$, τ being the maximal time for U ; moreover

$$V_n \rightarrow V \quad \text{in } C([0, \tau[; \mathfrak{L}_s(E)).$$

It is easy to see that $V_n = ZU_nZ^{-1}$, therefore

$$U_n \rightarrow U \quad \text{in } C([0, \tau[; \mathfrak{L}_s(E)), \quad ZU_nZ^{-1} \rightarrow V \quad \text{in } \mathfrak{L}_s(E)$$

it follows $U \in \mathfrak{L}(F)$, $V = ZUZ^{-1}$. $\neq$

REMARK. If A and B are independent of t we have the following result (cf. [3]).

THEOREM 4.2. *Assume that the hypotheses of Theorem 3.4 are fulfilled. Suppose moreover that $f \in C^1([0, T], \mathfrak{L}_s(E))$ and $U_0 \in D(L)$. Then the maximal solution of (3.1) is classical.*

5. Examples.

1) Let $f \in C^2(\mathbf{R})$, put:

$$(5.1) \quad f(T) = \int_{-\infty}^{+\infty} f(\lambda) dE_{\lambda}, \quad \forall T \in H(E),$$

E_{λ} being the spectral projector attached to T .

If we choose $Q = H(E)$, $B = A$ then f fulfils (3.11) (cf. Tartar [8]) and (3.1) has a unique maximal solution.

Assume now

$$(5.2) \quad Q = \{T \in H(E); a \leq T \leq b\}, \quad a, b \in \mathbf{R}.$$

LEMMA 5.1. If $f(a) \leq 0$ and $f(b) \geq 0$ then $\forall r > 0$, $\exists \beta_r > 0$ such that:

$$(5.3) \quad |x| \leq r, \quad x \geq a, \quad \beta \in]0, \beta_r[\Rightarrow x - \beta f(x) \geq a.$$

PROOF. If $f(a) < 0$ the thesis is evident. Assume $f(a) = 0$; then it is $f(x) = (x - a)\psi(x)$ and if $x \geq a$ it is

$$x - a - \beta f(x) = (x - a)(1 - \beta\psi(x)) \geq 0$$

for suitable β . $\neq$

The following proposition is now evident

PROPOSITION 5.2. Assume that (2.1) is fulfilled with $B = A$. Assume moreover that $f \in C^2(R)$, $f(a) \leq 0$, $f(b) \geq 0$. Then if $a \leq U_0 \leq b$ there exists a unique global solution U such that $a \leq U(t) \leq b$.

2) Riccati equation.

Assume $Q = H_+(E)$, $B = A$, $|e^{tA}| \leq 1$ and (2.1) fulfilled; assume $f(T) = TPT - F(t)$ where $P \geq 0$, $F(t) \geq 0$; then it is easy to see that f verifies (3.11); therefore (3.1) has a maximal solution in Q . Moreover it is

$$|T| \leq |T + \alpha TPT|, \quad \forall \alpha > 0, \quad \forall T \geq 0,$$

because

$$((T + \alpha TPT)x, x) \geq (Tx, x)$$

therefore if $U_0 \geq 0$ (3.1) has a global solution.

Finally assume $P \in \mathcal{L}(F)$, put $\bar{P} = ZPZ^{-1}$ then $f_z(V) = V\bar{P}V$ and the hypotheses of the Theorem 4.1 are fulfilled and the solution is classical.

REFERENCES

- [1] A. BOVE - G. DA PRATO - G. FANO, *On the Hartree-Fock time-dependent problem*, Commun. Math. Phys., **49** (1976), pp. 25-33.
- [2] R. F. CURTAIN - A. J. PRITCHARD, *Infinite dimensional linear systems theory*, Lecture notes in Control and Information sciences, Springer, 1978.
- [3] G. DA PRATO, *Equations d'évolutions dans des algèbres d'opérateurs et applications à des équations quasi-linéaires*, J. Math. Pures et Appl., **48** (1969), pp. 59-107; *Somme d'applications non linéaires dans des cônes et équations d'évolution dans des espaces d'opérateurs*, Ibid., **49** (1970), pp. 289-340; *Quelques résultats d'existence unicité et régularité pour un problème de la théorie du contrôle*, Ibid., **52** (1973), pp. 353-375.
- [4] G. DA PRATO - M. IANNELLI, *On a method for studying abstract evolution equations in the hyperbolic case*, Comm. Partial Diff. Eq., **1** (6) (1976), pp. 585-608.
- [5] T. KATO, *Nonlinear semi-groups and evolution equations*, J. Math. Soc. Japan, **19** (1967), pp. 508-520.
- [6] T. KATO, *Linear evolution equations of « hyperbolic » type I*, J. Fac. Sci. Univ. Tokio, **17** (1970), pp. 241-258; and *II*, J. Math. Soc. Japan, **25** (1973), pp. 648-666.
- [7] J. L. LIONS, *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*, Dunod-Gauthier-Villars, 1968.
- [8] L. TARTAR, *Sur l'étude directe d'équations non linéaires intervenant en théorie du contrôle optimal*, Journ. Functional Analysis, **6** (1974), pp. 1-47.
- [9] R. TEMAM, *Sur l'équation de Riccati associée à des opérateurs non bornés en dimension infinie*, Journ. Functional Analysis (1971).

Manoscritto pervenuto in redazione il 10 dicembre 1978.