

RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

LUIGI SALCE

The λ -inductive topology on abelian p -groups

Rendiconti del Seminario Matematico della Università di Padova,
tome 59 (1978), p. 167-177

http://www.numdam.org/item?id=RSMUP_1978__59__167_0

© Rendiconti del Seminario Matematico della Università di Padova, 1978, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

The λ -Inductive Topology on Abelian p -Groups (*).

LUIGI SALCE (*)

1. Introduction.

Let G be an abelian p -group. The p -adic topology on G has the family of subgroups $\{p^n G\}_{n \in \mathbb{N}}$ as a basis of neighborhoods of 0. The Hausdorff completion $L_\omega(G) = \varprojlim G/p^n G$ of G in this topology is not a p -group if G is unbounded; the maximal torsion subgroup of $L_\omega(G)$, which is a p -group, is denoted by $T_\omega(G)$ and is called the torsion completion of G ; G is said torsion complete if the canonical homomorphism $\delta: G \rightarrow L_\omega(G)$ sends G isomorphically upon $T_\omega(G)$. The class of torsion complete p -groups is one of the best known in the theory of abelian p -groups (see [4, Chap. XI]).

B. Charles made in 1967 the important observation that the torsion completion $T_\omega(G)$ of G can be realized as the completion of G in a suitable topology, which is the inductive topology; a basis of neighborhoods of 0 for this topology is the family of large subgroups, introduced by Pierce [11], which are those fully invariant subgroups L of G such that $G = L + B$ for every basic subgroup B of G ; for the proof of this fact see [2] or [4, 70.2]. Large subgroups have a nice description: in fact Pierce proved ([11]; see also [4, 67.2]) that, if G is unbounded, every large subgroup L of G can be realized as $L = G(\mathbf{u})$, where $\mathbf{u} = (r_n)_{n \in \mathbb{N}}$ is an increasing sequence of integers, and $G(\mathbf{u}) = \{x \in G \mid h_a(p^n x) \geq r_n, \forall n \in \mathbb{N}\}$ (h_a denotes the height in G); note that

(*) Indirizzo dell'A.: Seminario Matematico, Università - Via Belzoni 7 - 35100 Padova.

Lavoro eseguito nell'ambito dei Gruppi di Ricerca Matematica del Consiglio Nazionale delle Ricerche.

the subgroups $G(\mathbf{u})$ described above are a natural generalization of the subgroups $p^n G$ ($n \in \mathbf{N}$).

The generalization of the p -adic topology from the ordinal ω to an arbitrary limit ordinal λ is the λ -adic topology, which has as a basis of neighborhoods of 0 the family of subgroups $\{p^\sigma G\}_{\sigma < \lambda}$; this topology has been studied by Mines [10]. The Hausdorff completion of G in this topology and its torsion part are denoted respectively by $L_\lambda(G)$ and $T_\lambda(G)$; if λ is cofinal with ω (we shall write $\lambda = \text{cf } \omega$), then, if $\delta_\lambda: G \rightarrow L_\lambda(G)$ is the canonical homomorphism, $L_\lambda(G)/\delta_\lambda(G)$ is divisible and the topology of the completion on $L_\lambda(G)$ coincides with the λ -adic topology; if λ is not cofinal with ω (we shall write $\lambda \neq \text{cf } \omega$), then $L_\lambda(G) = T_\lambda(G)$ and in general $L_\lambda(G)/\delta_\lambda(G)$ is not divisible, and the two above topologies do not coincide. The generalization of torsion complete groups is given by those p -groups G such that δ_λ sends G isomorphically upon $T_\lambda(G)$; these groups coincide, if $\lambda = \text{cf } \omega$, with the p^λ -high injective groups, which have been studied by Megibben [9] and Dubois [3].

It is natural to try to generalize the inductive topology from the ordinal ω to an arbitrary limit ordinal λ ; recall that, given an increasing sequence of ordinals and symbols $\infty: \mathbf{u} = (\sigma_n)_{n \in \mathbf{N}}$ ($\sigma < \infty$ for each ordinal σ and $\infty < \infty$), the subgroup of G

$$G(\mathbf{u}) = \{x \in G \mid h_\sigma(p^n x) \geq \sigma_n, \forall n \in \mathbf{N}\}$$

is a fully invariant subgroup of G . We define the λ -inductive topology on G by means of a basis \mathcal{B}_λ of neighborhoods of 0: \mathcal{B}_λ consists of those subgroups $G(\mathbf{u})$ defined above such that $\sigma_n < \lambda$ for every $n \in \mathbf{N}$. In section 2 we shall study some properties of the λ -inductive topology and we shall prove the analogue of Charle's result, i.e. that the completion $L_\lambda(G)$ of G in the λ -inductive topology is canonically isomorphic to $T_\lambda(G)$; it follows that, if $\lambda = \text{cf } \omega$, a reduced p -group is p^λ -high injective if and only if it is complete in the λ -inductive topology.

A more natural definition of λ -inductive topology would get as a basis of neighborhoods of 0 the family of λ -large subgroups of G , i.e. those fully invariant subgroups of G such that $G = L + B$ for every λ -basic subgroup B of G ; this definition makes sense only for those p -groups which have λ -basic subgroups; in the non trivial cases, these groups are, as Wallace proved in [12], the C_λ -groups with $\lambda = \text{cf } \omega$ (for the definitions of λ -basic subgroup and C_λ -group see Megibben [8] and Wallace [12]). In section 3 we shall prove that for a C_λ -group,

with $\lambda = \text{cf } \omega$, the λ -inductive topology has as a basis of neighborhoods of 0 the family of λ -large subgroups, by generalizing the result of Pierce which characterize large subgroups.

2. The λ -inductive topology.

All groups considered are abelian groups. Notation and terminology do not depart from those in [4]. In particular, if G is a reduced p -group, $l(G)$ denotes the length of G , i.e. the minimal ordinal σ such that $p^\sigma G = 0$; to every element $x \in G$ is associated an increasing sequence of ordinals and ∞ 's: $H(x) = \{h_\sigma(p^n x) | n \in \mathbb{N}\}$, which is said the indicator of x . Let λ be a limit ordinal; let \mathcal{B}_λ be the family of subgroups $G(\mathbf{u})$ of G , where $\mathbf{u} = (\sigma_n)_{n \in \mathbb{N}}$, such that $\sigma_n < \lambda$ for every $n \in \mathbb{N}$; if \mathbf{u} and \mathbf{v} are two such sequences of ordinals, the equality $G(\mathbf{u}) \cap G(\mathbf{v}) = G(\mathbf{u} \vee \mathbf{v})$, where $\mathbf{u} \vee \mathbf{v}$ denotes the pointwise supremum, ensures that \mathcal{B}_λ is a basis of neighborhoods of 0 for a topology, which is said the λ -inductive topology. We shall consider now some properties of the λ -inductive topology.

1) Every $G(\mathbf{u}) \in \mathcal{B}_\lambda$ is closed in the λ -adic topology.

It follows from the easily proved equality: $G(\mathbf{u}) = \bigcap_{\tau < \lambda} (p^\tau G + G(\mathbf{u}))$.

2) G is discrete in the λ -inductive topology if and only if $l(G) < \lambda$ and it is Hausdorff if and only if $l(G) \leq \lambda$.

G is discrete if and only if there exists $G(\mathbf{u}) \in \mathcal{B}_\lambda$ such that $G(\mathbf{u}) = 0$; this implies that $p_{\sigma_0} G[p] = 0$, where σ_0 is the first ordinal of \mathbf{u} , hence $p^{\sigma_0} G = 0$ and $l(G) \leq \sigma_0 < \lambda$. The viceversa is obvious. The second claim follows from the following property.

3) $p^\lambda G = \bigcap \{G(\mathbf{u}) | G(\mathbf{u}) \in \mathcal{B}_\lambda\}$.

If $\mathbf{u} = (\sigma_n)_{n \in \mathbb{N}}$ with $\sigma_n < \lambda$ for every $n \in \mathbb{N}$, then obviously $p^\lambda G \subseteq G(\mathbf{u})$, hence $p^\lambda G \subseteq \bigcap \{G(\mathbf{u}) | G(\mathbf{u}) \in \mathcal{B}_\lambda\}$; the opposite inclusion follows from the fact that, if $\sigma < \lambda$, then $p^\sigma G \in \mathcal{B}_\lambda$: in fact $p^\sigma G = G(\sigma, \sigma + 1, \sigma + 2, \dots)$.

4) The λ -adic topology is coarser than the λ -inductive topology; the two topologies coincide if $\lambda \neq \text{cf } \omega$.

We have already seen that, if $\sigma < \lambda$, then $p^\sigma G \in \mathcal{B}_\lambda$; if $\lambda \neq \text{cf } \omega$, given $G(\mathbf{u}) \in \mathcal{B}_\lambda$ let $\sigma = \sup \{\sigma_n | n \in \mathbb{N}\}$; then $\sigma < \lambda$ and $G(\mathbf{u}) \supseteq p^\sigma G$ imply that $G(\mathbf{u})$ is open in the λ -adic topology.

The above property, with the following theorem 2.1, clarify the role of the cofinality with ω of the ordinal λ in the λ -adic topology. Let now A be a subgroup of the reduced p -group G . We say that A satisfies the λ -th *Pierce's condition* if, for every $n \in \mathbb{N}$, there exists an ordinal $\tau_n < \lambda$ such that

$$(1) \quad p^{\tau_n} G[p^n] \subseteq A .$$

5) *A subgroup A of G is open in the λ -inductive topology if and only if it satisfies the λ -th Pierce's condition.*

If A is open there exists $G(\mathbf{u}) \in \mathfrak{B}_\lambda$, $\mathbf{u} = (\sigma_n)_{n \in \mathbb{N}}$, such that $G(\mathbf{u}) \subseteq A$; then A satisfies (1): just take $\tau_n = \sigma_n$. Viceversa, let $\sigma_n = \tau_{n+1} + n$ for each $n \in \mathbb{N}$. If $\mathbf{u} = (\sigma_n)_{n \in \mathbb{N}}$ and $x \in G(\mathbf{u}) \cap G[p]$, then $px = 0$ and $h_\alpha(x) \geq \sigma_0 = \tau_1$ imply $x \in A$; now we suppose by induction that, given $k \geq 1$, $G(\mathbf{u}) \cap G[p^{k-1}] \subseteq A$ and let $p^k x = 0$, $x \in G(\mathbf{u})$; then $h_\alpha(p^{k-1}x) \geq \sigma_{k-1} = \tau_k + k - 1$, therefore there exists $y \in G$ such that $p^{k-1}x = p^{k-1}y$ and $h_\alpha(y) \geq \tau_k$; it follows that $y \in A$ and $h_\alpha(p^n(x-y)) \geq \inf \{ \sigma_n, \tau_k + n \} = \sigma_n$ for $n = 0, 1, \dots, k-2$; hence $x-y \in G(\mathbf{u}) \cap G[p^{k-1}] \subseteq A$ and $x \in A$.

6) *The λ -inductive topology on $G/p^\lambda G$ coincides with the quotient topology of the λ -inductive topology on G .*

It follows easily from the equality $G(\mathbf{u})/p^\lambda G = (G/p^\lambda G)(\mathbf{u})$, which holds if $\mathbf{u} = (\sigma_n)_{n \in \mathbb{N}}$ with $\sigma_n < \lambda$ for every $n \in \mathbb{N}$.

Let now $I_\lambda(G)$ be the Hausdorff completion of G in the λ -inductive topology; if $l(G) < \lambda$, then $I_\lambda(G) = G$; if $l(G) > \lambda$, from property 6 it follows that $I_\lambda(G) = I_\lambda(G/p^\lambda G)$; therefore we can suppose $l(G) = \lambda$. Recall that $I_\lambda(G) = \varprojlim_{G(\mathbf{u}) \in \mathfrak{B}_\lambda} G/G(\mathbf{u})$ and $L_\lambda(G) = \varprojlim_{\sigma < \lambda} G/p^\sigma G$; but $p^\sigma G \in \mathfrak{B}_\lambda$ if $\sigma < \lambda$, therefore we can define a homomorphism

$$\varphi: I_\lambda(G) \rightarrow L_\lambda(G)$$

in the following way: if $(g_u + G(\mathbf{u}))_u \in I_\lambda(G)$, we consider only the cosets modulo the subgroups $p^\sigma G$, where $\sigma < \lambda$, denote them by $g_\sigma + p^\sigma G$ and put

$$(2) \quad \varphi((g_u + G(\mathbf{u}))_u) = (g_\sigma + p^\sigma G)_\sigma;$$

with this notation we can prove the following theorem.

THEOREM 2.1. *The homomorphism φ sends $I_\lambda(G)$ isomorphically upon $T_\lambda(G)$.*

PROOF. We prove first that, if in (2) $g_\sigma \in p^\sigma G$ for each $\sigma < \lambda$ and if $G(\mathbf{u}) \in \mathcal{B}_\lambda$, then $g_{\mathbf{u}} \in G(\mathbf{u})$, hence φ is injective. Let τ be an arbitrary ordinal less than λ and $\mathbf{v} = p^\tau G = (\tau, \tau + 1, \tau + 2, \dots)$; put $\mathbf{w} = \mathbf{u} \vee \mathbf{v}$; then $g_{\mathbf{w}} - g_{\mathbf{v}} \in p^\tau G$, therefore $g_{\mathbf{w}} \in p^\tau G$; it follows that $g_{\mathbf{u}} \in p^\tau G + G(\mathbf{u})$, for every $\tau < \lambda$; by property 1) $g_{\mathbf{u}} \in G(\mathbf{u})$. We shall prove now that $I_\lambda(G)$ is a p -group; let exist, by way of contradiction, $(g_{\mathbf{u}} + G(\mathbf{u}))_{\mathbf{u}} \in I_\lambda(G)$ such that, for a suitable increasing sequence of indexes: $\mathbf{u}_1 < \mathbf{u}_2 < \mathbf{u}_3 < \dots$ the orders p^{m_k} of $g_{\mathbf{u}_k} + G(\mathbf{u}_k)$ give an increasing sequence of positive integers; we can suppose that $m_{k+1} \geq m_k + k$ for every $k \in \mathbb{N}$. Let \mathbf{u}_k be fixed; then $p^{m_k-1}g_{\mathbf{u}_k} + G(\mathbf{u}_k)$ has order p and, if $G(\mathbf{u}) \subseteq G(\mathbf{u}_k)$, then $p^{m_k-1}g_{\mathbf{u}} + G(\mathbf{u}_k)$ has also order p . Then $p^{m_k-1}g_{\mathbf{u}_k} \notin G(\mathbf{u}_k)$, therefore, by property 1), there exists $\tau_k < \lambda$ such that

$$\tau_k = \sup \{h_\sigma(p^{m_k-1}g_{\mathbf{u}}) | \mathbf{u} \geq \mathbf{u}_k, G(\mathbf{u}) \in \mathcal{B}_\lambda\}.$$

For every $k \in \mathbb{N}$ let ϱ_k be ordinals such that $\tau_{k+1} < \varrho_k < \lambda$ and $\varrho_k < \varrho_{k+1}$; consider now $G(\mathbf{v}) \in \mathcal{B}_\lambda$ where $\mathbf{v} = (\varrho_n)_{n \in \mathbb{N}}$. If $g_{\mathbf{v}} + G(\mathbf{v})$ has order p^n , then $p^n g_{\mathbf{u}} \in G(\mathbf{v})$ for every $\mathbf{u} \geq \mathbf{v}$ with $G(\mathbf{u}) \in \mathcal{B}_\lambda$. Let $r \in \mathbb{N}$ be such that $m_r \geq n$. If $\mathbf{u} \geq \mathbf{v} \vee \mathbf{u}_{k+1}$ and $G(\mathbf{u}) \in \mathcal{B}_\lambda$, then $p^{m_r}g_{\mathbf{u}} \in G(\mathbf{v})$ implies

$$h_\sigma(p^{m_{r+1}-1}g_{\mathbf{u}}) \geq \varrho_{m_{r+1}-m_r-1};$$

but $m_{r+1} - m_r - 1 \geq r$ implies $\varrho_{m_{r+1}-m_r-1} \geq \varrho_r > \tau_{r+1}$ and $\mathbf{u} \geq \mathbf{u}_{r+1}$ implies

$$h_\sigma(p^{m_{r+1}-1}g_{\mathbf{u}}) \leq \tau_{r+1}$$

which is absurd. From the above proof we get the inclusion $\varphi(I_\lambda(G)) \subseteq T_\lambda(G)$; let now $(g'_\sigma + p^\sigma G)_\sigma \in T_\lambda(G)$ be an element of order m ; if $G(\mathbf{u}) \in \mathcal{B}_\lambda$ let $g_{\mathbf{u}} = g'_{\sigma_m}$; it is easy to see that $\varphi((g_{\mathbf{u}} + G(\mathbf{u}))_{\mathbf{u}}) = (g'_\sigma + p^\sigma G)_\sigma$ and that $(g_{\mathbf{u}} + G(\mathbf{u}))_{\mathbf{u}} \in I_\lambda(G)$, therefore the image of φ is the whole of $T_\lambda(G)$.

An obvious consequence of theorem 2.1 is the following

COROLLARY 2.2. *A reduced p -group G is complete in the λ -inductive topology if and only if $p^\lambda G = 0$ and $\delta_\lambda(G) = T_\lambda(G)$.*

By property 4) we are interested in the λ -inductive topology on a reduced p -group only if $\lambda = \text{cf } \omega$; in this case the λ -adic topology

on $I_\lambda(G) \cong T_\lambda(G)$ coincides with the topology of the completion of G endowed with the λ -adic topology, hence the subgroups $p^\sigma I_\lambda(G)$ with $\sigma < \lambda$, are open in the topology of the completion of G endowed with the λ -inductive topology; it follows that, if $G(\mathbf{u}) \in \mathcal{B}_\lambda$, then $I_\lambda(G)(\mathbf{u})$ is closed in the above topology, hence it contains the closure of $\delta_\lambda(G(\mathbf{u}))$ which is open; therefore $I_\lambda(G)(\mathbf{u})$ is open in the topology of the completion of G endowed with the λ -inductive topology. A subbase of neighborhoods of 0 for this topology is given by the subgroups $I_\lambda(G) \cap \text{Ker } \pi_{\mathbf{u}}$, where $G(\mathbf{u}) \in \mathcal{B}_\lambda$ and

$$\pi_{\mathbf{u}}: \prod_{G(\mathbf{v}) \in \mathcal{B}_\lambda} G/G(\mathbf{v}) \rightarrow G/G(\mathbf{u}),$$

is the canonical surjection; we shall prove that $I_\lambda(G)(\mathbf{u})$ is contained in $I_\lambda(G) \cap \text{Ker } \pi_{\mathbf{u}}$; let $(g_{\mathbf{v}} + G(\mathbf{v}))_{\mathbf{v}} \in I_\lambda(G)(\mathbf{u}) \subseteq \prod_{G(\mathbf{v}) \in \mathcal{B}_\lambda} (G/G(\mathbf{v}))(\mathbf{u})$, let r

be the order of $g_{\mathbf{u}}$ and, if $\mathbf{u} = (\sigma_n)_{n \in \mathbb{N}}$, put $\mathbf{w} = (\sigma_r, \sigma_{r+1}, \sigma_{r+2}, \dots)$. If $\sigma < \sigma_r$ we have; $p^\sigma(G/G(\mathbf{w})) = p^\sigma G/G(\mathbf{w})$, therefore, for $i = 0, 1, \dots, \dots, r-1$, the following inequalities hold: $h_G(p^i g_{\mathbf{w}}) \geq \sigma_i$. But $\mathbf{w} \geq \mathbf{u}$ implies $g_{\mathbf{w}} - g_{\mathbf{u}} \in G(\mathbf{u})$, hence $h_G(p^i g_{\mathbf{u}}) \geq \sigma_i$ for $i = 0, 1, \dots, r-1$, which means that $g_{\mathbf{u}} \in G(\mathbf{u})$. Thus we have proved the following

THEOREM 2.3. *Let G be a p -group and λ a limit ordinal cofinal with ω . Then the topology of the completion (of G endowed with the λ -inductive topology) on $I_\lambda(G)$ coincides with its own λ -inductive topology.*

3. λ -large subgroups of C_λ -groups.

A p -group G is a C_λ -group, where λ is a limit ordinal, if $G/p^\sigma G$ is totally projective for every ordinal $\sigma < \lambda$. The C_λ -groups have been introduced by Megibben [8] with the restriction on λ to be countable; Wallace studied [12] the general case; in particular, he proved that a reduced p -group G of length $\geq \lambda$ contains a proper λ -basic subgroup B if and only if G is a C_λ -group and $\lambda = \text{cf } \omega$ (recall that B is a λ -basic subgroup of G if it is totally projective of length $\leq \lambda$ and isotype and dense in the λ -adic topology in G). We are interested on groups with proper λ -basic subgroups, therefore from now on G will denote a C_λ -group with $\lambda = \text{cf } \omega$. A subgroup L of G is said λ -large if it is fully invariant and $G = L + B$ for every λ -basic subgroup B of G ; this definition has been introduced by Linton (see [6]). If $l(G) < \lambda$, then

the unique λ -basic subgroup of G is G itself and every fully invariant subgroup is λ -large; to exclude this trivial case, we suppose that $l(G) \geq \lambda$. If L is λ -large in G , then $p^n L$ is also λ -large for every $n \in \mathbb{N}$.

LEMMA 3.1. *Let L be a λ -large subgroup of the C_λ -group G , with $\lambda = \text{cf } \omega$. Then $p^\lambda G$ is contained in L .*

PROOF. We can suppose $l(G) > \lambda$. Let $x \in p^\lambda G$; if $G = L + B$ for a λ -basic subgroup B of G , then $x = b + y$, with $b \in B$, $y \in L$. There exists a direct summand B' of B such that $b \in B'$ and $l(B') < \lambda$. By [1, 2.3] we have: $G = B' \oplus G'$; if π is the projection of G upon G' with respect to this decomposition, then $\pi(x) = \pi(y) \in L$ because L is fully invariant, and $(1 - \pi)(x) = 0$, because $h_a(x) = h_B(x) \geq \lambda$; it follows that $x = \pi(y) \in L$.

We will need in the following some results on C_λ -groups.

LEMMA 3.2. *Let G be a C_λ -group, $\lambda = \text{cf } \omega$; let $x \in G$ such that $\langle x \rangle \cap p^\lambda G = 0$; then there exists a direct summand X of G containing x such that $l(X) < \lambda$.*

PROOF. The lemma is proved in [1, 2.7] if $l(G) = \lambda$; if $l(G) > \lambda$ the proof is quite similar and it is omitted.

The fact that every C_λ -group of length $\leq \lambda$ is fully transitive, which is proved by Le Borgne [5] (see also Linton [6]), is generalized in the following

LEMMA 3.3. *Let G be a C_λ -group, $\lambda = \text{cf } \omega$. Let $x, y \in G$ such that $H(x) \leq H(y)$ and $\langle x \rangle \cap p^\lambda G = 0 = \langle y \rangle \cap p^\lambda G$. Then there exists an endomorphism $f \in \text{End}(G)$ such that $f(x) = y$.*

PROOF. From lemma 3.2 one easily deduces a decomposition $G = X \oplus G'$, with $x, y \in X$ and $l(X) < \lambda$; since X is totally projective, hence fully transitive, the claim is now obvious.

LEMMA 3.4. *Let C be a C_λ -group, $\lambda = \text{cf } \omega$, and L a fully invariant subgroup of G such that $p^\lambda G \subseteq p^\omega L$. If $\sigma_n = \inf \{h_a(p^n x) | x \in L\} < \lambda$ for every $n \in \mathbb{N}$, there exists, for each $r \in \mathbb{N}$, an element $g \in L$ such that $h_a(p^i g) = \sigma_i$ for $i = 0, 1, \dots, r$.*

PROOF. We induct on r . If $r = 0$ the claim is obvious; suppose $r \geq 1$; there exists an element $g' \in L$ such that

$$H(g') = (\sigma_0, \sigma_1, \dots, \sigma_{r-1}, \tau_r, \tau_{r+1}, \dots)$$

where we can suppose $\tau_r > \sigma_r$, otherwise g' is the wanted element. There exists an element $t \in L$ such that $h_g(p^r t) = \sigma_r$. Let $k(t)$ be the minimal non negative integer such that $h_g(p^i t) = \sigma_i$ if $k(t) \leq i \leq r$. We choose $t \in L$ such that $k(t)$ is minimal; let us observe that $k(t) \leq r-1$ (otherwise consider $t + g'$). Our goal is to prove that $k(t) = 0$. Among the elements t with $k(t)$ minimal, choose one of minimal order $m+1$ ($m \geq r$); then $h_g(p^m t) < \lambda$ otherwise $p^m t \in p^\lambda G \subseteq p^\omega L$, hence $p^m t = p^{m+1} t'$ for a $t' \in L$; in this case $p^m(t - pt') = 0$ and $h_g(p^i(t - pt')) = \sigma_i$ if $k(t) \leq i \leq r$, because $h_g(p^i t) = \sigma_i$ and $h_g(p^{i+1} t') \geq \sigma_{i+1} > \sigma_i$. Thus, without loss of generality, we can assume, if $k = k(t)$:

$$H(t) = (\sigma'_0, \sigma'_1, \dots, \sigma'_{k-1}, \sigma_k, \sigma_{k+1}, \dots, \sigma_r, \sigma'_{r+1}, \dots, \sigma'_m, \infty, \dots)$$

where $\sigma'_{k-1} > \sigma_{k-1}$, $\sigma'_m < \lambda$ and $k \leq r-1$; suppose, by way of contradiction, that $k > 0$. There exist elements $b \in L$ such that $h_g(p^{k-1} b) = \sigma_{k-1}$ and $h_g(p^r b) = \sigma_r$ (consider for instance $t + g'$); choose b among these elements such that $k(b)$ is minimal; it is enough to prove that $k(b) \leq k-1$. Let, by absurd, $k(b) \geq k$ and put $k(b) = k'$. Then $h_g(p^{k'} b) = \sigma_{k'}$ and $h_g(p^{k'-1} b) > \sigma_{k'-1}$ imply that $\sigma_{k'} > \sigma_{k'-1} + 1$; therefore there exists an element $w \in G$ such that $p^{k'} t = p^{k'} w$ and $h_g(p^{k'-1} w) > \sigma_{k'-1}$; of course we can choose w in such a way that $H(w) \geq H(t)$; because of $\langle t \rangle \cap p^\lambda G = 0 = \langle w \rangle \cap p^\lambda G$, we can apply lemma 3.3 and deduce that $w \in L$. Then we have: $p^{k'}(t - w) = 0$ and $h_g(p^{k'-1}(t - w)) = \sigma_{k'-1}$. Consider now the element $b - t + w \in L$; then $k(b - t + w) < k(b) = k'$, because $h_g(p^{k'-1}(b - t + w)) = \sigma_{k'-1}$, which contradicts the choice of b .

We can now prove the main result of this section, which generalizes the characterization of large subgroups given by Pierce [11].

THEOREM 3.5. *Let G be a C_λ -group, $\lambda = \text{cf } \omega$, with $l(G) \geq \lambda$. Let L be a fully invariant subgroup of G . Then the following are equivalent:*

- 1) $L = G(\mathbf{u})$, where $\mathbf{u} = (\sigma_n)_{n \in \mathbb{N}}$ is an increasing sequence of ordinals and $\sigma_n < \lambda$ for every $n \in \mathbb{N}$.
- 2) L satisfies the λ -th Pierce's condition.
- 3) L is λ -large.

PROOF. 1) \rightarrow 2) See property 5) of section 2.

2) \rightarrow 3) Let B be a λ -basic subgroup of G and let $g \in G$ be an element of order $k \in \mathbb{N}$; there exists an ordinal $\sigma < \lambda$ such that $p^\sigma G[p^k] \subseteq L$;

but $G = p^\sigma G + B$, hence $g = g' + b$ with $g' \in p^\sigma G$, $b \in B$; then $0 = p^k g = p^k g' - p^k b$, and $p^k b \in p^{\sigma+k} G \cap B = p^{\sigma+k} B$; therefore there exists a $b' \in p^\sigma B$ such that $p^k b = p^k b'$; thus we obtain: $g = (g' + b') + (b - b')$ where $b - b' \in B$ and $p^k(b - b') = 0$; it follows that $g' + b' \in p^\sigma G[p^k] \subseteq L$ and $g \in L + B$.

3) \rightarrow 1) If $l(G) = \lambda$, then G is fully transitive, therefore every fully invariant subgroup L is equal to $G(\mathbf{u})$, for a suitable increasing sequence $\mathbf{u} = (\sigma_n)_{n \in \mathbb{N}}$ of ordinals and ∞ 's. If $\sigma_n \geq \lambda$ for some n , then $p^n L \subseteq p^\lambda G = 0$, which is absurd, because $p^n L$ is λ -large while 0 is not λ -large. Suppose now that $l(G) > \lambda$. For every $n \in \mathbb{N}$ we put:

$$\sigma_n = \inf \{h_\alpha(p^n x) \mid x \in L\};$$

obviously $\sigma_n < \sigma_{n+1}$ and $\sigma_n < \lambda$ for each n , otherwise $p^n L = p^\lambda G$, which is absurd because $p^\lambda G$ is not λ -large; the inclusion $L \subseteq G(\mathbf{u})$, where $\mathbf{u} = (\sigma_n)_{n \in \mathbb{N}}$ is obvious; we suppose by absurd that this inclusion is proper; let y be an element of minimal order among the elements of $G(\mathbf{u}) \setminus L$; if $H(y) = (\tau_0, \tau_1, \dots, \tau_k, \infty, \infty, \dots)$ with $\tau_k < \infty$, then $\tau_k < \lambda$: in fact, if $\tau_k \geq \lambda$, then $p^k y \in p^\lambda G$, which is contained in $\bigcap_{n \in \mathbb{N}} p^n L = p^\omega L$,

by lemma 3.1; therefore there exists an element $z \in L$ such that $p^k(y - z) = 0$ and $y - z \in G(\mathbf{u}) \setminus L$, contradicting the choice of y . Let now x be an element of minimal order among the elements of L such that $h_\alpha(p^i x) = \sigma_i$ for $i = 0, 1, \dots, k$; such an element does exist, as lemma 2.4 shows; let $H(x) = (\sigma_0, \sigma_1, \dots, \sigma_k, \sigma'_{k+1}, \dots, \sigma'_n, \infty, \dots)$, with $\sigma'_n < \infty$; then $\sigma'_n < \lambda$, otherwise, fixed an integer $m \geq 1$, there exists an element $a \in L$ such that $p^n x = p^{n+m} a$, which implies that $p^n(x - p^m a) = 0$; but if $i = 0, 1, \dots, k$, then $h_\alpha(p^i(x - p^m a)) = \inf \{\sigma_i, h_\alpha(p^{m+i} a)\} = \sigma_i$, contradicting the choice of x . We are now in the hypotheses of lemma 3.3, therefore there exists an endomorphism f of G such that $f(x) = y$; but $x \in L$ and L is fully invariant, hence $y \in L$ and $L = G(\mathbf{u})$.

Theorem 3.5 enables us to improve theorems 4 and 5 of [6] to C_λ -groups of arbitrary length.

COROLLARY 3.6. *Let G be a C_λ -group, $\lambda = \text{cf } \omega$, and $L = G(\mathbf{u})$ a λ -large subgroup, where $\mathbf{u} = (\sigma_n)_{n \in \mathbb{N}}$ and $\sigma_n < \lambda$ for each $n \in \mathbb{N}$. Then G/L is totally projective and, if $\sigma = \sup \sigma_n$ and $\lambda = \sigma + \tau$, then L is a $C_{\omega+\tau}$ -group.*

PROOF. If B is a λ -basic subgroup of G , the first fact follows from [6, Thm. 1] and the following isomorphism

$$G/L = G/G(\mathbf{u}) = G(\mathbf{u}) + B/G(\mathbf{u}) \cong B/G(\mathbf{u}) \cap B = B/B(\mathbf{u}).$$

It is easy to show that $p^\omega L = p^\sigma G$; then the second fact follows from the easily proved equalities, which hold for every ordinal ϱ

$$L/p^{\omega+\varrho} L = G(\mathbf{u})/p^{\omega+\varrho} G(\mathbf{u}) = G(\mathbf{u})/p^{\sigma+\varrho} G = (G/p^{\sigma+\varrho} G)(\mathbf{u}).$$

COROLLARY 3.7. *Let G be a C_λ -group, $\lambda = \text{cf } \omega$, and L a λ -large subgroup of G . Then L is totally projective if and only if G is totally projective.*

PROOF. The sufficiency follows from [6, Thm. 1]. The necessity follows from [7, Thm. 3], because G/L is totally projective by corollary 3.6.

Let us finally remark that, in the class of C_λ -groups, $\lambda = \text{cf } \omega$, two groups complete in the λ -inductive topology, i.e. two reduced p^λ -high injective groups, are isomorphic if and only if they have the same Ulm-Kaplansky invariants. This result has been proved by Megibben [8, Thm. 7] in the case of λ countable and by Crawley [1] in the general case; it has only to be pointed out that, if $\lambda = \text{cf } \omega$, the C_λ -groups torsion complete defined by Crawley are essentially the same as the reduced p^λ -high injective C_λ -groups.

Added in proof. While this paper was submitted for publication, appeared R. C. LINTON's paper: λ -large subgroups of C_λ -groups, in Pacific J. Math., **78**, no. 2 (1978), in which the equivalence of 1) and 3) of our theorem 3.5 and corollary 3.6 are proved. However his proof is quite different and is based on topological methods.

REFERENCES

- [1] P. CRAWLEY, *On the structure of a class of abelian p -groups*, Symposia Math., **20** (1978).
- [2] D. O. CUTLER - R. W. STRINGALL, *A topology for primary abelian groups*, Studies on Abelian Groups, Paris (1968), pp. 93-100.

- [3] P. F. DUBOIS, *Generally p^α -torsion complete abelian groups*, Trans. Amer. Math. Soc., **139** (1971), pp. 245-255.
- [4] L. FUCHS, *Infinite Abelian Groups*, vol. 2, New York-London, 1973.
- [5] M. M. LE BORGNE, *Groupes λ -separables*, C. R. Acad. Sci. Paris, **281** (1975), pp. 415-417.
- [6] R. C. LINTON, *On fully invariant subgroups of primary abelian groups*, Michigan Math. J., **22** (1975), pp. 281-284.
- [7] R. C. LINTON - C. MEGIBBEN, *Extensions of totally projective groups*, Proc. Amer. Math. Soc., **61-1** (1977), pp. 35-38.
- [8] C. MEGIBBEN, *A generalization of the classical theory of primary groups*, Tohoku Math. J., **22** (1970), pp. 347-356.
- [9] C. MEGIBBEN, *On p^α -high injectives*, Math. Z., **122** (1971), pp. 104-110.
- [10] R. MINES, *A family of functor defined on generalized primary groups*, Pacific J. Math., **26** (1968), pp. 349-360.
- [11] R. S. PIERCE, *Homomorphisms of primary groups*, Topics in Abelian Groups, Chicago (1963), pp. 215-310.
- [12] K. D. WALLACE, *C_λ -groups and λ -basic subgroups*, Pacific J. Math., **43** (1972), pp. 799-809.

Manoscritto pervenuto in redazione il 7 ottobre 1978.