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MARCO BIROLI

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problem for variational inequalities**

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An estimate on convergence for an homogeneisation problem for variational inequalities.

MARCO BIROLI (*)

1. - Introduction and results.

Let be $\Omega \subset R^n$ an open bounded set with smooth boundary Γ , $Y = \prod_{i=1}^n [0, y_i]$, $a_{ij}(y) \in L^\infty(Y)$ $i, j = 1, \dots, n$, such that

$$\sum_{i,j=1}^n a_{ij}(y) \xi_i \xi_j \geq \alpha |\xi|^2, \alpha > 0, a. e. \text{ in } Y$$

Let be $\forall u, v \in H_0^1(\Omega)$

$$\langle A^\varepsilon u, v \rangle = \int_{\Omega} \sum_{i,j=1}^n a_{ij} \left(\frac{x}{\varepsilon} \right) \frac{\partial u}{\partial x_j} (x) \frac{\partial v}{\partial x_i} (x) dx$$

(where \langle, \rangle is the duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ and $a_{ij}(y)$ are prolonged to R^n by periodicity) and

$$\langle A u, v \rangle = \int_{\Omega} \sum_{i,j=1}^n q_{ij} \frac{\partial u}{\partial x_j} (x) \frac{\partial v}{\partial x_i} (x) dx$$

where q_{ij} are constants defined as in [3], [9], [10].

(*) Indirizzo dell'A: Istituto di Matematica del Politecnico di Milano - Via Bonardi, 9 Milano.

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Let be $f \in H^{-1}(\Omega)$, we have :

Th. 1 — We consider the problems

$$(1,1_\varepsilon) \quad A^\varepsilon u = f$$

$$(1,1) \quad A u = f$$

let be u_ε and u the solution of (1, 1_ε) and (1, 1), we have

$$\lim_{\varepsilon \rightarrow 0}^* u_\varepsilon = u \text{ in } H_0^1(\Omega)$$

Moreover if $a_{ij}(y) \in C^1(R^n)$, $f \in W^{1,p}(\Omega)$, $n < p \leq +\infty$, we have

$$(1,2) \quad \|u_\varepsilon - u\|_{L^\infty} \leq C \|f\|_{W^{1,p}} \varepsilon.$$

The first part of Th. 1 is shown in [7] for the symmetric case and in [8], [12] in the general case by « local energy » method, the second part of Th. 1 is shown by « multiple scales » method in [8] [9] (*). We consider now the problems

$$(1,3_\varepsilon) \quad \begin{aligned} \langle A^\varepsilon u_\varepsilon, v - u_\varepsilon \rangle &\geq \langle f, v - u_\varepsilon \rangle \\ \forall v \in H_0^1(\Omega) &, \quad v \leq \psi \\ u \in H_0^1(\Omega) &, \quad u \geq \psi \end{aligned}$$

$$(1,3) \quad \begin{aligned} \langle A u, v - u \rangle &\geq \langle f, v - u \rangle \\ \forall v \in H_0^1(\Omega) &, \quad v \leq \psi \\ u \in H_0^1(\Omega) &, \quad u \leq \psi \end{aligned}$$

where ψ is a measurable function, such that $\{v \in H_0^1(\Omega), v \leq \psi\} \neq \emptyset$.

Th. 2 Let be u_ε and u solution of (1,3_ε) and (1,3) we have

$$\lim_{\varepsilon \rightarrow 0}^* u_\varepsilon = u \text{ in } H_0^1(\Omega)$$

Moreover if $\psi \in C(\bar{\Omega})$, $f \in L^\infty(\Omega)$

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon = u \text{ in } C(\bar{\Omega}).$$

(*) Cf. also [1], [2].

The first part of Th. 2 is shown in [11] by the method of « local energy » (cfr. also [5] and [6]), the second part of Th. 2 is shown in [4] by penalisation method and Hölder-estimates. The aim of this work is show an estimate analogous to (1, 2) in the case of variational inequalities.

Th. 3. Let be $a_{ij}(y) \in C^1(\mathbb{R}^n)$, $f \in L^\infty(\Omega)$, $u_\varepsilon(u)$ the solution

of (1, 3_ε) ((1, 3)), α the De Giorgi's exponent, $n \geq 2$

(I) If $\|A^\varepsilon \psi\|_{L^\infty}, \|A \psi\|_{L^\infty} \leq C$, we have

$$\|u_\varepsilon - u\|_{L^\infty} \leq K \varepsilon^{\frac{\alpha}{n-2+3\alpha}}$$

(II) If $\psi \in W^{1,p}(\Omega)$, $n < p < +\infty$, we have

$$\|u_\varepsilon - u\|_{L^\infty} \leq K \varepsilon^{\frac{\alpha}{n-2+3\alpha}}$$

The Th. 2 is shown by penalisation and « multiple scales » method. In § 2 we show some preliminary results, in § 3 we study the convergence for the penalised equations and in § 4 we give the proof of Th. 3.

Rem. 1 (a) The hypothesis of the part (I) are verified if $\psi = 0$:

(b) The result of Th. 3 can be generalized to the case $a_{ij}\left(x, \frac{x}{\varepsilon}\right)$

(c) We don't know if the estimates in Th. 3 are optimal ;

2. - Some preliminary results.

We consider the penalised problem

$$(2,1) \quad A u + \frac{1}{\lambda} (u - \psi)^+ = f, \quad \lambda > 0.$$

LEMMA 1. *The problem (2.1) has a unique solution u^λ ; if $A \psi \in L^\infty(\Omega)$ we have*

$$(2,2) \quad \frac{1}{\lambda} \|(u^\lambda - \psi)^+\|_{L^\infty} \leq \|f\|_{L^\infty} + \|A\psi\|_{L^\infty}$$

$$(2,3) \quad \|A u^\lambda\|_{L^\infty} \leq 2 \|f\|_{L^\infty} + \|A\psi\|_{L^\infty}$$

From (2,1) we have

$$\frac{1}{\lambda^P} \|(u^\lambda - \psi)^+\|_{L^p}^P \leq \frac{1}{\lambda^{P-1}} \|(u^\lambda - \psi)^+\|_{L^p}^{P-1} (\|f\|_{L^p} + \|A\psi\|_{L^p})$$

then

$$(2,4) \quad \frac{1}{\lambda} \|(u^\lambda - \psi)^+\|_{L^p} \leq C_1 + C_2 \|A \psi\|_{L^p}$$

$$(2,5) \quad \|A u^\lambda\|_{L^p} \leq C_1 + C_2 \|A \psi\|_{L^p}$$

From (2,5) we have $u^\lambda \in C(\bar{\Omega})$.

We show now (2,2) « ab absurdum ».

We write

$$(2,6) \quad K = \|f\|_{L^\infty} + \|A \psi\|_{L^\infty}.$$

Let be

$$\Omega' = \left\{ x \mid \frac{1}{\lambda} (u^\lambda - \psi)^+ > K \right\} \neq \emptyset$$

(2,7)

$$\Omega'_\eta = \left\{ x \mid \frac{1}{\lambda} (u^\lambda - \psi)^+ > K + \eta \right\} \neq \emptyset, \quad \eta > 0$$

Ω' and Ω'_η are open set and $\Omega'_\eta \subset \bar{\Omega}'_\eta \subset \Omega'$.

Let be now Ω'' an open set with smooth boundary such that $\bar{\Omega}'_\eta \subset \Omega'' \subset \bar{\Omega}'' \subset \Omega'$.

We have

$$A(u^\lambda - \psi) < 0 \text{ in } \Omega''$$

then from the maximum principle, [13],

$$(2,8) \quad \frac{1}{\lambda} (u^\lambda - \psi)^+ \leq K + \eta \text{ in } \Omega'' \supset \Omega'_\eta$$

From (2,8) we have

$$(2,9) \quad \Omega'_\eta = \emptyset ,$$

then from (2,7) (2,9) we have a contradiction.

From (2,1) (2,2) we have also (2,3).

LEMMA 2. *If $f \in L^\infty(\Omega)$, $\psi \in W^{1,p}(\Omega)$, $n < p \leq +\infty$, we have*

$$(2,10) \quad \|u^\lambda - u\|_{L^\infty} \leq K \lambda^{\frac{\alpha}{n-2(1-\alpha)}}$$

Moreover if $A\psi \in L^\infty(\Omega)$ we have

$$(2,11) \quad \|u^\lambda - u\|_{L^\infty} \leq K \lambda^{\frac{\alpha}{n-2(1-\alpha)}}$$

We show at first (2.11).

We observe that if $A\psi \in L^\infty(\Omega)$ from the lemma 1 we have $\|A u^\lambda\|_{L^\infty}, \|A u\|_{L^\infty} \leq \tilde{C}$ and repeating the calculation of [6] pg. 76-88 or [13] 177-197 we have

$$\|u^\lambda\|_{C^\alpha}, \|u\|_{C^\alpha} \leq C$$

where C depend only on \tilde{C} .

From (1,6_e) (2,1) we have

$$\langle A u^\lambda, u - u^\lambda \rangle \geq \langle f, u - u^\lambda \rangle$$

$$\langle A u, u^\lambda - (u^\lambda - \psi)^+ - u \rangle \geq \langle f, u^\lambda - (u^\lambda - \psi)^+ - u \rangle$$

then

$$(2,12) \quad \|u^\lambda - u\|_{H_0^1} \leq \lambda^{1/2} (C_1 + C_2 \|A\psi\|_{L^\infty})$$

From (2,12) we have

$$(2,13) \quad \|u^\lambda - u\|_{L^{\frac{2n}{n-2}}} \leq \lambda^{1/2} (C'_1 + C'_2 \|A\psi\|_{L^\infty}).$$

We prolongate u^λ and u by 0 and we denote again by u^λ and u these new functions.

From (2,13) we have

$$\int_{B(\bar{x}, \varrho)} |u^\lambda(x) - u(x)| dx \leq \lambda^{1/2} (C'_1 + C'_2 \|A\psi\|_{L^\infty}) |B(\bar{x}, \varrho)|^{1 - \frac{n-2}{2n}}$$

then

$$\begin{aligned} \frac{1}{|B(\bar{x}, \varrho)|} \int_{B(\bar{x}, \varrho)} |u^\lambda(x) - u(x)| dx &\leq \lambda^{1/2} (C'_1 + \\ &+ C'_2 \|A\psi\|_{L^\infty}) |B(\bar{x}, \varrho)|^{-\frac{n-2}{2n}} \leq \lambda^{1/2} (C''_1 + C''_2 \|A\psi\|_{L^\infty}) \varrho^{-\frac{n-2}{2n}} \end{aligned}$$

then there is $\tilde{x} \in B(\bar{x}, \varrho)$ such that

$$(2,14) \quad |u^\lambda(\tilde{x}) - u(\tilde{x})| \leq \lambda^{1/2} (C''_1 + C''_2 \|A\psi\|_{L^\infty}) \varrho^{-\frac{n-2}{2n}}.$$

From (2,14) we have

$$(2,15) \quad |u^\lambda(x) - u(x)| \leq \lambda^{1/2} (C''_1 + C''_2 \|A\psi\|_{L^\infty}) \varrho^{-\frac{n-2}{2n}} + C \varrho^\alpha$$

for $x \in B(\bar{x}, \varrho)$.

We choose $\varrho = \frac{1}{\lambda^{n-(1-\alpha)}}$ then

$$(2,16) \quad |u^\lambda(x) - u(x)| \leq \overline{\lambda^{n-2(1-\alpha)}} (C''_1 + C''_2 \|A\psi\|_{L^\infty}) + C \lambda^{\frac{\alpha}{n-2(1-\alpha)}}.$$

We show now (2,10).

From the lemma 2.1 of [7] (the hypothesis of symmetry is unnecessary for this lemma) if $\psi \in W^{1,p}(\Omega)$, $n < p \leq +\infty$, there is a sequence $\{\psi_\eta\}$ such that

$$(2,17) \quad \|A \psi_\eta\|_{L^\infty} \leq C_3 \eta^{-1/2}$$

$$(2,18) \quad \|\psi_\eta - \psi\|_{L^\infty} \leq C_3 \eta^{1/2}$$

where C_3 is a constant dependent only on $\|\psi\|_{W^{1,p}}$.

Let be $u_\eta^\lambda (u_\eta)$ the solution of the problem

$$(2,19) \quad A u_\eta^\lambda + \frac{1}{\lambda} (u_\eta^\lambda - \psi_\eta)^+ = f$$

$$(2,20) \quad \begin{aligned} \langle A u_\eta, v - u_\eta \rangle &\geq \langle f, v - u_\eta \rangle \\ \forall v \in H_0^1(\Omega), v &\leq \psi_\eta \\ u_\eta &\in H_0^1(\Omega), u_\eta \leq \psi_\eta. \end{aligned}$$

From [12] pg. 22, part III, we have

$$(2,21) \quad \|u_\eta^\lambda - u^\lambda\|_{L^\infty} \leq C_5 \eta^{1/2}$$

$$(2,22) \quad \|u_\eta - u\|_{L^\infty} \leq C_5 \eta^{1/2}.$$

From (2,16) (2,21) (2,22) we have

$$\begin{aligned} |u^\lambda(x) - u(x)| &\leq |u^\lambda(x) - u_\eta^\lambda(x)| + |u_\eta^\lambda(x) - u_\eta(x)| + |u_\eta(x) - u(x)| \\ &\leq \lambda^{\frac{\alpha}{n-2(1-\alpha)}} (C_1'' + C_2'' \eta^{-1/2}) + C \lambda^{\frac{\alpha}{n-2(1-\alpha)}} + 2 \eta^{1/2} \end{aligned}$$

We choose $\eta = \lambda^{\frac{\alpha}{2[n-2(1-\alpha)]}}$ then

$$|u^\lambda(x) - u(x)| \leq C_6 \lambda^{\frac{\alpha}{2[n-2(1-\alpha)]}}$$

Rem. 1 — We observe that the constants in the lemma 2 depend only on L^∞ — norm of the coefficients of A .

Lemma 3. *We consider the problems*

$$A^\varepsilon u_\varepsilon = f, \quad A u = f$$

for $f \in L^\infty(\Omega)$ we have

$$\|u_\varepsilon - u\|_{L^\infty} \leq C \varepsilon^{1/2}.$$

We consider $\theta = -\Delta^{-1} f$, $\theta \in W^{2,p}(\Omega)$. $\forall p$, for the lemma 2.1 of [7] there is a sequence θ_η such that

$$\|\theta_\eta - \theta\|_{W^{1,\infty}} \leq C_1 \eta^{1/2}$$

$$\|\Delta \theta_\eta\|_{W^{1,\infty}} \leq C_1 \eta^{-1/2}.$$

Let be $f_\eta = -\Delta \theta_\eta$: we have

$$\|f_\eta - f\|_{W^{-1,\infty}} \leq C_2 \eta^{1/2}$$

$$\|f_\eta\|_{W^{1,\infty}} \leq C_2 \eta^{-1/2}.$$

Let be $u_\varepsilon^\eta (u^\eta)$ the solution of the problem

$$A^\varepsilon u_\varepsilon^\eta = f_\eta \quad (A u^\eta = f_\eta).$$

We have

$$\begin{aligned} \|u_\varepsilon - u\|_{L^\infty} &\leq \|u_\varepsilon - u_\varepsilon^\eta\|_{L^\infty} + \|u_\varepsilon^\eta - u^\eta\|_{L^\infty} + \|u^\eta - u\|_{L^\infty} \leq \\ &\leq 2 C_1' \eta^{1/2} + C_2' \varepsilon \eta^{-1/2} \end{aligned}$$

We choose $\eta = \varepsilon$ and we have the result.

3. - The penalised equation.

We suppose for the moment $f = 0$.

We consider the penalised problems

$$(3,1_\varepsilon) \quad A^\varepsilon u_\varepsilon^\lambda + \frac{1}{\lambda} (u_\varepsilon^\lambda - \psi)^+ = 0$$

$$(3.1) \quad A u^\lambda + \frac{1}{\lambda} (u^\lambda - \psi)^+ = 0.$$

Lemma 4. *We have*

$$\|u_\varepsilon^\lambda - u^\lambda\|_{L^\infty} \leq (K_1 + K_2 \lambda^{-1}) \|A \psi\|_{L^p} \varepsilon$$

$$n < p < +\infty.$$

We use for the proof the « multiple scales » method.

Let be $y = \frac{x}{\varepsilon}$ we have, [8] [9],

$$(3.2) \quad A^\varepsilon = \varepsilon^{-2} A_1 + \varepsilon^{-1} A_2 + A_3$$

where

$$A_1 = - \sum_{ij=1}^n a_{ij}(y) \frac{\partial^2}{\partial y_i \partial y_j} - \sum_{ij=1}^n \left(\frac{\partial}{\partial y_i} a_{ij}(y) \right) \frac{\partial}{\partial y_j} =$$

$$= - \sum_{ij=1}^n \frac{\partial}{\partial y_i} \left(a_{ij}(y) \frac{\partial}{\partial y_j} \right)$$

$$A_2 = - \sum_{ij=1}^n a_{ij}(y) \left(\frac{\partial^2}{\partial x_i \partial y_j} + \frac{\partial^2}{\partial x_j \partial y_i} \right) - \sum_{ij=1}^n \left(\frac{\partial}{\partial y_i} a_{ij}(y) \right) \frac{\partial}{\partial x_j} =$$

$$= - \sum_{i,j=1}^n \frac{\partial}{\partial y_j} \left(a_{ij}(y) \frac{\partial}{\partial x_i} \right) - \sum_{i,j=1}^n a_{ij}(y) \frac{\partial^2}{\partial x_i \partial y_j}$$

$$A_3 = - \sum_{i,j=1}^n a_{ij}(y) \frac{\partial^2}{\partial x_i \partial x_j}$$

Let be

$$A_1 \chi^j(y) = \sum_{i=1}^n \frac{\partial}{\partial y_i} (a_{ij}(y))$$

$$\chi^j(y) \text{ Y-periodic, } \int_{\gamma} \chi^j(y) dy = 0$$

we have $\chi^j(y) \in C^2(\bar{\Omega})$.

Let be

$$\chi_1(x; y) = \sum_{i=1}^n \chi^i(y) \frac{\partial u^\lambda}{\partial x_i}; \quad v_\varepsilon = u^\lambda + \varepsilon \chi_1 + \varepsilon^2 \chi_2$$

where $\chi_2(x; y)$ is not given for the moment.

We consider $w = u_\varepsilon^\lambda - v_\varepsilon$,

$$\begin{aligned} A^\varepsilon(w) + \frac{1}{\lambda} (u_\varepsilon^\lambda - \psi)^+ - \frac{1}{\lambda} (u^\lambda + \varepsilon \chi_1 + \varepsilon^2 \chi_2 - \psi)^+ &= \\ &= - (A_1 \chi_2 + A_2 \chi_1 + A_3 u^\lambda + \frac{1}{\lambda} (u^\lambda - \lambda)^+) - \\ &\quad - \varepsilon (A_3 \chi_1 + A_2 \chi_2) - \varepsilon^2 A_3 \chi_2 + \frac{1}{\lambda} (u^\lambda - \psi)^+ \\ &\quad - \frac{1}{\lambda} (u^\lambda + \varepsilon \chi_1 + \varepsilon^2 \chi_2 - \psi)^+ \end{aligned}$$

We choose $\chi_2(x; y)$ such that

$$(3,4) \quad A_1 \chi_2 + A_2 \chi_1 + A_3 u^\lambda + \frac{1}{\lambda} (u^\lambda - \psi)^+ = 0$$

$\chi_2(x; y)$ Y -periodic in y .

We observe that we have

$$\int_Y \left(A_2 \chi_1 + A_3 u^\lambda + \frac{1}{\lambda} (u^\lambda - \psi)^+ \right) dy = 0$$

then the problem (3,4) has a solution.

We observe that

$$\begin{aligned} (3,5) \quad \|A \psi\|_{L^p} \leq C_1 &\Rightarrow \frac{1}{\lambda} \|(u^\lambda - \psi)^+\|_{L^p} \leq C_1 \Rightarrow \\ &\Rightarrow \|u^\lambda\|_{w^2, P} \leq C_2 \cdot C_1 \Rightarrow \frac{1}{\lambda} \|(u^\lambda - \psi)^+\|_{w^1, P} \leq \frac{C_3 C_1}{\lambda} \\ &\Rightarrow \|u^\lambda\|_{w^3, P} \leq \frac{C_3 \cdot C_1}{\lambda} \quad (n < p \leq +\infty). \end{aligned}$$

From (3,5) we have

$$\|\chi_1(x; y)\|_{c^1(Y; w^2, P)} \leq \frac{C_5}{\lambda} \|A\psi\|_{L^P}$$

then

$$(3,6) \quad \|A_3 \chi_1(x; y)\|_{c(Y; L^P)} \leq \frac{C_5}{\lambda} \|A\psi\|_{L^P}$$

$$(3,7) \quad \|A_2 \chi_1(x; y)\|_{c(Y; W^1, P)} \leq \frac{C_7}{\lambda} \|A\psi\|_{L^P}.$$

From (3,4) (3,5) (3,7) we have

$$(3,8) \quad \|\chi_2(x; y)\|_{c^2(Y; w^1, P)} \leq \frac{C_8}{\lambda} \|A\psi\|_{L^P}$$

and, by multiplication for $v \in W^{1, p'}(\Omega)$ $1/p + 1/p' = 1$, we can easily show

$$(3,9) \quad \|\varepsilon A_3 \chi_2\left(x; \frac{x}{\varepsilon}\right)\|_{w^{-1, P}} \leq \frac{C_9}{\lambda} \|A\psi\|_{L^P}.$$

We observe now that

$$(3,10) \quad \begin{aligned} \left\| \frac{1}{\lambda} (w^\lambda - \psi)^+ - \frac{1}{\lambda} (w^\lambda + \varepsilon \chi_1 + \varepsilon^2 \chi_2 - \psi)^+ \right\|_{L^\infty} &\leq \\ &\leq \frac{\varepsilon}{\lambda} \|\chi_1\|_{L^\infty} + \frac{\varepsilon^2}{\lambda} \|\chi_2\|_{L^\infty} \\ &\leq C_{10} \left(\frac{\varepsilon + \varepsilon^2}{\lambda} \right) \|A\psi\|_{L^\infty} \end{aligned}$$

From (3,6) (3,7) (3,8) (3,9) (3,10) we have

$$(3,11) \quad A^\varepsilon(w) + \frac{1}{\lambda} (w^\lambda - \psi)^+ - \frac{1}{\lambda} (w^\lambda + \varepsilon \chi_1 + \varepsilon^2 \chi_2 - \psi)^+ = \theta$$

where

$$\|\theta\|_{w^{-1},P} \leq \left(C_{11} + \frac{C_{12}}{\lambda} \right) \|A\psi\|_{L^P} \varepsilon$$

By the same method used in [10], [13] for the proof of L^∞ – estimate in elliptic problems we have

$$(3,12) \quad \|w\|_{L^\infty} \leq \left(C'_{11} + \frac{C'_{12}}{\lambda} \right) \|A\psi\|_{L^P} \varepsilon$$

then from (3,12)

$$\|u_\varepsilon^\lambda - u^\lambda\|_{L^\infty} \leq \left(C''_{11} + \frac{C''_{12}}{\lambda} \right) \|A\psi\|_{L^P} \varepsilon.$$

We consider now the case $f \neq 0$, $f \in L^P$, $n < p \leq +\infty$.
Let be

$$\sigma = A^{-1}f \quad \sigma^\varepsilon = (A^\varepsilon)^{-1}f$$

we have

$$\|\sigma^2 - \sigma\|_{L^\infty} \leq C_{13} \varepsilon^{1/2}$$

We observe that the problems

$$(3,13_\varepsilon) \quad A^\varepsilon u_\varepsilon^\lambda + \frac{1}{\gamma} (u_\varepsilon^\lambda - \psi)^+ = f$$

$$(3,13) \quad Au^\lambda + \frac{1}{\lambda} (u^\lambda - \psi)^+ = f$$

are equivalent to the problems

$$(3,14_\varepsilon) \quad A^\varepsilon v_\varepsilon^\lambda + \frac{1}{\lambda} (v_\varepsilon^\lambda - (\psi - \sigma^\varepsilon))^+ = 0$$

$$(3,14) \quad Av^\lambda + \frac{1}{\lambda} (v^\lambda - (\psi - \sigma))^+ = 0$$

where

$$(3,15) \quad v^\lambda = u^\lambda - \sigma \quad v_\varepsilon^\lambda = u_\varepsilon^\lambda - \sigma^\varepsilon .$$

We consider now the problem

$$(3,14') \quad A^\varepsilon w_\varepsilon^\lambda + \frac{1}{\lambda} (w_\varepsilon^\lambda - (\psi - \sigma))^+ = 0 .$$

From [12] pg 22. part III we have

$$\|v_\varepsilon^\lambda - w_\varepsilon^\lambda\|_{L^\infty} \leq C_{13} \varepsilon^{1/2}$$

and from (3,12) we have

$$\|w_\varepsilon^\lambda - v^\lambda\|_{L^\infty} \leq \left(C_{14} + \frac{C_{15}}{\lambda} \right) (\|A\psi\|_{L^P} + \|f\|_{L^P}) \varepsilon$$

then

$$(3,16) \quad \|u_\varepsilon^\lambda - u^\lambda\|_{L^\infty} \leq (C_{14} + \frac{C_{15}}{\lambda}) (\|A\psi\|_{L^P} + \|f\|_{L^P}) \varepsilon + 3 C_{13} \varepsilon^{1/2} .$$

4. - Prof. of Th. 3.

We consider the problem

$$(4,1_\varepsilon) \quad \langle A^\varepsilon u_\varepsilon, v - u_\varepsilon \rangle \geq \langle f, v - u_\varepsilon \rangle$$

$$\forall v \in H_0^1(\Omega) \quad v \leq \psi$$

$$u_\varepsilon \in H_0^1(\Omega) \quad v_\varepsilon \leq \psi$$

$$(4,1) \quad \langle A u, v - u \rangle \geq \langle f, v - u \rangle$$

$$\forall v \in H_0^1(\Omega) \quad v \leq \psi$$

$$u \in H_0^1(\Omega) \quad u \leq \psi$$

and the corresponding penalised problems

$$(4,2) \quad A^\varepsilon u_\varepsilon^\lambda + \frac{1}{\lambda} (u_\varepsilon^\lambda - \psi)^+ = f$$

$$(4,2) \quad A u^\lambda + \frac{1}{\lambda} (u^\lambda - \psi)^+ = f$$

We suppose for the moment

$$\|A^\varepsilon \psi\|_{L^\infty}, \|A \psi\|_{L^\infty} \leq C.$$

We have

$$\|u_\varepsilon^\lambda - u_\varepsilon\|_{L^\infty} \leq C_1 \lambda^{\frac{1}{n-2(1-\alpha)}}$$

$$\|u^\lambda - u\|_{L^\infty} \leq C_1 \lambda^{\frac{1}{n-2(1-\alpha)}}$$

$$\|u_\varepsilon^\lambda - u^\lambda\|_{L^\infty} \leq \left(C_2 + \frac{C_3}{\lambda}\right) \varepsilon + C_4 \varepsilon^{1/2}$$

then

$$\|u_\varepsilon - u\|_{L^\infty} \leq 2C_1 \lambda^{\frac{1}{n-2(1-\alpha)}} + \left(C_2 + \frac{C_3}{\lambda}\right) \varepsilon + C_4 \varepsilon^{1/2}.$$

We choose $\lambda = \frac{n-2+2\alpha}{\varepsilon^{n-2+3\alpha}}$ then

$$\|u_\varepsilon - u\|_{L^\infty} \leq C_5 \varepsilon^{\frac{\alpha}{n-2+3\alpha}}.$$

Let be now $\psi \in W^{1,p}(\Omega)$, $p > n$; there is $\psi_\varepsilon^\eta, \psi^\eta$ ($\eta > 0$) such that, [7],

$$\|\psi_\varepsilon^\eta - \psi\|_{L^\infty} \leq C_6 \eta^{1/2}$$

$$\|\psi^\eta - \psi\|_{L^\infty} \leq C_6 \eta^{1/2}$$

$$\|A^\varepsilon \psi_\varepsilon^\eta\|_{L^\infty} \leq C_7 \eta^{-1/2}$$

$$\|A \psi^\eta\|_{L^\infty} \leq C_7 \eta^{1/2}.$$

We consider the problems

$$(4,3_\eta) \quad \begin{aligned} \langle A^\varepsilon u_\varepsilon^\eta, v - u_\varepsilon^\eta \rangle &\geq \langle f, v - u_\varepsilon^\eta \rangle \\ \forall v &\in H_0^1(\Omega), \quad v \leq \psi_\varepsilon^\eta \\ u_\varepsilon^\eta &\in H_0^1(\Omega), \quad u_\varepsilon^\eta \leq \psi_\varepsilon^\eta \end{aligned}$$

$$(4,3) \quad \begin{aligned} \langle A u^\eta, v - u^\eta \rangle &\geq \langle f, v - u^\eta \rangle \\ \forall v &\in H_0^1(\Omega), \quad v \leq \psi^\eta \\ u^\eta &\in H_0^1(\Omega), \quad u^\eta \leq \psi^\eta. \end{aligned}$$

From the above we have

$$\begin{aligned} \|u_\varepsilon^\eta - u^\eta\|_{L^\infty} &\leq \left(C_8 + \frac{C_9}{\eta^{1/2}} \right) \varepsilon^{\frac{\alpha}{n-2+3\alpha}} \\ \|u_\varepsilon^\eta - u_\varepsilon\|_{L^\infty}, \|u^\eta - u\|_{L^\infty} &\leq C_7 \eta^{1/2} \end{aligned}$$

then

$$\|u_\varepsilon - u\|_{L^\infty} \leq (C_8 + C_9/\eta^{1/2}) \varepsilon^{\frac{\alpha}{n-2+3\alpha}} + 2 C_7 \eta^{1/2}.$$

We choose $\eta = \frac{\alpha}{\lambda^{n-2+3\alpha}}$ then

$$\|u_\varepsilon - u\|_{L^\infty} \leq C_{10} \varepsilon^{\frac{\alpha}{2[n-2+3\alpha]}}$$

Rem. 1 — We observe that the dependence on n of the estimate of th. 3 derive from the dependence on n of the estimate in the lemma 2 ; then from (2,13) repeating our calculations we have

$$\|u_\varepsilon - (u + \varepsilon \chi_1)\|_{H_0^1} \leq C \varepsilon^{1/4}$$

$$\text{if } \|A^\varepsilon \psi\|_{L^2}, \|A \psi\|_{L^2} \leq C.$$

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