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## A Watsonian theorem for multiple series \*\*

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### 1. - Introduction

Put

$$(1) \quad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1) \dots (\lambda+n-1), & \forall n \in \{1, 2, 3, \dots\}, \end{cases}$$

and let  $F_{l:m;n}^{p;q;k}$  denote Kampé de Fériet's double hypergeometric function [1, p. 150] in the (modified) notation of Burchinal and Chaundy [2, p. 112], defined by

$$(2) \quad F_{l:m;n}^{p;q;k} \left[ \begin{matrix} (a_p) : (b_q) ; (c_k) ; \\ (a_l) : (\beta_m) ; (\gamma_n) ; \end{matrix} \middle| x, y \right]$$

$$= \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_{r+s} \prod_{j=1}^q (b_j)_r \prod_{j=1}^k (c_j)_s}{\prod_{j=1}^l (a_j)_{r+s} \prod_{j=1}^m (\beta_j)_r \prod_{j=1}^n (\gamma_j)_s} \frac{x^r}{r!} \frac{y^s}{s!},$$

where, for convenience,  $(a_p)$  abbreviates the sequence of  $p$  parameters  $a_1, \dots, a_p$ , with similar interpretations for  $(b_q)$ ,  $(c_k)$ , etc.

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Recently, B. L. Sharma [4] gave a double-series analogue of the familiar Watson's summation theorem (*cf.*, *e. g.*, [6, p. 54, Eq. (2.3.3.13)]; see also [8])

$$(3) \quad {}_3F_2 \left[ a, b, c; \frac{1}{2}(1+a+b), 2c; 1 \right] \\ = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+c\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}a+c\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}b+c\right)},$$

which holds true when  $\operatorname{Re}(2c-a-b) > -1$ ; indeed, making use of the notation (2), we have Sharma's result (*cf.* [4, p. 95, Eq. (3)])

$$(4) \quad F_{2;0;0}^{2;1;1} \left[ \begin{matrix} a, \beta : \gamma; \delta; \\ 2a, (1+\beta+\gamma+\delta)/2 : -; -; \end{matrix} \quad \begin{matrix} 1, 1 \end{matrix} \right] \\ = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+a\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta+\frac{1}{2}\gamma+\frac{1}{2}\delta\right)\Gamma\left(\frac{1}{2}+a-\frac{1}{2}\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\gamma+\frac{1}{2}\delta\right)\Gamma\left(\frac{1}{2}+a-\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}+a-\frac{1}{2}\gamma-\frac{1}{2}\delta\right)},$$

provided that  $\operatorname{Re}(2a-\beta-\gamma-\delta) > -1$ .

In his long and involved proof of the summation formula (4), Sharma [*loc. cit.*] applies a number of results including, for example, the well-known Gaussian summation theorem for  ${}_2F_1[a, b; c; 1]$ , a formula of his own [5, p. 105, Eq. (5)], and indeed, the Watsonian theorem (3). The object of the present note is first to observe that the summation formula (4) is essentially equivalent to, and *not* a generalization of, Watson's theorem (3); we then show how readily one can derive a similar multiple-series analogue of (3) and (4).

## 2. - Equivalence of (3) and (4)

We begin by recalling the known result (*cf.* [7, p. 297, Eq. (16)])

$$(5) \quad \sum_{m,n=0}^{\infty} \Delta_{m+n}(\gamma)_m(\delta)_n \frac{x^{m+n}}{m!n!} = \sum_{n=0}^{\infty} \Delta_n(\gamma+\delta)_n \frac{x^n}{n!},$$

where  $\gamma, \delta$  are arbitrary parameters, real or complex, and  $\{\Delta_n\}$  is a sequence of arbitrary complex numbers, it being assumed that the series involved are absolutely convergent.

Of our concern here is merely a special case of (5), involving hypergeometric functions, which was indeed given by Appell and Kampé de Fériet [1, p. 155] as long ago as 1926. Thus, by further specializing this 1926 result or by setting  $x = 1$  and

$$(6) \quad \Delta_n = \frac{(\alpha)_n (\beta)_n}{(2\alpha)_n \left( \frac{1}{2} + \frac{1}{2} \beta + \frac{1}{2} \gamma + \frac{1}{2} \delta \right)_n}, \quad \forall n \in \{0, 1, 2, \dots\},$$

in (5), we readily have

$$(7) \quad {}_2F_{2;0;0}^{2;1;1} \left[ \begin{matrix} \alpha, \beta : \gamma; \delta; \\ 2\alpha, (1 + \beta + \gamma + \delta)/2 : -; -; \end{matrix} \right. \left. \begin{matrix} 1, 1 \end{matrix} \right] \\ = {}_3F_2 [\alpha, \beta, \gamma + \delta; 2\alpha, (1 + \beta + \gamma + \delta)/2; 1],$$

which exhibits the fact that the first member of (4) is just the hypergeometric  ${}_3F_2$  series occurring on the left-hand side of Watson's theorem (3) with, of course,  $a = \beta$ ,  $b = \gamma + \delta$ ,  $c = \alpha$ , and the equivalence of (3) and (4) evidently follows.

We remark in passing that, in view of the well-known formula [1, p. 23, Eq. (25)]

$$(8) \quad {}_1F_1[a, b, b'; c; x, x] = {}_2F_1[a, b + b'; c; x], \quad |x| < 1,$$

also contained in the general result (5), Sharma's formula [4, p. 96, Eq. (6)] for the Appell function  $F_1$  is, in fact, equivalent to (and *not* a generalization of) Gauss's second summation theorem [6, p. 32, Eq. (1.7.1.9)], *viz*

$$(9) \quad {}_2F_1 \left[ a, b; \frac{1}{2}(1 + a + b); \frac{1}{2} \right] = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}a + \frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}a\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}b\right)}.$$

### 3. - A multiple-series analogue

A multiple-series analogue of the summation formulas (3) and (4) would follow readily from a generalization of (5) considered recently by R. Panda [3], who also gave the hypergeometric form [*op. cit.*, p. 168, Eq. (12)]

$$(10) \quad {}_qF_q^{p;1;\dots;1; \dots;1}_{q;0;\dots;0} \left( \begin{matrix} \alpha_1, \dots, \alpha_p : \gamma_1; \dots; \gamma_n; \\ \beta_1, \dots, \beta_q : -; \dots; -; \end{matrix} ; x, \dots, x \right) \\ = {}_{p+1}F_q \left[ \begin{matrix} \alpha_1, \dots, \alpha_p, \gamma_1 + \dots + \gamma_n; \\ \beta_1, \dots, \beta_q; \end{matrix} ; x \right],$$

where, for convergence,  $p < q$  and  $|x| < \infty$ , or  $p = q$  and  $|x| < 1$ , or  $p = q$ ,  $x = 1$ , and

$$(11) \quad \operatorname{Re} \left( \sum_{j=1}^q \beta_j - \sum_{j=1}^p \alpha_j - \sum_{j=1}^n \gamma_j \right) > 0.$$

If, in the reduction formula (10), we set  $p = q = 2$ ,  $\alpha_1 = \alpha$ ,  $\alpha_2 = \beta$ ,  $\beta_1 = 2\alpha$ ,  $\beta_2 = (1 + \beta + \gamma_1 + \dots + \gamma_n)/2$ , and  $x = 1$ , and apply Watson's theorem (3), we shall at once get the following multiple-series analogue :

$$(12) \quad {}_2F_2^{2;1;\dots;1; \dots;1}_{2;0;\dots;0} \left( \begin{matrix} \alpha, \beta : \gamma_1; \dots; \gamma_n; \\ 2\alpha, (1 + \beta + \gamma_1 + \dots + \gamma_n)/2 : -; \dots; -; \end{matrix} ; 1, \dots, 1 \right) = \\ = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2} + \alpha\right)\Gamma\left(\frac{1}{2} + \frac{1}{2}\beta + \frac{1}{2}\gamma_1 + \dots + \frac{1}{2}\gamma_n\right)\Gamma\left(\frac{1}{2} + \alpha - \frac{1}{2}\beta - \frac{1}{2}\gamma_1 - \dots - \frac{1}{2}\gamma_n\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2} + \frac{1}{2}\gamma_1 + \dots + \frac{1}{2}\gamma_n\right)\Gamma\left(\frac{1}{2} + \alpha - \frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2} + \alpha - \frac{1}{2}\gamma_1 - \dots - \frac{1}{2}\gamma_n\right)},$$

provided that  $\operatorname{Re}(2\alpha - \beta - \gamma_1 - \dots - \gamma_n) > -1$ .

Evidently, the reduction formula (10) can be applied to derive similar multiple-series analogues of several other known hypergeometric summation theorems including, for example, Gauss's second summation theorem (9) above.

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