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Abstract: The investigation of $U_n$-submodules yields a generalization of Ulm subgroups. Through this generalization, we study the $\alpha$-th Ulm submodule of $M$, $M^\alpha$, and the functor $M \mapsto M^\alpha$, in relation to its preserving sums, products and certain quotients. We also discover conditions for which every module is the first Ulm submodule of some module. Finally, the well-known result that the first Ulm subgroup of an algebraically compact abelian group is divisible is extended to semi-hereditary rings.

1. - Introduction.

This paper deals with certain submodules, called $U_n$-submodules, of arbitrary left $R$-modules. When $n = 1$, we get a generalization of the concept of Ulm subgroup of an Abelian group. We are chiefly concerned with this generalization.

It is well-known in Abelian Group Theory that the first Ulm subgroup of an algebraically compact group is divisible. See Fuchs [1]. One aim of this paper is to extend this theorem to semi-hereditary rings. We will also show that the map $M \mapsto \alpha$-th Ulm submodule of $M$ preserves sums, products and certain quotients. A theorem relating purity to the first Ulm submodule is proved. In addition, necessary and sufficient conditions for an arbitrary

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module to be equal to the first Ulm submodule of some module are
given.
Throughout this paper \( R \) will be an associative ring with identity. All
modules considered will be unitary left \( R \)-modules. The symbol \( \dim R \)
will denote the left homological dimension of \( R \), and hereditary
and noetherian will mean left hereditary and left noetherian.
The letters \( k, n, m \) are always taken from the set \( \{1, 2, 3, \ldots \} \cup \{0\} \).

We will let \( L_n = \{P \rightarrow Q | \text{where } Q/P \text{ is finitely presented of}
dimension \leq n \text{ and } Q \text{ is finitely generated free}\} \).

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2. - \( U_n \)-submodules.

A submodule \( A \) of \( M \) will be called a \( U_n \)-submodule if given a
monomorphism \( P \rightarrow Q \in L_n \), every map \( f: P \rightarrow A \) can be extended
to a map \( g: Q \rightarrow M \).

We can interpret \( U_n \)-submodules in terms of systems of equations. A system of equations

\[
\sum_{j=1}^{s} r_{ij} x_j = a_i \quad (i \in I)
\]

will be called an \( n \)-system in \( A \) if the \( a_i \)'s are in \( A \) and if \( P \rightarrow Q \in L_n \)
where \( Q \) is free with generators \( x_j \) and \( P \) is generated by the \( \Sigma r_{ij} x_j \)
for \( i \in I \). Thus a submodule \( A \) of \( M \) is a \( U_n \)-submodule if and only
if any finite compatible \( n \)-system of equations in \( A \) is solvable in \( M \).

Some easy relations between the various \( U_n \)-submodules are
given below:

A. A submodule of a \( U_n \)-submodule of \( M \) is again a \( U_n \)-submodule
of \( M \).

B. Every \( U_n \)-submodule is a \( U_k \)-submodule for all \( k \leq n \).

C. The union of an ascending chain of \( U_n \)-submodules is again
one. Hence, every \( U_n \)-submodule is contained in a maximal one.

D. Every injective submodule is a \( U_n \)-submodule for all \( n \).
E. If \( \text{dim } R = n \) then every \( U_n \)-submodule is also a \( U_k \)-submodule for all \( k \geq n \).

The particular case when \( R \) is a P.I.D. reduces the definition of a \( U_1 \)-submodule to solutions of \( rx = a \) for all \( r \in R \) which are not zero divisors. That is, \( A \) is a \( U_1 \)-submodule of \( M \) if \( rx = a \) is solvable in \( M \) for each \( a \) in \( A \) and each non-zero divisor \( r \) in \( R \). In particular when \( R = \) the ring of integers, \( A \) is a \( U_1 \)-subgroup of an abelian group \( M \) if and only if \( A \) is contained in the first Ulm subgroup of \( M \). See Fuchs [1].

We can generalize this somewhat in the theorem following lemma 1:

**Lemma 1:** An arbitrary sum of \( U_1 \)-submodules is again one.

**Proof:** Let \( A_i, i \in I \), be \( U_1 \)-submodules of \( M \) and let \( P \rightarrow Q \in I \)
with \( f: P \rightarrow A_i \). As \( P \) is finitely generated, we may assume that \( I \) is finite. It suffices to verify lemma 1 for \( |I| = 2 \). Consider the following diagram where \( \alpha: A_1 \oplus A_2 \rightarrow A_1 + A_2 \) is the epimorphism given by \( \alpha(a_1, a_2) = a_1 + a_2 \) and \( \nabla \) is the codiagonal map:

As \( P \) is projective, \( f \) factors through \( \alpha \). As \( A_1 \) and \( A_2 \) are \( U_1 \)-submodules, there exists a map \( g: Q \rightarrow M \oplus M \) making the outside rectangle commute. Thus \( \nabla \circ g \) makes the inside rectangle commute. Hence, \( A_1 + A_2 \) is a \( U_1 \)-submodule of \( M \).

The proof of theorem 1 is immediate.

**Theorem 1:** Every module contains a unique maximal \( U_1 \)-submodule. This maximal \( U_1 \)-submodule of \( M \) will be called the first Ulm submodule of \( M \) and will be denoted \( M^1 \).

In abelian groups, the first Ulm subgroup of a group \( A \) is the intersection of all subgroups \( nA \) where \( n \) ranges over all positive
integers. Clearly then, if $R$ is the ring of integers, the first Ulm submodule is exactly the first Ulm subgroup. In fact, for a P.I.D. both concepts coincide.

Just as for abelian groups, we define for each ordinal $\alpha$, $M_{\alpha+1} = \bigcap_{\beta < \alpha} M_\beta$ and for each limit ordinal $\beta$, set $M_\beta = \bigcap_{\alpha < \beta} M_\alpha$. By this process we get a descending sequence of Ulm submodules of $M$. We call the sequence

$$M \supset M^1 \supset M^2 \supset \ldots \supset M^\beta \supset \ldots$$

the *Ulm sequence* for $M$. A cardinality argument gives us that $M^\gamma = M^\gamma+1$ for some ordinal $\gamma$. We will denote this Ulm submodule by $M^\gamma$.

It is important to note that as an injective module is a $U_1$-submodule in any module containing it, any injective submodule of $M$ is contained in $M^\gamma$.

For a fixed ordinal $\alpha$, the function $M \mapsto M^\alpha$ is functorial; i.e. if $f : M \to N$ then $f(M^\alpha) \subseteq N^\alpha$. Thus $M^\alpha$ is a fully invariant submodule of $M$. We then have:

**Theorem 2**: If $C \subset M^\alpha$ then $(M/C)^\alpha = M^\alpha/C$.

Proof by transfinite induction: the previous discussion shows that $M^\alpha/C$ is a $U_1$-submodule of $M/C$ and hence is contained in $(M/C)^\alpha$. For the reverse inclusion, let $A/C$ be a $U_1$-submodule of $M/C$. We must show that $A \subset M^\alpha$. Consider the following diagram for arbitrary $P \to Q \in \mathcal{L}_1$ and $f : P \to A$:

![Diagram]

where $h$ and $k$ are the canonical maps. By assumption, $h \circ f$ can be extended to $t : Q \to M/C$. As $Q$ is free, there exists a map $s : Q \to M$.
such that $k \circ s = t$. Now $s \circ i = j \circ f : P \to \ker k = C$. As $C \subseteq M^1$, we get an extension $g : Q \to M$ of $s \circ i = j \circ f$. Then $s + g$ extends $f$.

It is clear that if $\beta$ is a limit ordinal or if $\beta = \alpha + 1$ and $M^\beta/C = (M/C)^\alpha$ for all $\alpha < \beta$, then we have $(M/C)^\beta = M^\beta/C$.

The function $M \to M^a$, for fixed $a$, also preserves direct sums and direct products. The proof of lemma 2 is straightforward.

**Lemma 2.**

$$(\bigoplus_i A_i)^a = \bigoplus_i A_i^a,$$

$$(\prod_i A_i)^a = \prod_i A_i^a.$$ 

It also follows immediately that the canonical map $M \to M/M^a$ preserves all Ulm submodules up to the $a$-th; i.e. the image of the $\beta$-th Ulm submodule is equal to the $\beta$-th Ulm submodule of $M/M^a$, for $\beta \leq a$.

The next theorem relates, to a certain extent, the first Ulm submodule to purity. Recall, $A$ is pure in $M$ if every finite system of equations which is solvable in $M$ is solvable in $A$. See Warfield [6]. The result generalizes that in abelian groups.

**Theorem 3:** A pure in $M$ implies $A^1 = M^1 \cap A$.

**Proof:** It is clear that $A^1 \subseteq M^1 \cap A$. For the reverse, we must show that $M^1 \cap A$ is a $U_1$-submodule of $A$. Given any compatible $n$-system of equations in $M^1 \cap A \subseteq M^1$, this system has a solution in $M$. As $A$ is pure in $M$, the system is solvable in $A$. Hence $M^1 \cap A$ is a $U_1$-submodule of $A$.

**Corollary:** Let $R$ be hereditary and noetherian. If $M^1$ is pure in $M$ then $M^1$ is injective.

**Proof:** The proof is immediate upon observing that $M^*$ is injective with respect to all elements of $L_1$. In closing this section, we would like to give necessary and sufficient conditions for an $R$-module to be the first Ulm submodule of some module.

**Theorem 4:** Every $R$-module $A$ is the first Ulm submodule of some module $M$ if and only if the first Ulm submodule of all quotients $Q/P \in L_1$, is of the form $X/P$ where $X = P \oplus Y$ for some $Y$. 
Proof: For necessity, let \( P \hookrightarrow Q \in L_1 \). Consider the diagram below where \( P = M^1 \):

\[
\begin{array}{ccc}
P & \xrightarrow{f} & Q & \xrightarrow{g} & Q/P \\
\downarrow & & \downarrow & & \downarrow \\
P & \xrightarrow{} & M & \xrightarrow{} & M/P
\end{array}
\]

The map \( f \) is from the universal property of the Ulm submodules and \( g \) is the natural extension of \( f \). Let \( (Q/P)^1 = X/P \subset Q/P \). As \( (M/P)^1 = M^1/P = 0 \), \( f(X) \subset P \). Hence \( P \hookrightarrow X \) splits.

For sufficiency, consider the set \( I \) of all diagrams

\[
\begin{array}{ccc}
P & \xrightarrow{f} & X & \xrightarrow{f \oplus 0} & Q \\
& \downarrow & & \downarrow & \\
& & A
\end{array}
\]

where \( X = P \oplus Y \) and \( P \hookrightarrow Q \) ranges over the set \( L_1 \). Now take \( M \) to be the pushout of the two maps \( \oplus X \hookrightarrow \oplus Q \) and \( \oplus f : \oplus X \to A \).

Clearly, \( A \) is a \( U_1 \)-submodule of \( M \). \( A = M^1 \) follows immediately as \( (Q/X)^1 = 0 \).

3 - \( n \)-algebraically Compact Modules.

Recall: a module \( M \) is called algebraically compact if any compatible system of equations in \( M \) which is finitely solvable is also solvable in \( M \). See Warfield [6]. This concept can be generalized as follows: call a module \( M \) \( n \)-algebraically compact if any compatible \( n \)-system of equations in \( M \) which is finitely solvable in \( M \) is solvable in \( M \). Any \( n \)-algebraically compact module is also \( k \)-algebraically compact for \( n \geq k \). If \( \dim R = k \) then any \( k \)-algebraically compact module is \( n \)-algebraically compact for \( n \geq k \).

It is known that if \( A \) is an algebraically compact abelian group then \( A^1 \) is the maximum injective subgroup of \( A \). The next theorem generalizes this.
THEOREM 5: Let be semi-hereditary. If \( M \) is 1-algebraically compact then \( M^1 \) is injective; hence, \( M^1 \) is the maximum injective submodule.

Proof: Let \( \hat{M} \) be an injective hull of \( M \). Let \( E \) be the injective hull of \( M^1 \) in \( \hat{M} \). As \( M^1 \) contains all injective submodules of \( M \), it suffices to show that an injective submodule of \( M \) contains \( M^1 \). Let \( E \) be generated by \( M^1 \cup \{ e_i : i \in I \} \). Consider all relations in \( E \) between \( e_i \) and the elements of \( M^1 \) (i.e., a presentation of \( E \mod M^1 \)),
\[
\sum_{finite} r_{ij} e_j = m_i \text{ where } r_{ij} \in R, m_i \in M^1 \text{ and } i \in I.
\]
As \( R \) is semi-hereditary, any finite subsystem is a 1-system. Thus, any finite subsystem in \( M^1 \) is solvable in \( M \). As \( M \) is 1-algebraically compact, the entire system is solvable in \( M \) i.e., there exists a map \( \varrho : E \rightarrow M \) which leaves the elements of \( M^1 \) fixed. As \( E \) is an essential extension of \( M^1 \), we have \( \ker \varrho = 0 \). Therefore \( M^1 \subset \varrho E \subset M \) where \( \varrho E \) is injective. Hence \( M^1 = \varrho E \) is injective.

Remark: for all modules \( M, M^* \) is injective if and only if \( R \) is hereditary and noetherian.

Proof: the necessity is obvious. For the sufficiency, let \( I \) be a left ideal of \( R \). As \( R \) is hereditary and noetherian, \( I \rightarrow \mathbb{R} \in L_I \). Thus, as \( M^* \) is injective with respect to all elements of \( L_I \), Baer's criterion implies \( M^* \) is injective.

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