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Abstract: The investigation of U_n-submodules yields a generalization of Ulm subgroups. Through this generalization, we study the α-th Ulm submodule of M, M^α, and the functor $M \mapsto M^\alpha$, in relation to its preserving sums, products and certain quotients. We also discover conditions for which every module is the first Ulm submodule of some module. Finally, the well-known result that the first Ulm subgroup of an algebraically compact abelian group is divisible is extended to semi-hereditary rings.

1. - Introduction.

This paper deals with certain submodules, called U_n-submodules, of arbitrary left R-modules. When $n = 1$, we get a generalization of the concept of Ulm subgroup of an Abelian group. We are chiefly concerned with this generalization.

It is well-known in Abelian Group Theory that the first Ulm subgroup of an algebraically compact group is divisible. See Fuchs [1]. One aim of this paper is to extend this theorem to semi-hereditary rings. We will also show that the map $M \mapsto \alpha$-th Ulm submodule of M preserves sums, products and certain quotients. A theorem relating purity to the first Ulm submodule is proved. In addition, necessary and sufficient conditions for an arbitrary

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module to be equal to the first Ulm submodule of some module are given.

Throughout this paper R will be an associative ring with identity. All modules considered will be unitary left R-modules. The symbol dim R will denote the left homological dimension of R, and hereditary and noetherian will mean left hereditary and left noetherian. The letters k, n, m are always taken from the set $\{1, 2, 3, \ldots\} \cup \{\infty\}$.

We will let $L_n = \{P \rightarrowtail Q \mid \text{where } Q/P \text{ is finitely presented of dimension} \leqq n \text{ and } Q \text{ is finitely generated free}\}$.

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2. - U_n-submodules.

A submodule A of M will be called a U_n-submodule if given a monomorphism $P \rightarrowtail Q \in L_n$, every map $f: P \rightarrow A$ can be extended to a map $g: Q \rightarrow M$.

We can interpret U_n-submodules in terms of systems of equations. A system of equations

$$\sum_{j=1}^{s} r_{ij} x_j = a_i \qquad (i \in I)$$

will be called an *n-system* in A if the a_i's are in A and if $P \rightarrowtail Q \in L_n$ where Q is free with generators x_j and P is generated by the $\Sigma r_{ij} x_j$ for i $\in$ I. Thus a submodule A of M is a U_n-submodule if and only if any finite compatible n-system of equations in A is solvable in M.

Some easy relations between the various U_n-submodules are given below:

A. A submodule of a U_n-submodule of M is again a U_n-submodule of M.

B. Every U_n-submodule is a U_k-submodule for all $k \leq n$.

C. The union of an ascending chain of U_n-submodules is again one. Hence, every U_n-submodule is contained in a maximal one.

D. Every injective submodule is *a* U_n-submodule for all n.

E. If $\dim R = n$ then every U_n-submodule is also a U_k-submodule for all $k \geq n$.

The particular case when R is *a* P.I.D. reduces the definition of a U_1-submodule to solutions of $rx = a$ for all $r \in R$ which are not zero divisors. That is, A is a U_1-submodule of M if $rx = a$ is solvable in M for each a in A and each non-zero divisor r in R. In particular when $R =$ the ring of integers, A is a U_1-subgroup of an abelian group M if and only if A is contained in the first Ulm subgroup of M. See Fuchs [1].

We can generalize this somewhat in the theorem following lemma 1 :

Lemma 1 : An arbitrary sum of U_1-submodules is again one.

Proof : Let. A_i, $i \in I$, be U_1-submodules of M and let $P \rightarrowtail Q \in L_1$ with $f : P \rightarrowtail A_i$. As P is finitely generated, we may assume that I is finite. It suffices to verify lemma 1 for $|I| = 2$. Consider the following diagram where $\alpha : A_1 \oplus A_2 \rightarrow A_1 + A_2$ is the epimorphism given by $\alpha\ (a_1 , a_2) = a_1 + a_2$ and ∇ is the codiagonal map :

$$\begin{tikzcd}
& P \arrow[r, tail] \arrow[d, "f"] \arrow[ddl, dashed] & Q \arrow[ddr, "g"] & \\
& A_1 + A_2 \arrow[r, tail] & M & \\
A_1 \oplus A_2 \arrow[ur, two heads, "\alpha"] \arrow[rrr, tail] & & & M \oplus M \arrow[ul, "\nabla"']
\end{tikzcd}$$

As P is projective, f factors through α. As A_1 and A_2 are U_1-submodules, there exists a map $g : Q \rightarrow M \oplus M$ making the outside rectangle commute. Thus $\nabla \circ g$ makes the inside rectangle commute. Hence, $A_1 + A_2$ is a U_1-submodule of M.

The proof of theorem 1 is immediate.

Theorem 1 : Every module contains a unique maximal U_1-submodule. This maximal U_1-submodule of M will be called the *first Ulm submodule* of M and will be denoted M^1.

In abelian groups, the first Ulm subgroup of a group A is the intersection of all subgroups nA where n ranges over all positive integers. Clearly then, if R is the ring of integers, the first Ulm submo dule is exactly the first Ulm subgroup. In fact, for a P.I.D. both concepts coincide.

Just as for abelian groups, we define for each ordinal α, $M^{\alpha+1} = (M^{\alpha})^1$ and for each limit ordinal β, set $M^{\beta} = \bigcap_{\alpha<\beta} M^{\alpha}$. By this process we get a descending sequence of Ulm submodules of M. We call the sequence

$$M \supset M^1 \supset M^2 \supset \ldots \supset M^{\alpha} \supset \ldots$$

the *Ulm sequence* for M. A cardinality argument gives us that $M^{\gamma} = M^{\gamma+1}$ for some ordinal γ. We will denote this Ulm submodule by $M^{\#}$.

It is important to note that as an injective module is a U_1-submodule in any module containing it, any injective submodule of M is contained in $M^{\#}$.

For a fixed ordinal α, the function $M \longmapsto M^{\alpha}$ is functorial; i. e. if $f: M \to N$ then $f(M^{\alpha}) \subset N^{\alpha}$. Thus M^{α} is a fully invariant submodule of M. We then have:

THEOREM 2: If $C \subset M^{\alpha}$ then $(M/C)^{\alpha} = M^{\alpha}/C$.

Proof by transfinite induction: the previous discussion shows that M^1/C is a U_1-submodule of M/C and hence is contained in $(M/C)^1$. For the reverse inclusion, let A/C be a U_1-submodule of M/C. We must show that $A \subset M^1$. Consider the following diagram for arbitrary $P \rightarrowtail Q \in L_1$ and $f: P \to A$:

$$\begin{tikzcd}
P \arrow[rr, tail, "i"] \arrow[d, "f"'] & & Q \arrow[dl, dashed, "s"'] \arrow[dd, dashed, "t"] \\
A \arrow[r, tail, "i"] \arrow[d, "h"'] & M \arrow[dr, "k"] & \\
A/C \arrow[rr, tail] & & M/C
\end{tikzcd}$$

where h and k are the canonical maps. By assumption, $h \circ f$ can be extended to $t: Q \to M/C$. As Q is free, there exists a map $s: Q \to M$

such that $k \circ s = t$. Now $s \circ i - j \circ f \colon P \to ker\ k = C$. As $C \subset M^1$, we get an extension $g \colon Q \to M$ of $s \circ i - j \circ f$. Then $s + g$ extends f.

It is clear that if β is a limit ordinal or if $\beta = \alpha + 1$ and $M^\alpha/C = (M/C)^\alpha$ for all $\alpha < \beta$, then we have $(M/C)^\beta = M^\beta/C$.

The function $M \to M^\alpha$, for fixed α, also preserves direct sums and direct products. The proof of lemma 2 is straightforward.

LEMMA 2.
$$(\oplus A_i)^\alpha = \oplus A_i^\alpha,$$
$$(\Pi A_i)^\alpha = \Pi A_i^\alpha,$$

It also follows immediately that the canonical map $M \to M/M^\alpha$ preserves all Ulm submodules up to the *α-th*; i.e. the image of the β-th Ulm submodule is equal to the β-th Ulm submodule of M/M^α, for $\beta \leq \alpha$.

The next theorem relates, to a certain extent, the first Ulm submodule to purity. Recall, A is pure in M if every finite system of equations which is solvable in M is solvable in A. See Warfield [6]. The result generalizes that in abelian groups.

THEOREM 3 : A pure in M implies $A^1 = M^1 \cap A$.

Proof : It is clear that $A^1 \subset M^1 \cap A$. For the reverse, we must show that $M^1 \cap A$ is a U_1-submodule of A. Given any compatible n-system of equations in $M^1 \cap A \subset M^1$, this system has a solution in M. As A is pure in M, the system is solvable in A. Hence $M^1 \cap A$ is a U_1-submodule of A.

COROLLARY : Let R be hereditary and noetherian. If M^1 is pure in M then M^1 is injective.

Proof : The proof is immediate upon observing that $M^\#$ is injective with respect to all elements of L_1.

In closing this section, we would like to give necessary and sufficient conditions for an R-module to be the first Ulm submodule of some module.

THEOREM 4 : Every R-module A is the first Ulm submodule of some module M if and only if the first Ulm submodule of all quotients $Q/P \in L_1$, is of the form X/P where $X = P \oplus Y$ for some Y.

Proof: For necessity, let $P \rightarrowtail Q \in L_1$. Consider the diagram below where $P = M^1$:

$$\begin{tikzcd}
P \arrow[r, tail] \arrow[d, equal] & Q \arrow[r, two heads] \arrow[d, "f"] & Q/P \arrow[d, "g"] \\
P \arrow[r, tail] & M \arrow[r, two heads] & M/P
\end{tikzcd}$$

The map f is from the universal property of the Ulm submodules and g is the natural extension of f. Let $(Q/P)^1 = X/P \subset Q/P$. As $(M/P)^1 = M^1/P = 0$, $f(X) \subset P$. Hence $P \rightarrowtail X$ splits.

For sufficiency, consider the set I of all diagrams

$$\begin{tikzcd}
P \arrow[r, tail] \arrow[dr, "f"'] & X \arrow[r, tail] \arrow[d, "f \oplus 0"] & Q \\
& A &
\end{tikzcd}$$

where $X = P \oplus Y$ and $P \rightarrowtail Q$ ranges over the set L_1. Now take M to be the pushout of the two maps $\underset{I}{\oplus} X \rightarrowtail \underset{I}{\oplus} Q$ and $\underset{I}{\oplus} f : \underset{I}{\oplus} X \to A$. Clearly, A is a U_1-submodule of M. $A = M^1$ follows immediately as $(Q/X)^1 = 0$.

3 - n-algebraically Compact Modules.

Recall: a module M is called algebraically compact if any compatible system of equations in M which is finitely solvable is also solvable in M. See Warfield [6]. This concept can be generalized as follows: call a module M *n-algebraically compact* if any compatible n-system of equations in M which is finitely solvable in M is solvable in M. Any n-algebraically compact module is also k-algebraically compact for $n \geq k$. If dim $R = k$ then any k-algebraically compact module is n-algebraically compact for $n \geq k$.

It is known that if A is an algebraically compact abelian group then A^1 is the maximum injective subgroup of A. The next theorem generalizes this.

THEOREM 5: Let R be semi-hereditary. If M is 1-algebraically compact then M^1 is injective; hence, M^1 is the maximum injective submodule.

Proof: Let $\hat{M}$ be an injective hull of M. Let E be the injective hull of M^1 in $\hat{M}$. As M^1 contains all injective submodules of M, it suffices to show that an injective submodule of M contains M^1. Let E be generated by $M^1 \cup \{e_i : i \in I\}$. Consider all relations in E between e_i and the elements of M^1 (*i. e.* a presentation of E mod M^1), $\sum_{finite} r_{ij} e_j = m_i$ where $r_{ij} \in R$, $m_i \in M^1$ and $i \in I$. As R is semi-hereditary, any finite subsystem is a l-system. Thus, any finite subsystem in M^1 is solvable in M. As M is l-algebraically compact, the entire system is solvable in M *i. e.* there exists a map $\varrho : E \to M$ which leaves the elements of M^1 fixed. As E is an essential extension of M^1, we have $\ker \varrho = 0$. Therefore $M^1 \subset \varrho E \subset M$ where ϱE is injective. Hence $M^1 = \varrho E$ is injective.

Remark: for all modules M, $M^{\#}$ is injective if and only if R is hereditary and noetherian.

Proof: the necessity is obvious. For the sufficiency, let I be a left ideal of R. As R is hereditary and noetherian, $I \rightarrowtail R \in L_1$. Thus, as $M^{\#}$ is injective with respect to all elements of L_1, Baer's criterion implies $M^{\#}$ is injective.

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