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Giuliano Bratti

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A remark about elliptic overdeterminated systems.

GIULIANO BRATTI (*)

1) After Lech's work about ideals of polynomials (2), Hormander, (1958), characterized hypoelliptic overdeterminated systems of partial differential equations with constant coefficients (3).

Of course, the same method permits also the characterization of elliptic systems of the same size, if, as usual (after I. G. Petrovskii (5)), we want that words like « elliptic » and « analytic solutions » are equivalent.

Following Lech-Hormander: let

$$\sum_j P_{i,j}(D)u_j = f_i\,,\quad I \leq i \leq N\,,\quad n \leq N\,, \tag{1}$$

a system of partial differential equations with $P_{i,j}(D) = \sum_{|\alpha| \leq m_{i,j}} a^{\alpha}_{i,j} D^{\alpha}$, $m_{i,j} \in N$, $a^{\alpha}_{i,j} \in C$.

If we consider the matrix $P = \|P_{i,j}(\xi)\|$, of kind (N, n) and the $D_k(\xi)$, $1 \leq k \leq \binom{N}{n}$, are the determinants of submatrix of P of size (n,n), the system (1) is hypoelliptic, (every solution of (1) is C^∞ where are C^∞ the f_i), if and only if the ideal in $C[\xi]$, whose generators are the $D_k(\xi)$, contains a polynomial Q which is hypoelliptic (3). (L'ech's work shows that: if I is an ideal in $C[\xi]$, there exits a polynomial $P \in$ I, and a constant $c \in R_+$ such that: for every real $a \in R^n$ we have: $d(a, P) \leq cd(a, I)$, where $d(a, P)$ is the distance, in C^n, of a from the variety of zeros of P in C^n, and $d(a,$ I) is the distance of a from the variety of zeros of I: if I $= (P_1, \ldots, P_n)$ the variety of zeros of I is $(z \in C^n : P_k(z) = 0,$

(*) Indirizzo dell'A.: Seminario Matematico, Via Belzoni, 7, 35100 PD.

$i \leq k \leq n$)). In the case of elliptic system, in (4) is proved that a system like (1) is elliptic if and only if the variety N of the zeros of P has the following propriety:

$$(2) \qquad z \in N \text{ implies : } |z| \leq c(1 + |Im\ z|)$$

where $c \in R_+$ and $Im\ z$ lies for the imaginary part of z.

(2) above, is valid if and only if: $a \in R^n$ gives: $|a| \leq c'$ $(1 + d(a, \mathrm{I}))$, with symbols like above; as a consequence of Leach's work: the system (1) is elliptic if and only if the ideal I, whose generators are the D_k, contains an elliptic polynomial.

2) As usual, to verify that a square system of partial differential equations with constant coefficients is elliptic, we calculate its determinant and we control, of it, its real characteristics [1].

In this note, I *like to give a method» by real characteristics*», to inquire if an overdeterminated system is elliptic, (it does not seem to the A. that such method is present in the literature).

Let (1) be the system; let I be the ideal of the system (1); for every P which belongs to I, we indicate with P_m its principal part; we also indicate with $C(P_m)$ the characteristic cone of P: $C(P_m) = (\xi \in R^n/\{0\} : P_m(\xi) = 0))$. We prove the following:

THEOREM, *The following two propositions, a) and b), are equivalent*:

a) the system (1) *is elliptic*;

b) $\wedge\, (C(P_m) : P \in \mathrm{I}) = \{o\}$, *(zero of* R^n*)*, [2].

PROOF. *a*) $\Rightarrow$ *b*). If the system (1) is elliptic, there exists $Q \in \mathrm{I}$ with Q elliptic; so we have the implication.

b) $\Rightarrow$ *a*). If we suppose the system (1) non elliptic, for every $P \in \mathrm{I}$, we have: $C(P_m) \supsetneqq \{o\}$, (by Lech's theorem). Let S_{n-1} the

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[1] In my opinion the first who gave this definition of square elliptic system is A. Avantaggiati in (1). (As he shows, this definition is more extensive of the last one before that, of Douglis-Nirenberg).

[2] It's not true that the propriety in *b*) proposition of the theorem is necessary for the ellipticity of the system if it is referred only to the generators of the ideal I of the system; a simple countrexample is fornished by the system: $\{D_x^3 - D_y^3,\ D_x^3 - D_y^3, -Q(D_x, D_y)\}$ if $Q(D_x, D_y)$ is the Laplace's operator.

sphere of R^n, and let $P_1, \dots, P_n$ are a finite number of polynomials of I.

We want to show that:

$$C(P_{1,m_1}) \wedge \dots \wedge (P_{n,m_n}) \wedge S_{n-1} \neq \varnothing .$$

In fact: if the intersection above is $\varnothing$, let k_i be a positive integer such that: $m_i + k_i = n$, $1 \leq i \leq n$.

The polynomial $R_{2_n} = \Sigma_i P_i \overline{P}_i |\xi|^{2k_i}$, where $\overline{P}$ is the coniugate of P, is, of course, in I; R is elliptic: in fact if $a \in R^n/\{o\}$ and $R_{2n}(a) = 0$, $|P_{i,m_i}(a)|^2 = 0$, $1 \leq i \leq n$, because: $R_{2n} = \Sigma_i P_i \overline{P}_i |\xi|^{2k_i}$. Absurd.

So far we have:

if $F = (C(P_m) \wedge S_{n-1})$ is the family of closed subset of S_{n-1}, for every $P \in \mathrm{I}$, F has the propriety of finite intersection if the system (1) is not elliptic.

For compactness of S_{n-1}, F has non empty intersection.

This implies: $b) \Rightarrow a)$.

To conclude this item, we consider the following example: in R^n, let the system S be:

$$S : P_i u = (D_i^{2k_i} + \text{l.o.t.})\, u = 0, \; k_i \geq 1, \; 1 \leq i \leq N \,(^3),$$

(where $D_i = D_{x_i}$).

With the method above it is immediate to see that S is an elliptic system, for: $C(P_i) = (\xi \in R^n/\{o\} : \xi_i = o)$. Not so easy is the application of the method in 1) formula (2).

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(3) l.o.t. = lower order terms.

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