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Aldo Ursini

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# A sequence of theories for arithmetic whose union is complete

ALDO URSINI (*)

SOMMARIO - Si studia una successione di teorie formali del primo ordine, secondo una proposta di R. Magari in [3] 8, n° 4. Si tratta di una successione numerabile crescente costruita a partire dall'aritmetica di PEANO, ed aggiungendo al passo $n+1$—*mo* come assiomi le proposizioni che sono, in un certo senso, dimostrabilmente falsificabili, se false, entro il passo precedente, e la cui falsità non è una tesi nel passo precedente (cioè : che siano indecidibili nella *n-ma* teoria). L'*n-ma* teoria $Q_n$ è un insieme di $\Sigma_{n+1}$ nella gerarchia aritmetica ; in $Q_n$ sono numerate — nel senso di S. Feferman,[1],— tutte e sole le relazioni di $\Sigma_{n+1}$ ; $Q_n$ è incompleta e la sua incompletezza è una tesi di $Q_{n+1}$ : inoltre $Q_{n+1}$ dimostra la formalizzazione « standard » della asserzione che $Q_n$ è consistente, la quale, invece, non è dimostrabile in $Q_n$ ; e $\bigcup_{n\in\omega} Q_n$ è l'insieme delle proposizioni dell'aritmetica al I° ordine vere nel modello standard.

SUMMARY - We study a sequence of formal theories of the first order, following a proposal of R. Magari's in [3], § 8, n° 4. It is a denumarable encreasing sequence starting from PEANO arithmetic, and taking as axioms at the $n+1$—*st* stage the set of those sentences whose negation is not provable in the $n$—*th* and such that, if false, they are provably falsifiable by the $n$—*th* theory. The $n$—*th* theory $Q_n$ is a set of $\Sigma_{n+1}$ in the arithmetical hyerarchy ; in $Q_n$ are numerated — in the sense of [1] — exactly the relations of $\Sigma_{n+1}$ ; $Q_n$ is incomplete and consistent (if PEANO arithmetic is consistent) and cannot prove the « standard » formalization of its own consistency ; $Q_{n+1}$ can prove the incompleteness and consistency of $Q_n$ ; $\bigcup_{n\in\omega} Q_n$ is the set of true sentences of first order arithmetic.

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(*) Indirizzo dell'A. : Istituto Matematico - Università di Siena.
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## Introduction.

The aim of the present paper is to investigate a proposal of Magari's ([3], § 8, N. 4). The present author found that the framework proposed there should be somehow modified in order to get the desired results (i. e. results generalizing those obtained in [3] when passing from $T$ to $V_0$), (cfr. also [7]).

I employ two sequences of theories: one « principal » $(Q_n)_{n\in\omega}$ and one « ancillary » $(T_n)_{n\in\omega}$; $Q_n$ would correspond to the set $V_n$ proposed in [3], loc. cit.; $T_n$ is a recursive extension of Peano Arithmetic, $T_n \subseteq Q_{n+1}$; and the rôle of $T_n$ is pretty strong: it has to prove a restricted form of the $\omega$—consitency of $Q_n$. This will be proved equivalent to:

i) $Tn$ proves a restricted form of reflection principle for $Q_n$; as well as to:

ii) $T_n$ proves that $Q_n$ has a truth definition for $\Pi_n$-formulas.

Such a construction may be obtained in many trivial ways: hence the interest of the one *I* give here, if any, lies in the way the passage from $Q_n$ to $Q_{n+1}$ is accomplished.

The principal result is Th. 21 below, which immediatly gives, the completeness of $\bigcup_{n\in\omega} Q_n$.

Open problems are:

— To compare this (highly non-constructive) completion with those achieved by Transfinite Recursive Progressions (see [2] and [6]);

— To prove (or disprove) the following:

« Each relation of $\sum_{n+1} \cap \Pi_{n+1}$ is binumerable in $Q_n$, and conversely ».

Apart from minor obvious changes in notation, *I* adopt the terminology, symbolism and results of [1], [2] and occasionally of [6]. $V$ is the set of the sentences of $K_0$ which are true in the standard model. A theory $\langle A, K_0\rangle$ will be denoted simply by $A$; a formula $\varphi$ with $Fv(\varphi) = \{v_0, \ldots, v_{n-1}\}$ is called a *semirepresentative* (resp. a *representative*) *in* $A$ (cfr. [3]) if it numerates, (resp. binumerates) in $A$ the relation $\hat{\varphi}$ defined by:

$$\langle a_0, \ldots, a_{n-1}\rangle \in \hat{\varphi} \text{ iff } \varphi(\bar{a}_0, \ldots, \bar{a}_{n-1}) \in V .$$

The following conventions will be used thorough.

1) *PRF* is the set of *PR*-formulas ; $\Sigma_n F$ is the set of the formulas which in prenex form have the matrix in *PRF* and a prefix which is $\Sigma_n$ ; $\Pi_n F$ is defined similarly.

2) If *I* claim that the formulas of some class $X$ belong to a class $Y$ and this has to hold independently of the number of free variables of the formulas involved, *I* assert something like the following :

« If $\varphi \in X$ , $Fv(\varphi) = \{x\}$ (or : $= \{x, y\}$) (*ahronov*), then $\varphi \in Y$ », where « ahronov » is the famous russian word meaning : « a harmless restriction on the number of free variables ».

3) Let $\varphi \in Fm_{K_0}$, with $x$ free ; for $\psi \in Fm_{K_0}$, $Fv(\psi) \subseteq \{x, y\}$ (ahronov), then $\varphi(\overline{\forall x\psi}(x, \dot{y}))$ stands for : $\varphi(\bar{g}(\dot{y}))$, where $g(y) = \forall x\psi(x, y)$. A similar convention for $\exists x$.

4) Let $\alpha$ be a formula with one free variable ; let $A \subseteq Fm_{K_0}$ ; then $A - \omega - \mathrm{con}_\alpha$ is the set of the generalizations of all formulas :

$$\underset{\cdot}{Pr}_\alpha(\overline{\neg \forall x\varphi}(x, \dot{y})) \rightarrow \neg \forall x \underset{\cdot}{Pr}_\alpha(\bar{\varphi}(\dot{\bar{x}}, \dot{y}))$$

where $\varphi \in A$, $Fv(\varphi) \subseteq \{x, y\}$ (ahronov).

5) If $A \subseteq Fm_{K_0}$, and $\varphi_0, \ldots, \varphi_k \in Fm_{K_0}$, then

$$\begin{aligned} \vdash_{\overline{A}} \varphi_0 &\rightarrow \varphi_1 \\ &\rightarrow \varphi_2 \\ &\cdots\cdots \\ &\rightarrow \varphi_k \end{aligned}$$

is an abbreviation of : « $\vdash_A \varphi_0 \rightarrow \varphi_1$, $\vdash_{\overline{A}} \varphi_1 \rightarrow \varphi_2$, ..., $\vdash_{\overline{A}} \varphi_{k-1} \rightarrow \varphi_k$ » ; and lastly, for $B \subseteq Fm_{K_0}$, $\vdash_{\overline{A}} B$ is an abbreviation *of* : « $\vdash_{\overline{A}} \varphi$ for each $\varphi \in B$ ».

I want to define : a sequence of sets of sentences of $K_0$

$$(R_n)_{n\in\omega}$$

*and* a sequence of formulae with only $x$ free,

$$(\alpha_n)_{n\in\omega}$$

with certain properties to be promptly specified.

I put : $Q_n = Pr_{R_n}$, and $\dot{Q}_n = \underset{\cdot}{Pr}_{\alpha_n}$. Moreover, let us define a sequence $(P_n)_{n\in\omega}$ of auxiliary theories in $K_0$:

$$P_0 = \text{Peano's Arithmetic } P ;$$

$$P_{n+1} = P_n \cup \Sigma_{n+1} F - \omega - \text{con}_{\alpha_n+1} .$$

Each $P_n$ is a recursive extention of $P$, admitting a natural binumeration $\pi_n$ in R. Robinson Arithmetic $Q$, $\pi_n \in PR\text{-}F$. Let us put:

$$T_n = Pr_{P_n}, \text{ (and } T_{-1} = T_0) ;$$

$$\dot{T}_n = \underset{\cdot}{Pr}_{\pi_n}$$

Hence $\dot{T}_n$ numerates $T_n$ in $Q$.

The properties $R_n$ and $a_n$ must satisfy, are the following:

$A_n$) Each formula of $\Sigma_{n+1} F$ is a semirepresentative in $R_n$;

$B_n$) For each $\psi \in \Sigma_{n+1} F$, with at most $x$ free —(ahronov)— :

i) $\vdash_{T_{n-1}} \psi(\dot{x}) \rightarrow \dot{Q}_n(\bar{\psi}(\dot{\bar{x}}))$,

ii) $\vdash_{T_n} \dot{Q}_n(\bar{\psi}(\dot{\bar{x}})) \rightarrow \psi(x)$.

$C_n$) A relation $R \in \Sigma_{n+1}$ iff it is numerable in $R_n$.

$D_n$) $\dot{Q}_n \in \Sigma_{n+1} F$, and $\dot{Q}_n$ numerates $Q_n$ in $R_n$.

$E_n$) $R_n \subseteq V$.

For the step $n = 0$, we let:

$$R_0 = \text{R.Robinson's Arithmetic } Q ;$$

$$a_0 = [Q] .$$

Then it is well known that $A_0, \ldots, E_0$ hold (cfr. [1], [2]).

Suppose now that $R_i, a_i$ be given for $i \leq n$, and that $A_i \div E_i$ hold for $i \leq n$. Then let us define:

$$R_{n+1} = \{a \in St_{K_0} \mid (a \in Q_n) \text{ or } (\neg\, a \notin Q_n \text{ and } \neg\, a \rightarrow \dot{Q}_n(\overline{\neg\, a}) \in T_n\} .$$

I will firstly give some properties of $R_{n+1}$. Let $U_{n+1}$ be the smallest set $Z \subseteq \omega$ such that:

(1) $(Q_n \cup T_n) \cap St_{K_0} \subseteq Z$ ;

(2) if $a \in St_{K_0}$, $\neg a \notin Q_n$ and $\neg a \rightarrow \dot{Q}_n(\overline{\neg a}) \in Z$, then $a \in Z$.

PROPOSITION 1. i) $T_n \cap St_{K_0} \subseteq V$.

ii) $T_n \cap St_{K_0} \subseteq R_{n+1} \subseteq U_{n+1}$ ;

iii) $R_{n+1} \subseteq V$ ;

iv) $Pr_{U_{n+1}} = Q_{n+1}$ ;

v) $R^{-}_{n+1} \subseteq \{a | a \in St_{K_0}, \neg a \notin Q_n$ and $\neg a \rightarrow \dot{Q}_n(\overline{\neg a}) \in R^{-}_{n+1}\} \subseteq \{a | a \in St_{K_0}, \neg a \notin Q_n$ and $\neg a \rightarrow \dot{Q}_n(\overline{\neg a}) \in Q_{n+1}\} = Q_{n+1} \cap St_{K_0}$.

PROOF. i) By induction on $h \leq n$; it is true for $h = 0$; hence it is enough to show that $\Sigma_{h+1} F - \omega - \text{con}_{\dot{Q}_{n+1}} \subseteq V$, assuming that $T_h \cap St_{K_0} \subseteq V$ $(h < n)$. Let $\varphi \in \Sigma_{h+1} F$, $Fv(\varphi) = \{x, y\}$ (ahronov). By absurd, let $b \in \omega$ such that:

$$\dot{Q}_{h+1}(\overline{\neg \forall x \varphi (x, \dot{\bar{b}})}) \wedge \forall x \dot{Q}_{h+1} (\overline{\varphi} (\dot{\bar{x}}, \dot{\bar{b}})) \in V;$$

then we would have:

$$\neg \forall x \varphi (x, \bar{b}) \in Q_{h+1} \text{ and for all } a \in \omega, \varphi (\bar{a}, \bar{b}) \in Q_{h+1},$$

which is absurd, because of $E_{h+1}$.

ii) Let a be a sentence of $T_n$; then $\neg$ $a$ is false and, by $E_n$, $\neg a \notin Q_n$; obviously, $\neg a \rightarrow \dot{Q}_n(\overline{\neg a}) \in T_n$; therefore : $a \in R_{n+1}$. Let $a \in R_{n+1}$, and let $Z$ satisfy (1) and (2); then $\neg$ $a \rightarrow \dot{Q}_n(\overline{\neg a}) \in Z$ because of (1), and $\neg$ $a \in Q_n$; therefore $a \in Z$, and also: $a \in U_{n+1}$.

iii) Obviously, $V$ is one of the $Z$ satisfying (1) and (2).

iv) It si enough to show that:

(3) $Q_{n+1} \cap St_{k_0}$ satisfies (1) and (2).

From ii) it follows that $Q_{n+1} \cap St_{K_0}$ satisfies (1). Now let $p$ be a sentence such that $\neg p \notin Q_n$ and $\neg p \to \dot{Q}_n(\overline{\neg p}) \in Q_{n+1} \cap St_{K_0}$. Then two cases are possible:

0) $p \in Q_n$; then $p \in Q_{n+1}$.

00) $p \notin Q_n$; then let $r = \neg p \to \neg \dot{Q}_n(\overline{\neg p})$: it is enough to show that $r \in Q_{n+1}$. Two cases are possible:

$00_1$) $r \in Q_n$, and then $r \in Q_{n+1}$;

$00_2$) $r \notin Q_n$; then observe that $\neg r \notin Q_n$, and moreover:

$$\vdash_{T_n} (\neg r \to \dot{Q}_n(\overline{\neg r})) \longleftrightarrow (\neg r \to \dot{Q}_n(\overline{\neg p \,\dot{\wedge}\, \dot{Q}_n(\overline{\neg p})}))$$
$$\longleftrightarrow (\neg r \to \dot{Q}_n(\overline{\neg p}))$$

(that the last equivalence holds follows from:

$$\vdash_{T_n} \dot{Q}_n(x) \to \dot{Q}_n(\overline{\dot{Q}_n(\dot{x})})$$

which follows, in turn, from $D_n$, $B_n$.)

But $\neg r \to \dot{Q}_n(\overline{\neg p})$ is an instance of a logical axiom, therefore $\neg r \to \dot{Q}_n(\overline{\neg r}) \in T_n$; hence $r \in Q_{n+1}$.

v) The only thing which requires a proof is the last equality. Let a be a sentence, $\neg a \notin Q_n$ and $\neg a \to \dot{Q}_n(\overline{\neg a}) \in Q_{n+1}$. Suppose that $\neg \dot{Q}_n(\overline{\neg a}) \notin R_{n+1}$; since $\dot{Q}_n(\overline{\neg a}) \to \dot{Q}_n(\overline{\dot{Q}_n \neg (a)}) \in T_n$, one should conclude that $\dot{Q}_n(\overline{\neg a}) \in Q_n$, and, by $D_n$, $\neg a \in Q_n$. Therefore $\neg \dot{Q}_n(\overline{\neg a}) \in R_{n+1}$; consequently $a \in Q_{n+1}$. The reverse inclusion is clear.

Corollary 2. If a is a sentence, $a \notin Q_n$ and $a \to \dot{Q}_n(\bar{a}) \in T_n$

(or: $a \to \dot{Q}_n(\bar{a}) \in Q_{n+1}$), then $\neg a \in Q_{n+1}$.

Proposition 3. i) $\dot{Q}_n$ binumerates $Q_n$ in $Q_{n+1}$.

ii) If $a, b$ are sentences, and $b \notin Q_n$ and $\neg a \to Q_n(\bar{b}) \in Q_{n+1}$, then $a \in Q_{n+1}$.

iii) If $\varphi \in R_{n+1}$ (or: $\varphi \in Q_{n+1}$) then $\neg \varphi \notin T_n$; hence is $\varphi \in T_n$, then $\neg \varphi \notin Q_n$.

PROOF. (i) That $\dot{Q}_n$ numerates $Q_n$ in $Q_{n+1}$ follows from $D_n$ and from Prop. 1 (iii). Let $a \notin Q_n$ ; then $\dot{Q}_n(\bar{a}) \notin Q_n$ ; but

$$\dot{Q}_n(\bar{a}) \rightarrow \dot{Q}_n(\overline{\dot{Q}_n(\bar{a})}) \in T_n\,,$$

therefore, by Cor. 2, $\neg\, \dot{Q}_n(\bar{a}) \in Q_{n+1}$.

(ii) follows from (i).

(iii) Is immediate.

PROPOSITION 4. For each $h \leq n$, $\vdash_{T_h} \underset{\bullet}{\mathrm{Con}}_{\alpha_h}$.

Proof. Observe that :

$$\vdash_{T_0} \underset{\bullet}{\mathrm{Con}}_{\alpha_h} \longleftrightarrow \neg\, \dot{Q}_h(\overline{\exists x\,(x \not\approx x)})\,.$$

But : $\vdash_{T_h} (x \approx x) \rightarrow \dot{Q}_h\,(\dot{\dot{x}} \approx \dot{\dot{x}})$, therefore

$$\vdash_{T_h} \forall x(x \approx x) \rightarrow \forall x\, \dot{Q}_h(\dot{\dot{x}} \approx \dot{\dot{x}})$$
$$\rightarrow \neg\, \dot{Q}_h(\overline{\neg\, \forall x\,(x \approx x)})$$
$$\rightarrow \neg\, \dot{Q}_h(\overline{\exists x(x \not\approx x)})\,.$$

And consequently $\vdash_{T_h} \underset{\bullet}{\mathrm{Con}}_{\alpha_h}$.

PROPOSITION 5. (i) Each $\psi \in \Sigma_{n+1}F \cup \Pi_{n+1}F$ is a representative in $Q_{n+1}$.

(ii) If $\vartheta$ is a semirepresentative in $Q_{n+1}$, then also $\exists x\vartheta$ is a semirepresentative in $Q_{n+1}$.

(iii) Each $\psi \in \Sigma_{n+2}\,F$ is a semirepresentative in $Q_{n+1}$.

PROOF. Let $\psi \in \Sigma_{n+1}F$, $Fv(\psi) = \{x\}$ (ahronov) ; let $a \in \omega$ ; if $\psi(\bar{a}) \in V$, then $\psi(\bar{a}) \in Q_n$, therefore $\psi(\bar{a}) \in Q_{n+1}$. If $\neg\, \psi(\bar{a}) \in V$, then $\psi(\bar{a}) \notin Q_n$, and $\psi(\bar{a}) \rightarrow \dot{Q}_n(\overline{\psi\,(\bar{a})}) \in T_n$: therefore $\neg\, \psi(\bar{a}) \in Q_{n+1}$. For $\psi \in \Pi_{n+1}$, apply the preceding result to $\neg\, \psi$ ; Therefore (i) holds.

(ii) Let $Fv(\vartheta) = \{x, y\}$ (ahronov) ; let $\zeta = \exists y\vartheta(x, y)$. If $\zeta(\bar{a}) \in V$, then for some $b \in \omega$, $\vartheta\,(\bar{a}, \bar{b}) \in V$; hence $\vartheta(\bar{a}, \bar{b}) \in Q_{n+1}$ ; by logic, one gets $\exists\, y\, \vartheta(\bar{a}, y) \in Q_{n+1}$.

(iii) is immediate from (i) and (ii).

PROPOSITION 6. For $i \leq n+1$, $Q_i \in \Sigma_{i+1}$.

PROOF. By induction on $i$: obviously $Q_0 \in \Sigma_1$. Let us suppose that $Q_j \in \Sigma_{j+1}$. Since $R_{j+1}$ is Turing reducible to $(T_j \times \bar{Q}_j) \cup Q_j$; then $R_{j+1}$ is in $\Delta_{j+2}$: therefore $Q_{j+1} \in \Sigma_{j+2}$.

PROPOSITION 7. A relation $R$ is in $\Sigma_{n+2}$ iff $R$ is numerable in $Q_{n+1}$.

PROOF. If $R \in \Sigma_{n+2}$, by Kleene's Enumeration and Normal Form Theorem, and by Prop. 5 (iii), $R$ is numerable in $Q_{n+1}$. If $R$ is numerable in $Q_{n+1}$, it is $1-1$ reducible to $Q_{n+1}$: by Prop. 6 $R$ is in $\Sigma_{n+2}$.

Now let $M_n$ be a p.r. extention of $P$, which has any term representing p.r. functions necessary for arithmetization (say: $M_n$ contains the set $\mathfrak{M}$ of § 4 of [1], and moreover $M_n$ has two unary terms $(\dot{n})$, $\ddot{Q}_n$, representing respectively the primitive recursive functions mapping $k \in \omega$ into $k \rightarrow \dot{Q}(\bar{k})$, and into: $\dot{Q}(\bar{k})$, respectively, and such that:

$$\vdash_{M_n} (\dot{n})(x) \approx x \mathbin{\dot{\rightarrow}} \ddot{Q}_n(x)$$

and

$$\vdash_{M_n} (\dot{n}) \mathbin{\dot{\neg}} \bar{\varphi}(\dot{\bar{x}}) \approx \bar{\varphi}_n(\dot{\bar{x}})$$

for each formula $\varphi$ with $x$ free (ahronov), where

$$\varphi_n = \neg\varphi(x) \rightarrow \dot{Q}_n(\overline{\neg\varphi}(\dot{\bar{x}})).$$

Then, to be pedantically precise, I put

$$\alpha_{n+1} = \underset{\cdot}{S}t_{K_0}(x) \wedge (\dot{Q}_n(x) \vee (\dot{T}_n((\dot{n}) \mathbin{\dot{\neg}} (x)) \wedge \neg \dot{Q}_n(\mathbin{\dot{\neg}} x)));$$

$$a_{n+1} = (\alpha_{n+1})^{M_n}.$$

PROPOSITION 8. $a_{n+1}$ binumerates $R_{n+1}$ in $Q_{n+1}$.

PROOF. Firstly observe that $a_{n+1} \in \Sigma_{n+2} F \cap \Pi_{n+2} F$; hence it is a semirepresentative in $Q_{n+1}$, and obviously it numerates $R_{n+1}$ in $Q_{n+1}$. Moreover, if $a \notin R_{n+1}$, then $a_{n+1}(\bar{a}) \rightarrow \dot{Q}_n(\overline{a_{n+1}(\bar{a})})$ is a $\Sigma_{n+2}$ and true sentence: hence it belongs to $Q_{n+1}$, and therefore $\neg a_{n+1}(\bar{a}) \in Q_{n+1}$.

COROLLARY 9. $\dot{Q}_{n+1}$ is a $\Sigma_{n+2}\, F$, and it numerates $Q_{n+1}$ in $Q_{n+1}$.

LEMMA 10. Let $\vartheta \in Fm_{K_0}$, then:

(i) $\vdash_{T_0} \underset{\cdot}{A}x_{K_0}\left(\overline{\forall v_0\, \vartheta} \underset{\cdot}{\rightarrow} \vartheta\,(\dot{\bar{v}}_0)\right)$;

(ii) $\vdash_{T_0} \underset{\cdot}{A}x_{K_0}\left(\bar{\vartheta}\,(\dot{\bar{v}}_0) \underset{\cdot}{\rightarrow} \overline{\exists\, v_0\, \vartheta}\right)$.

Let $\alpha$ be a formula of $K_0$, $Fv(\alpha) = \{x\}$; then for each $\varphi \in Fm_{K_0}$, with $Fv(\varphi) = \{x, y\}$ (ahronov);

(iii) $\vdash_{T_0} \exists x\, \underset{\cdot}{P}r_\alpha\left(\bar{\varphi}\,(\dot{\bar{x}}, \dot{\bar{y}})\right) \rightarrow \underset{\cdot}{P}r_\alpha\left(\overline{\exists x\, \varphi}\,(x, \dot{\bar{y}})\right)$;

(iv) $\vdash_{T_0} \underset{\cdot}{P}r_\alpha\left(\overline{\forall\, x\, \varphi}\,(x, \dot{\bar{y}})\right) \rightarrow \forall x\, \underset{\cdot}{P}r_\alpha\left(\bar{\varphi}\,(\dot{\bar{x}}, \dot{\bar{y}})\right)$.

PROOF. This is routine of arithmetization. For (i), remember that:

$$\vdash_{T_0} \left(\underset{\cdot}{V}r(x) \wedge \underset{\cdot}{F}m_{K_0}(y) \wedge \underset{\cdot}{T}m_{K_0}(z)\right) \rightarrow \underset{\cdot}{A}x_{K_0}\left(\underset{x}{\underset{\cdot}{\wedge}}\, y \underset{\cdot}{\rightarrow} \underset{\cdot}{S}b\, {}^{x}_{z}\, y\right)$$

and that: $\vdash_{T_0} \underset{\cdot}{v}r\bar{o} \approx \bar{v}_0$;

and hence: $\vdash_{T_0} \underset{\cdot}{V}r\,(\underset{\cdot}{v}r\bar{o})$.

By induction one shows that $\vdash_{T_0} \forall x\; \underset{\cdot}{T}m_{K_0}\,(\underset{\cdot}{n}m_x)$.

Therefore $\vdash_{T_0} \underset{\cdot}{A}x_{K_0}\,\left(\underset{\bar{v}_0}{\underset{\cdot}{\wedge}}\, \vartheta \underset{\cdot}{\rightarrow} \underset{\cdot}{S}b\, {}^{\underset{\cdot}{v}r\bar{o}}_{\underset{\cdot}{n}m_{v_0}}\, \bar{\theta}\right)^{-}$.

But: $\vdash_{T_0} \underset{\underset{\cdot}{v}r\bar{o}}{\underset{\cdot}{\wedge}}\, \bar{\vartheta} \approx \overline{\forall v_0 \vartheta}$; and (i) follows; (ii) is proved quite similarly, and (iii), (iv) follow immediatly.

LEMMA 11. For any formula $\varphi$, with $x$ free (ahronov), one has:

$$\vdash_{T_0} \neg\, \dot{Q}_n\left(\overline{\neg\, \varphi}(\dot{\bar{x}})\right) \wedge \dot{T}_n(\bar{\varphi}_n(\dot{\bar{x}})) \rightarrow \dot{Q}_{n+1}\left(\bar{\varphi}(\dot{\bar{x}})\right)$$

(where $\varphi_n$ is as above).

PROOF. This follows from [1], Th. 4.6(iii), and from the fact that

$$\vdash_{T_0} \underset{\cdot}{S}t_{K_0}\left(\overline{\neg\, \varphi}\,(\dot{\bar{x}})\right).$$

LEMMA 12. For $h \leq n$,

$$\vdash_{T_0} \dot{Q}_h(x) \wedge \dot{T}_h(\dot{\neg}\, x \underset{\cdot}{\to} \ddot{Q}_h(\dot{\neg}\, x)) \to \dot{Q}_{h+1}(x)$$

PROPOSITION 13. For $h \leq n$,

$$\vdash_{T_h} \dot{Q}_h(x) \to \dot{Q}_{n+1}(x)$$

REMARK. Here, and in similar cases, one should add to the premiss : « $Fm_{K_0}(x)$ » or something like that. This may easily supplied by the reader in each case.

PROOF. By induction on $h$. Let $h = 0$ ; then remember that :

$$\vdash_{T_0} \dot{Q}_0\,(x) \to \neg\; \dot{Q}_0(\dot{\neg}\, x)$$

and that

$$\vdash_{T_0} \dot{Q}_0(x) \to \dot{T}_0(x) \;\; ; \;\; \vdash_{T_0} \dot{Q}_0(x) \to \dot{T}_0(\dot{\neg}\, x \underset{\cdot}{\to} \ddot{Q}_0(\dot{\neg}\, x)) \, ;$$

therefore, by lemma 12,

$$\vdash_{T_0} \dot{Q}_0(x) \to \dot{Q}_1(x) \, .$$

Let us suppose that the theorem holds for $h < n$. We have :

$$\vdash_{T_{h+1}} \dot{Q}_h(\dot{\neg}\, x) \to \dot{Q}_{h+1}\,(\dot{\neg}\, x) \, ;$$

and hence :

$$\vdash_{T_0} \dot{T}_{h+1}\,(\ddot{Q}_h\,(\dot{\neg}\, x) \underset{\cdot}{\to} \ddot{Q}_{h+1}\,(\dot{\neg}\, x)) \, .$$

$$\vdash_{T_0} a_{h+1}(x) \to (\neg\, \dot{Q}_h(x) \to \dot{T}_h\,(\dot{\neg}\, x \underset{\cdot}{\to} \ddot{Q}_h(\dot{\neg}\, x)))$$
$$\to (\neg\, \dot{Q}_h(x) \to \dot{T}_{h+1}(\dot{\neg}\, x \underset{\cdot}{\to} \ddot{Q}_{h+1}(\dot{\neg}\, x))) \, ;$$

but one has :

$$\vdash_{T_0} \neg\, a_{h+2}(x) \to (\dot{T}_{h+1}\,(\dot{\neg}\, x \underset{\cdot}{\to} \ddot{Q}_{h+1}\,(\dot{\neg}\, x)) \to \dot{Q}_{h+1}\,(\dot{\neg}\, x))$$

Therefore :

$$\vdash_{T_0} (\alpha_{h+1} \wedge \neg \dot{Q}_h(x) \wedge \neg \alpha_{h+2}(x)) \to \neg \operatorname{Con}_{\alpha_{h+1}};$$

therefore :

$$\vdash_{T_{h+1}} \alpha_{h+1}(x) \to (\dot{Q}_h(x) \dot{\vee} \alpha_{h+2}(x))$$
$$\to \alpha_{h+2}(x) .$$

By Prop. 4. , one concludes :

$$\vdash_{T_{h+1}} \alpha_{h+1}(x) \to \alpha_{h+2}(x)$$

which is something more then required to show the theorem.

COROLLARY 14. $\vdash_{T_{n+1}} \operatorname{Con}_{\alpha_{n+1}}$ .

PROOF. We have :

$$\vdash_{T_{n+1}} x \approx x \to \dot{Q}_n(\dot{\bar{x}} \approx \dot{\bar{x}})$$
$$\to \dot{Q}_{n+1}(\dot{\bar{x}} \approx \dot{\bar{x}})$$

and from there on, the proof is quite similar to that of Prop. 4.

PROPOSITION 15. For each $\psi \in \Sigma_{n+1}F \cup \Pi_{n+1}F$, with $x$ free (ahronov), one has :

$$\text{(i)} \vdash_{T_{n+1}} \dot{Q}_{n+1}(\overline{\neg \psi} (\dot{\bar{x}})) \to \neg \dot{Q}_{n+1} (\bar{\psi} (\dot{\bar{x}}));$$

$$\text{(ii)} \vdash_{T_n} \neg \dot{Q}_{n+1}(\bar{\psi} (\dot{\bar{x}})) \to \dot{Q}_{n+1}(\overline{\neg \psi}(\dot{\bar{x}})) .$$

PROOF. (i) follows from Cor. 14.

(ii) Let $\psi \in \Sigma_{n+1}F$ ; then

$$\vdash_{T_n} \psi(x) \to \dot{Q}_n(\bar{\psi}(\dot{\bar{x}}))$$

whence :

$$\vdash_{T_0} \dot{T}_n ((\dot{n}) (\bar{\psi}(\dot{\bar{x}})))$$

But, by prop. 13,

$$\vdash_{T_n} \neg \dot{Q}_{n+1} (\bar{\psi}(\dot{x})) \rightarrow \neg \dot{Q}_n(\bar{\psi}(\dot{x})) .$$

therefore (ii) holds. To get (ii) for $\psi \in \pi_{n+1}F$, apply it to $\neg \psi$.

PROPOSITION 16. The following are equivalent (with *ahronov* of formulas involved, when suitable):

*a)* $\vdash_{T_n} \forall x(\dot{Q}_n(\bar{\vartheta}(\dot{x}))) \rightarrow \forall x\vartheta(x)$ , for $\vartheta \in \Sigma_{n+1} F$ ;
*b)* $\vdash_{T_n} \forall x\vartheta(x)$ , for $\vartheta \in \Sigma_{n+1}F$ and such that $\vdash_{T_{n-1}} \forall x\dot{Q}_n(\bar{\vartheta}(\dot{x}))$ ;
*c)* $\vdash_{T_n} \dot{Q}_n(\bar{\vartheta}(\dot{x})) \rightarrow \vartheta(x)$ , for $\vartheta \in \Sigma_{n+1}F$ ;
*d)* $\vdash_{T_n} \xi(x) \rightarrow \dot{Q}_{n+1}(\bar{\xi}(\dot{x}))$ , for $\xi \in \Pi_{n+1}F$ ;
*e)* $\vdash_{T_n} \zeta(x) \rightarrow \dot{Q}_{n+1} (\bar{\zeta}(\dot{x}))$ , for $\zeta \in \Sigma_{n+2}F$ ;
*f)* $\vdash_{T_n} \Sigma_{n+1}F$ — $\omega$-con$_{\alpha_n}$ ;
*g)* $\vdash_{T_n} \Pi_{n+1}F$ — $\omega$-con$_{\alpha_n}$ .

PROOF.

$(a) \Rightarrow (b)$ : this is obvious.

$(b) \Rightarrow (c)$ . To prove this, one employs an analogue of Lemma 2,18 of [2]; namely:

LEMMA. Let $\varphi \in Fm_{K_0}$, with $x$ free (ahronov); put

$$\psi (x , y) = \dot{\mathrm{Prf}}_{\alpha_n} (\bar{\varphi} (\dot{x}) , y) \rightarrow \varphi (x)$$

then one has:

$$\vdash_{T_{n-1}} \forall x \ \forall y \ \dot{Q}_n (\bar{\psi} (\dot{x} , \dot{y}))$$

The proof of the lemma is quite analogous to Feferman's: only observe that $\mathrm{Prf}_{\alpha_n}$ is in $\Pi_{n+1} F$.

Having this lemma, one concludes just as in the proof of Th. 2.19 of [2].

$(c) \Rightarrow (a)$ : This is obvious.

$(c) \Rightarrow (d)$ . By hypothesis,

$$\vdash_{T_n} \xi (x) \rightarrow \neg \dot{Q}_n (\overline{\neg \xi} (\dot{x})) ;$$

moreover :

$$\vdash_{T_n} \neg\ \xi(x) \to \dot{Q}_n(\overline{\neg\ \xi}(\dot{\bar{x}}))\,;$$

whence

$$\vdash_{T_0} \dot{T}_n(\overline{\neg\ \xi}(\dot{\bar{x}}) \dot{\to} \ddot{Q}_n(\overline{\neg\ \xi}(\dot{\bar{x}})))\,.$$

Therefore (by Lemma (1) :

$$\vdash_{T_n} \xi(x) \to \dot{Q}_{n+1}(\bar{\xi}(\dot{\bar{x}}))\,.$$

$(d) \Rightarrow (e)$ ; this is immediate, after lemma 10 (iii).

$(e) \Rightarrow (c)$ is obvious, by $B_n$ (ii) .

$(f) \Rightarrow (g)$ . Let $\psi(x, y) \in \Pi_{n+1} F$ (ahronov) ; one has :

$$\vdash_{T_n} \dot{Q}_n(\overline{\neg \forall x \psi}(x, y)) \to \dot{Q}_n(\overline{\neg \forall x \forall z \varphi}(x, \dot{\bar{y}}, z))$$

(where $\psi = \forall z\, \psi(x, y, z)$ , and $\varphi \in \Sigma_n F$)

$$\to \neg \forall x \forall z\, \dot{Q}_n(\bar{\varphi}(\dot{\bar{x}}, \dot{\bar{y}}, \dot{\bar{z}}))$$
$$\to \quad \exists x \neg \forall z\, \dot{Q}_n(\bar{\varphi}(\dot{\bar{x}}, \dot{\bar{y}}, \dot{\bar{z}}))$$
$$\to \quad \exists x \neg\, \dot{Q}_n(\overline{\neg \forall z \varphi}(\dot{\bar{x}}, \dot{\bar{y}}, z))\,,$$

where the last implication follows from Lemma 10(iv).

$(g) \Rightarrow (f)$ : this follows from : $\Sigma_n F \subseteq \pi_{n+4} F$ .

$(c) \Rightarrow (g)$ . Let $\psi \in \Pi_{n+1}F$ ($\psi = \forall y\, \varphi(x, y, z)$ , $\varphi \in \Sigma_n F$) ; then :

$$\vdash_{T_n} \dot{Q}_n(\overline{\neg \forall x \forall y \varphi}(x, y, \dot{\bar{z}})) \to$$
$$\to \neg \forall x \forall y\, \varphi(x, y, z)$$
$$\to \exists x \neg\ \dot{Q}_n(\overline{\forall x \varphi}(\dot{\bar{x}}, y, \dot{\bar{z}}))$$

(this implication follows from $B_n$(i) and from Prop. 4)

$$\to \neg\ \forall x\, \dot{Q}_n(\bar{\psi}(\dot{\bar{x}}, \dot{\bar{z}}))$$

$(g) \Rightarrow (c)$. Let $\vartheta = \neg \forall x\, \psi(x, y)$, with $\psi \in \Sigma_n F$: then

$$\vdash_{T_n} Q_n(\overline{\neg \forall x \psi}(x, \dot{y})) \rightarrow \neg \forall x \dot{Q}_n(\overline{\psi}(\dot{x}, \dot{y}))$$
$$\rightarrow \neg \forall x\, \psi(x, y)\ 0\ ;$$

(The last implication, by $B_n$(i)).

REMARKS 1. In the preceding proof, any implication having $(c)$ or $(f)$ as a consequent, would be obvious from the induction hypothesis; I have tried to use the latter the less possible; many of the implications follow simply from: $\vdash_{T_n} \dot{\mathrm{Con}}_{\alpha_n}$; in particular, $B_n$(ii) was only used in the proof of: $(e) \Rightarrow (c)$.

2. This kind of analysis leads to the following:

COROLLARY 17. Let $(a'), \ldots, (g')$ be obtained from $(a), \ldots, (g)$ of Prop. 16, by substituting $T_{n+1}$ to $T_n$; let $(h)$ be the following:

$$(h) \quad \vdash_{T_{n+1}} \dot{Q}_{n+1}(\overline{\vartheta}(\dot{x})) \rightarrow \vartheta(x)\,, \text{ for } \vartheta \in \Sigma_{n+1}F\,.$$

Then, under the only hypothesis that: $\vdash_{T_h} \dot{\mathrm{Con}}_{\alpha_h}$, for $h \leq n+1$ (i.e. without using that $\underset{n+1}{\Sigma F} - \omega - \mathrm{con}_{\alpha_{n+1}}$ is provable in $T_{n+1}$), one has:

$$(a')\,,\ (b')\,,\ (c')\,,\ (d')\,,\ (e')\,,\ (f')\,,\ (g')\,,\ (h)$$

are pairwise equivalent.

PROOF. For the most part, the proof of Prop. 16 works here also. The only implication that deserves attention is: $(c') \Rightarrow (h)$.
One has:

$$\vdash_{T_{n+1}} \dot{Q}_{n+1}(\overline{\vartheta}(\dot{x})) \rightarrow \neg \dot{Q}_{n+1}(\overline{\neg \vartheta}(\dot{x}))$$
$$\rightarrow \neg\, a_{n+1}(\overline{\neg \vartheta}(\dot{x}))$$
$$\rightarrow (\dot{T}_n(\overline{\vartheta}(\dot{x}) \dot{\rightarrow} \dot{Q}_n(\overline{\vartheta}(\dot{x}))) \rightarrow \dot{Q}_n(\overline{\vartheta}(\dot{x})))\,.$$

Therefore:

$$\vdash_{T_{n+1}} \dot{Q}_{n+1}(\overline{\vartheta}(\dot{x})) \longleftrightarrow \dot{Q}_n(\overline{\vartheta}(\dot{x}))\,.$$

From proposition 16 there follows immediatly:

COROLLARY 18. $B_{n+1}$ (i) holds.

PROPOSITION 19. The following are equivalent:

(0) $\vdash_{T_{n+1}} \dot{Q}_{n+1}(\bar{\vartheta}(\dot{\bar{x}})) \to \vartheta(x)$, for $\vartheta \in \Sigma_{n+2}F$ (ahronov);
(00) $\vdash_{T_{n+1}} \Sigma_{n+1}F - \omega - \mathrm{con}_{\alpha_{n+1}}$;
(000) $\vdash_{T_{n+1}} \Pi_{n+2}F - \omega - \mathrm{con}_{\alpha_{n+1}}$.

PROOF. (000) $\Rightarrow$ (00) is immediate, and (00) $\Rightarrow$ (000) is proved quite similarly to the proof of $(f) \Rightarrow (g)$ in Prop. 16; (0) $\Rightarrow$ (000) follows from Cor. 18 and 14; (00) $\Rightarrow$ (0) follows Cor. 18.

COROLLARY 20. $B_{n+1}$(ii) holds.

THEOREM 21. There exist two sequences $(R_n)_{n\in\omega}$ and $(\alpha)_{\alpha\in\omega}$ which satisfy $A_n$, $B_n$, $C_n$, $D_n$ and $E_n$.

Among the properties of these sequences, I list the following. Firstly, one cam mimeck the trick of Löb's in [4], to prove:

THEOREM 22. (i) Let $g(x)$ be any formula such that, if $a \in St_{K_0}$ and $\vdash_{Q_n} g(\bar{a})$, then $a \in Q_n$; then for any $p \in St_{K_0}$, if $\vdash_{Q_n} \dot{Q}_n(\overline{g(\bar{p})}) \to g(\bar{p})$, then $p \in Q_n$.
(ii) For $p \in St_{K_0}$, if $\vdash_{Q_n} \dot{Q}_n(\bar{p}) \to p$, then $p \in Q_n$.

PROOF. By diagonalization, let $b \in St_{K_0}$ be such that:

$$\vdash_{T_0} (\dot{Q}_n(\bar{b}) \to g(\bar{p})) \longleftrightarrow b .$$

Then:

$$\vdash_{T_0} \dot{Q}_n(\bar{b}) \to (\dot{Q}_n(\overline{\dot{Q}_n(\bar{b})}) \to \dot{Q}_n(\overline{g(\bar{p})})) .$$

But:

$$\vdash_{T_{n-1}} \dot{Q}_n(\bar{b}) \to \dot{Q}_n(\overline{\dot{Q}_n(\bar{b})}) .$$

(This is true by Th. 5.4. of [1] for $n = 0$, and follows from $B_n$(i) for $n > 0$).

Then we get:

$$\vdash_{Q_n} b\,;$$

whence:

$$\vdash_{Q_n} g(\bar{p})$$

and finally: $p \in Q_n$. ii) is proved in the same way.

THEOREM 23. $Q_{n+1}$ does not belong to $\Sigma_{n+1}$.

PROOF. By the proof of Prop. 7, if a set $S$ is numerable in $Q_n$, it is numerable in $Q_n$ by a formula of $\Sigma_{n+1}F$. Now let us suppose, by absurd, that some formula $g_n \in \Sigma_{n+1}F$ numerates $Q_{n+1}$ in $Q_n$. Let $p$ be any sentence; by considering the sentence $b$ which is equivalent (in $T_0$) to: $\dot{Q}_n(\bar{b}) \rightarrow g_n(\bar{p})$, one would get, as before,

$$\vdash_{Q_n} \dot{Q}_n(\bar{b}) \rightarrow \dot{Q}_n(\overline{g_n(\bar{p})})$$

but, by $B_n$ (ii):

$$\vdash_{Q_{n+1}} \dot{Q}_n(\overline{g_n(\bar{p})}) \rightarrow g_n(\bar{p})$$

whence one could get:

$$\vdash_{Q_{n+1}} b$$

whence:

$$\vdash_{Q_{n+1}} g_n(\bar{p})$$

and hence:

$$\vdash_{Q_n} g_n(\bar{p})$$

and finally:

$$p \in Q_{n+1}$$

Therefore each sentence would be in $Q_{n+1}$, which is absurd.

THEOREM 24. (i) $Q_n$ is not complete, for any $n$; in particular $\mathrm{C\underset{\cdot}{o}n}_{\alpha_n}$ is undecidable in $Q_n$; $\mathrm{C\underset{\cdot}{o}n}_{\alpha_n} \in Q_{n+1}$, and also

$$\text{Not-C}\underset{\cdot}{\text{o}}\text{mp}_{\alpha_n} = \exists x(\neg\, \dot{Q}_n(x) \wedge \neg\, \dot{Q}_n(\dot{\neg}\, x) \wedge \underset{\cdot}{St}_{K_0}(x))$$

is in $Q_{n+1}$; and lastly: $\vdash_{Q_{n+1}} \neg\, \dot{Q}_n(\overline{\mathrm{C\underset{\cdot}{o}n}_{\alpha_n}})$.

(ii) $\bigcup_{n\in\omega} (Q_n \cap St_{K_0}) = V$.

(iii) $T_n \; Q_n \subsetneqq$ ; $T_{n+1} \nsubseteq T_n$.

(iv) (Hilbert-Bernays; Kucnecov; Trahtenbrot-see [5], Ch. XII). $\{V\} \in \Pi^0_2$.

PROOF. (i) and (ii) are immediate.

(iii) $\mathrm{C\underset{\cdot}{o}n}_{\alpha_n} \in T_n$ but $\notin Q_n$; $\mathrm{C\underset{\cdot}{o}n}_{\alpha_{n+1}} \in T_{n+1}$, but $\notin T_n$, (otherwise it would $\in Q_{n+1}$).

(iv) follows from (ii).

Finally, it would be easy to prove (cfr. e.g. [7]) that the set $V_0$ of [3] is exactly $Q_1 \cap St_{K_0}$.

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—Oh me dolente! Come mi riscossi
quando mi prese dicendomi: 'Forse
tu non pensavi ch'io loico fossi'!—

Dante, Inf. XXVII, 120-123.

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