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A sequence of theories for arithmetic whose union is complete

ALDO URSINI (*)

SOMMARIO - Si studia una successione di teorie formali del primo ordine, secondo una proposta di R. Magari in [3] 8, n° 4. Si tratta di una successione numerabile crescente costruita a partire dall'aritmetica di PEANO, ed aggiungendo al passo $n+1$ —*mo* come assiomi le proposizioni che sono, in un certo senso, dimostrabilmente falsificabili, se false, entro il passo precedente, e la cui falsità non è una tesi nel passo precedente (cioè: che siano indecidibili nella n -*ma* teoria). L' n -*ma* teoria Q_n è un insieme di Σ_{n+1} nella gerarchia aritmetica; in Q_n sono numerate — nel senso di S. Feferman,[1],— tutte e sole le relazioni di Σ_{n+1} ; Q_n è incompleta e la sua incompletezza è una tesi di Q_{n+1} : inoltre Q_{n+1} dimostra la formalizzazione «standard» della asserzione che Q_n è consistente, la quale, invece, non è dimostrabile in Q_n ; e $\bigcup_{n \in \omega} Q_n$ è l'insieme delle proposizioni dell'aritmetica al I° ordine vere nel modello standard.

SUMMARY - We study a sequence of formal theories of the first order, following a proposal of R. Magari's in [3], § 8, n° 4. It is a denumerable encreasing sequence starting from PEANO arithmetic, and taking as axioms at the $n+1$ —*st* stage the set of those sentences whose negation is not provable in the n —*th* and such that, if false, they are provably falsifiable by the n —*th* theory. The n —*th* theory Q_n is a set of Σ_{n+1} in the arithmetical hyerarchy; in Q_n are numerated — in the sense of [1] — exactly the relations of Σ_{n+1} ; Q_n is incomplete and consistent (if PEANO arithmetic is consistent) and cannot prove the «standard» formalization of its own consistency; Q_{n+1} can prove the incompleteness and consistency of Q_n ; $\bigcup_{n \in \omega} Q_n$ is the set of true sentences of first order arithmetic.

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Introduction.

The aim of the present paper is to investigate a proposal of Magari's ([3], § 8, N. 4). The present author found that the framework proposed there should be somehow modified in order to get the desired results (i. e. results generalizing those obtained in [3] when passing from T to V_0), (cfr. also [7]).

I employ two sequences of theories: one « principal » $(Q_n)_{n \in \omega}$ and one « ancillary » $(T_n)_{n \in \omega}$; Q_n would correspond to the set V_n proposed in [3], loc. cit.; T_n is a recursive extension of Peano Arithmetic, $T_n \subseteq Q_{n+1}$; and the rôle of T_n is pretty strong: it has to prove a restricted form of the ω -consistency of Q_n . This will be proved equivalent to:

i) T_n proves a restricted form of reflection principle for Q_n ; as well as to:

ii) T_n proves that Q_n has a truth definition for Π_n -formulas.

Such a construction may be obtained in many trivial ways: hence the interest of the one I give here, if any, lies in the way the passage from Q_n to Q_{n+1} is accomplished.

The principal result is Th. 21 below, which immediatly gives, the completeness of $\bigcup_{n \in \omega} Q_n$.

Open problems are:

— To compare this (highly non-constructive) completion with those achieved by Transfinite Recursive Progressions (see [2] and [6]);

— To prove (or disprove) the following:

« Each relation of $\sum_{n+1} \cap \Pi_{n+1}$ is binumerable in Q_n , and conversely ».

Apart from minor obvious changes in notation, I adopt the terminology, symbolism and results of [1], [2] and occasionally of [6]. V is the set of the sentences of K_0 which are true in the standard model. A theory $\langle A, K_0 \rangle$ will be denoted simply by A ; a formula φ with $Fv(\varphi) = \{v_0, \dots, v_{n-1}\}$ is called a *semirepresentative* (resp. a *representative*) in A (cfr. [3]) if it numerates, (resp. binumerates) in A the relation $\hat{\varphi}$ defined by:

$$\langle a_0, \dots, a_{n-1} \rangle \in \hat{\varphi} \text{ iff } \varphi(\bar{a}_0, \dots, \bar{a}_{n-1}) \in V.$$

The following conventions will be used thorough.

1) PRF is the set of PR -formulas ; $\Sigma_n F$ is the set of the formulas which in prenex form have the matrix in PRF and a prefix which is Σ_n ; $\Pi_n F$ is defined similarly.

2) If I claim that the formulas of some class X belong to a class Y and this has to hold independently of the number of free variables of the formulas involved, I assert something like the following :

« If $\varphi \in X$, $Fv(\varphi) = \{x\}$ (or $:= \{x, y\}$) (ahronov), then $\varphi \in Y$ », where « ahronov » is the famous russian word meaning : « a harmless restriction on the number of free variables ».

3) Let $\varphi \in Fm_{K_0}$, with x free ; for $\psi \in Fm_{K_0}$, $Fv(\psi) \subseteq \{x, y\}$ (ahronov), then $\varphi(\overline{\forall x \psi(x, \dot{y})})$ stands for : $\varphi(\overline{g(\dot{y})})$, where $g(y) = \forall x \psi(x, y)$. A similar convention for $\exists x$.

4) Let α be a formula with one free variable ; let $A \subseteq Fm_{K_0}$; then $A - \omega - \text{con}_\alpha$ is the set of the generalizations of all formulas :

$$Pr_\alpha(\overline{\neg \forall x \varphi(x, \dot{y})}) \rightarrow \neg \forall x Pr_\alpha(\overline{\varphi(\dot{x}, \dot{y})})$$

where $\varphi \in A$, $Fv(\varphi) \subseteq \{x, y\}$ (ahronov).

5) If $A \subseteq Fm_{K_0}$, and $\varphi_0, \dots, \varphi_k \in Fm_{K_0}$, then

$$\begin{aligned} \overline{A} \varphi_0 &\rightarrow \varphi_1 \\ &\rightarrow \varphi_2 \\ &\dots\dots \\ &\rightarrow \varphi_k \end{aligned}$$

is an abbreviation of : « $\overline{A} \varphi_0 \rightarrow \varphi_1, \overline{A} \varphi_1 \rightarrow \varphi_2, \dots, \overline{A} \varphi_{k-1} \rightarrow \varphi_k$ » ; and lastly, for $B \subseteq Fm_{K_0}$, $\overline{A} B$ is an abbreviation of : « $\overline{A} \varphi$ for each $\varphi \in B$ ».

I want to define : a sequence of sets of sentences of K_0

$$(R_n)_{n \in \omega}$$

and a sequence of formulae with only x free,

$$(\alpha_n)_{n \in \omega}$$

with certain properties to be promptly specified.

I put: $Q_n = Pr_{R_n}$, and $\dot{Q}_n = Pr_{\alpha_n}$. Moreover, let us define a sequence $(P_n)_{n \in \omega}$ of auxiliary theories in K_0 :

$$P_0 = \text{Peano's Arithmetic } P;$$

$$P_{n+1} = P_n \cup \Sigma_{n+1} F - \omega - \text{con}_{\alpha_{n+1}}.$$

Each P_n is a recursive extension of P , admitting a natural binumeration π_n in R. Robinson Arithmetic Q , $\pi_n \in PR\text{-}F$. Let us put:

$$T_n = Pr_{P_n}, \text{ (and } T_{-1} = T_0);$$

$$\dot{T}_n = Pr_{\pi_n}$$

Hence \dot{T}_n numerates T_n in Q .

The properties R_n and α_n must satisfy, are the following:

- A_n) Each formula of $\Sigma_{n+1} F$ is a semirepresentative in R_n ;
 B_n) For each $\psi \in \Sigma_{n+1} F$, with at most x free $-(\text{ahronov})-$:

- i) $\vdash_{\bar{T}_{n-1}} \psi(\dot{x}) \rightarrow \dot{Q}_n(\bar{\psi}(\dot{x})),$
 ii) $\vdash_{\bar{T}_n} \dot{Q}_n(\bar{\psi}(\dot{x})) \rightarrow \psi(x).$

C_n) A relation $R \in \Sigma_{n+1}$ iff it is numerable in R_n .

D_n) $\dot{Q}_n \in \Sigma_{n+1} F$, and \dot{Q}_n numerates Q_n in R_n .

E_n) $R_n \subseteq V$.

For the step $n = 0$, we let:

$$R_0 = \text{R. Robinson's Arithmetic } Q;$$

$$\alpha_0 = [Q].$$

Then it is well known that A_0, \dots, E_0 hold (cfr. [1], [2]).

Suppose now that R_i, α_i be given for $i \leq n$, and that $A_i \div E_i$ hold for $i \leq n$. Then let us define:

$$R_{n+1} = \{a \in St_{K_0} \mid (a \in Q_n) \text{ or } (\neg a \notin Q_n \text{ and } \neg a \rightarrow \dot{Q}_n(\overline{\neg a}) \in T_n)\}.$$

I will firstly give some properties of R_{n+1} . Let U_{n+1} be the smallest set $Z \subseteq \omega$ such that :

- (1) $(Q_n \cup T_n) \cap St_{K_0} \subseteq Z$;
- (2) if $a \in St_{K_0}$, $\neg a \notin Q_n$ and $\neg a \rightarrow \dot{Q}_n(\overline{\neg a}) \in Z$, then $a \in Z$.

- PROPOSITION 1. i) $T_n \cap St_{K_0} \subseteq V$.
- ii) $T_n \cap St_{K_0} \subseteq R_{n+1} \subseteq U_{n+1}$;
 - iii) $R_{n+1} \subseteq V$;
 - iv) $Pr_{U_{n+1}} = Q_{n+1}$;
 - v) $R_{n+1} \subseteq \{a \mid a \in St_{K_0}, \neg a \notin Q_n \text{ and } \neg a \rightarrow \dot{Q}_n(\overline{\neg a}) \in R_{n+1}\} \subseteq \{a \mid a \in St_{K_0}, \neg a \notin Q_n \text{ and } \neg a \rightarrow \dot{Q}_n(\overline{\neg a}) \in Q_{n+1}\} = Q_{n+1} \cap St_{K_0}$.

PROOF. i) By induction on $h \leq n$; it is true for $h = 0$; hence it is enough to show that $\Sigma_{h+1} F - \omega - \text{con}_{\dot{Q}_{n+1}} \subseteq V$, assuming that $T_h \cap St_{K_0} \subseteq V$ ($h < n$). Let $\varphi \in \Sigma_{h+1} F$, $Fv(\varphi) = \{x, y\}$ (ahronov). By absurd, let $\bar{b} \in \omega$ such that :

$$\dot{Q}_{h+1}(\overline{\neg \forall x \varphi(x, \bar{b})}) \wedge \forall x \dot{Q}_{h+1}(\overline{\varphi(x, \bar{b})}) \in V ;$$

then we would have :

$$\neg \forall x \varphi(x, \bar{b}) \in Q_{h+1} \text{ and for all } a \in \omega, \varphi(\bar{a}, \bar{b}) \in Q_{h+1},$$

which is absurd, because of E_{h+1} .

ii) Let a be a sentence of T_n ; then $\neg a$ is false and, by E_n , $\neg a \notin Q_n$; obviously, $\neg a \rightarrow \dot{Q}_n(\overline{\neg a}) \in T_n$; therefore : $a \in R_{n+1}$. Let $a \in R_{n+1}$, and let Z satisfy (1) and (2) ; then $\neg a \rightarrow \dot{Q}_n(\overline{\neg a}) \in Z$ because of (1), and $\neg a \in Q_n$; therefore $a \in Z$, and also : $a \in U_{n+1}$.

iii) Obviously, V is one of the Z satisfying (1) and (2).

iv) It is enough to show that :

$$(3) Q_{n+1} \cap St_{K_0} \text{ satisfies (1) and (2).}$$

From ii) it follows that $Q_{n+1} \cap St_{K_o}$ satisfies (1). Now let p be a sentence such that $\neg p \notin Q_n$ and $\neg p \rightarrow \dot{Q}_n(\overline{\neg p}) \in Q_{n+1} \cap St_{K_o}$. Then two cases are possible:

0) $p \in Q_n$; then $p \in Q_{n+1}$.

00) $p \notin Q_n$; then let $r = \neg p \rightarrow \neg \dot{Q}_n(\overline{\neg p})$: it is enough to show that $r \in Q_{n+1}$. Two cases are possible:

00₁) $r \in Q_n$, and then $r \in Q_{n+1}$;

00₂) $r \notin Q_n$; then observe that $\neg r \notin Q_n$, and moreover:

$$\begin{aligned} \overline{T}_n(\neg r \rightarrow \dot{Q}_n(\overline{\neg r})) &\longleftrightarrow (\neg r \rightarrow \dot{Q}_n(\overline{\neg p} \wedge \overline{\dot{Q}_n(\overline{\neg p})})) \\ &\longleftrightarrow (\neg r \rightarrow \dot{Q}_n(\overline{\neg p})) \end{aligned}$$

(that the last equivalence holds follows from:

$$\overline{T}_n \dot{Q}_n(x) \rightarrow \dot{Q}_n(\overline{\dot{Q}_n(x)})$$

which follows, in turn, from D_n, B_n .)

But $\neg r \rightarrow \dot{Q}_n(\overline{\neg p})$ is an instance of a logical axiom, therefore $\neg r \rightarrow \dot{Q}_n(\overline{\neg r}) \in T_n$; hence $r \in Q_{n+1}$.

v) The only thing which requires a proof is the last equality. Let a be a sentence, $\neg a \notin Q_n$ and $\neg a \rightarrow \dot{Q}_n(\overline{\neg a}) \in Q_{n+1}$. Suppose that $\neg \dot{Q}_n(\overline{\neg a}) \notin R_{n+1}$; since $\dot{Q}_n(\overline{\neg a}) \rightarrow \dot{Q}_n(\overline{\dot{Q}_n(\overline{\neg a})}) \in T_n$, one should conclude that $\dot{Q}_n(\overline{\neg a}) \in Q_n$, and, by D_n , $\neg a \in Q_n$. Therefore $\neg \dot{Q}_n(\overline{\neg a}) \in R_{n+1}$; consequently $a \in Q_{n+1}$. The reverse inclusion is clear.

COROLLARY 2. If a is a sentence, $a \notin Q_n$ and $a \rightarrow \dot{Q}_n(\overline{a}) \in T_n$

(or: $a \rightarrow \dot{Q}_n(\overline{a}) \in Q_{n+1}$), then $\neg a \in Q_{n+1}$.

PROPOSITION 3. i) \dot{Q}_n binumerates Q_n in Q_{n+1} .

ii) If a, b are sentences, and $b \notin Q_n$ and $\neg a \rightarrow Q_n(\overline{b}) \in Q_{n+1}$, then $a \in Q_{n+1}$.

iii) If $\varphi \in R_{n+1}$ (or: $\varphi \in Q_{n+1}$) then $\neg \varphi \notin T_n$; hence is $\varphi \in T_n$, then $\neg \varphi \notin Q_n$.

PROOF. (i) That \dot{Q}_n numerates Q_n in Q_{n+1} follows from D_n and from Prop. 1 (iii). Let $a \notin Q_n$; then $\dot{Q}_n(\bar{a}) \notin Q_n$; but

$$\dot{Q}_n(\bar{a}) \rightarrow \dot{Q}_n(\overline{\dot{Q}_n(\bar{a})}) \in T_n,$$

therefore, by Cor. 2, $\neg \dot{Q}_n(\bar{a}) \in Q_{n+1}$.

(ii) follows from (i).

(iii) Is immediate.

PROPOSITION 4. For each $h \leq n$, $\vdash_{T_h} \text{Con}_{\alpha_h}$.

Proof. Observe that :

$$\vdash_{T_0} \text{Con}_{\alpha_h} \longleftrightarrow \neg \dot{Q}_h(\overline{\exists x (x \approx x)}).$$

But : $\vdash_{T_h} (x \approx x) \rightarrow \dot{Q}_h(\dot{x} \approx \dot{x})$, therefore

$$\begin{aligned} \vdash_{T_h} \forall x (x \approx x) &\rightarrow \forall x \dot{Q}_h(\dot{x} \approx \dot{x}) \\ &\rightarrow \neg \dot{Q}_h(\overline{\neg \forall x (x \approx x)}) \\ &\rightarrow \neg \dot{Q}_h(\overline{\exists x (x \approx x)}). \end{aligned}$$

And consequently $\vdash_{T_h} \text{Con}_{\alpha_h}$.

PROPOSITION 5. (i) Each $\psi \in \Sigma_{n+1}F \cup \Pi_{n+1}F$ is a representative in Q_{n+1} .

(ii) If ϑ is a semirepresentative in Q_{n+1} , then also $\exists x \vartheta$ is a semirepresentative in Q_{n+1} .

(iii) Each $\psi \in \Sigma_{n+2}F$ is a semirepresentative in Q_{n+1} .

PROOF. Let $\psi \in \Sigma_{n+1}F$, $Fv(\psi) = \{x\}$ (ahronov); let $a \in \omega$; if $\psi(\bar{a}) \in V$, then $\psi(\bar{a}) \in Q_n$, therefore $\psi(\bar{a}) \in Q_{n+1}$. If $\neg \psi(\bar{a}) \in V$, then $\psi(\bar{a}) \notin Q_n$, and $\psi(\bar{a}) \rightarrow \dot{Q}_n(\overline{\psi(\bar{a})}) \in T_n$: therefore $\neg \psi(\bar{a}) \in Q_{n+1}$. For $\psi \in \Pi_{n+1}$, apply the preceding result to $\neg \psi$; Therefore (i) holds.

(ii) Let $Fv(\vartheta) = \{x, y\}$ (ahronov); let $\zeta = \exists y \vartheta(x, y)$. If $\zeta(\bar{a}) \in V$, then for some $b \in \omega$, $\vartheta(\bar{a}, \bar{b}) \in V$; hence $\vartheta(\bar{a}, \bar{b}) \in Q_{n+1}$; by logic, one gets $\exists y \vartheta(\bar{a}, y) \in Q_{n+1}$.

(iii) is immediate from (i) and (ii).

PROPOSITION 6. For $i \leq n + 1$, $Q_i \in \Sigma_{i+1}$.

PROOF. By induction on i : obviously $Q_0 \in \Sigma_1$. Let us suppose that $Q_j \in \Sigma_{j+1}$. Since R_{j+1} is Turing reducible to $(T_j \times \bar{Q}_j) \cup Q_j$; then R_{j+1} is in Δ_{j+2} : therefore $Q_{j+1} \in \Sigma_{j+2}$.

PROPOSITION 7. A relation R is in Σ_{n+2} iff R is numerable in Q_{n+1} .

PROOF. If $R \in \Sigma_{n+2}$, by Kleene's Enumeration and Normal Form Theorem, and by Prop. 5 (iii), R is numerable in Q_{n+1} . If R is numerable in Q_{n+1} , it is 1-1 reducible to Q_{n+1} : by Prop. 6 R is in Σ_{n+2} .

Now let M_n be a p.r. extension of P , which has any term representing p.r. functions necessary for arithmetization (say: M_n contains the set \mathcal{M} of § 4 of [1], and moreover M_n has two unary terms (\dot{n}) , \dot{Q}_n , representing respectively the primitive recursive functions mapping $k \in \omega$ into $k \rightarrow \dot{Q}(\bar{k})$, and into: $\dot{Q}(\bar{k})$, respectively, and such that:

$$\overline{M}_n(\dot{n})(x) \approx x \rightarrow \dot{Q}_n(x)$$

and

$$\overline{M}_n(\dot{n}) \neg \bar{\varphi}(\dot{x}) \approx \bar{\varphi}_n(\dot{x})$$

for each formula φ with x free (ahronov), where

$$\varphi_n = \neg \varphi(x) \rightarrow \dot{Q}_n(\overline{\neg \varphi}(\dot{x})).$$

Then, to be pedantically precise, I put

$$\alpha_{n+1} = St_{K_0}(x) \wedge (\dot{Q}_n(x) \vee (\dot{T}_n((\dot{n}) \neg (x)) \wedge \neg \dot{Q}_n(\neg x)));$$

$$\alpha_{n+1} = (\alpha_{n+1})^{M_n}.$$

PROPOSITION 8. α_{n+1} binumerates R_{n+1} in Q_{n+1} .

PROOF. Firstly observe that $\alpha_{n+1} \in \Sigma_{n+2} \mathcal{F} \cap \Pi_{n+2} \mathcal{F}$; hence it is a semirepresentative in Q_{n+1} , and obviously it numerates R_{n+1} in Q_{n+1} . Moreover, if $a \notin R_{n+1}$, then $\alpha_{n+1}(\bar{a}) \rightarrow \dot{Q}_n(\overline{\alpha_{n+1}(\bar{a})})$ is a Σ_{n+2} and true sentence: hence it belongs to Q_{n+1} , and therefore $\neg \alpha_{n+1}(\bar{a}) \in Q_{n+1}$.

COROLLARY 9. \dot{Q}_{n+1} is a Σ_{n+2} F , and it numerates Q_{n+1} in Q_{n+1} .

LEMMA 10. Let $\vartheta \in Fm_{K_0}$, then :

$$(i) \quad |\overline{T}_0 \ Ax_{K_0} (\overline{\forall v_0} \vartheta \rightarrow \vartheta (\dot{v}_0)) ;$$

$$(ii) \quad |\overline{T}_0 \ Ax_{K_0} (\overline{\vartheta} (\dot{v}_0) \rightarrow \overline{\exists v_0} \vartheta) .$$

Let α be a formula of K_0 , $Fv(\alpha) = \{x\}$; then for each $\varphi \in Fm_{K_0}$, with $Fv(\varphi) = \{x, y\}$ (ahronov);

$$(iii) \quad |\overline{T}_0 \ \exists x Pr_\alpha (\overline{\varphi} (\dot{x}, \dot{y})) \rightarrow Pr_\alpha (\overline{\exists x} \varphi (x, \dot{y})) ;$$

$$(iv) \quad |\overline{T}_0 \ Pr_\alpha (\overline{\forall x} \varphi (x, \dot{y})) \rightarrow \forall x Pr_\alpha (\overline{\varphi} (\dot{x}, \dot{y})) .$$

PROOF. This is routine of arithmetization. For (i), remember that :

$$|\overline{T}_0 \ (\overline{Vr}(x) \wedge \overline{Fm}_{K_0}(y) \wedge \overline{Tm}_{K_0}(z)) \rightarrow Ax_{K_0} (\bigwedge_x y \rightarrow Sb^x_z y)$$

and that : $|\overline{T}_0 \ vr\bar{0} \approx \bar{v}_0 ;$

and hence : $|\overline{T}_0 \ \overline{Vr} (vr\bar{0}) .$

By induction one shows that $|\overline{T}_0 \ \forall x Tm_{K_0} (nm_x) .$

Therefore $|\overline{T}_0 \ Ax_{K_0} (\bigwedge_{v_0} \vartheta \rightarrow Sb^{vr\bar{0}}_{nm_{v_0}} \bar{\vartheta}) .$

But : $|\overline{T}_0 \ \bigwedge_{vr\bar{0}} \bar{\vartheta} \approx \overline{\forall v_0} \vartheta ;$ and (i) follows; (ii) is proved quite similarly, and (iii), (iv) follow immediatly.

LEMMA 11. For any formula φ , with x free (ahronov), one has :

$$|\overline{T}_0 \ \rightarrow \dot{Q}_n (\overline{\neg \varphi} (\dot{x})) \wedge \dot{T}_n (\overline{\varphi_n} (\dot{x})) \rightarrow \dot{Q}_{n+1} (\overline{\varphi} (\dot{x}))$$

(where φ_n is as above).

PROOF. This follows from [1], Th. 4.6(iii), and from the fact that

$$|\overline{T}_0 \ St_{K_0} (\overline{\neg \varphi} (\dot{x})) .$$

LEMMA 12. For $h \leq n$,

$$|\overline{T}_o \dot{Q}_h(x) \wedge \dot{T}_h(\neg x \rightarrow \ddot{Q}_h(\neg x)) \rightarrow \dot{Q}_{h+1}(x)$$

PROPOSITION 13. For $h \leq n$,

$$|\overline{T}_h \dot{Q}_h(x) \rightarrow \dot{Q}_{h+1}(x)$$

REMARK. Here, and in similar cases, one should add to the premiss: « $Fm_{K_o}(x)$ » or something like that. This may easily be supplied by the reader in each case.

PROOF. By induction on h . Let $h = 0$; then remember that:

$$|\overline{T}_o \dot{Q}_0(x) \rightarrow \neg \dot{Q}_0(\neg x)$$

and that

$$|\overline{T}_o \dot{Q}_0(x) \rightarrow \dot{T}_0(x); \quad |\overline{T}_o \dot{Q}_0(x) \rightarrow \dot{T}_0(\neg x \rightarrow \ddot{Q}_0(\neg x));$$

therefore, by lemma 12,

$$|\overline{T}_o \dot{Q}_0(x) \rightarrow \dot{Q}_1(x).$$

Let us suppose that the theorem holds for $h < n$. We have:

$$|\overline{T}_{h+1} \dot{Q}_h(\neg x) \rightarrow \dot{Q}_{h+1}(\neg x);$$

and hence:

$$|\overline{T}_o \dot{T}_{h+1}(\ddot{Q}_h(\neg x) \rightarrow \ddot{Q}_{h+1}(\neg x)).$$

$$\begin{aligned} |\overline{T}_o \alpha_{h+1}(x) &\rightarrow (\neg \dot{Q}_h(x) \rightarrow \dot{T}_h(\neg x \rightarrow \ddot{Q}_h(\neg x))) \\ &\rightarrow (\neg \dot{Q}_h(x) \rightarrow \dot{T}_{h+1}(\neg x \rightarrow \ddot{Q}_{h+1}(\neg x))); \end{aligned}$$

but one has:

$$|\overline{T}_o \neg \alpha_{h+2}(x) \rightarrow (\dot{T}_{h+1}(\neg x \rightarrow \ddot{Q}_{h+1}(\neg x)) \rightarrow \dot{Q}_{h+1}(\neg x))$$

Therefore :

$$\overline{|T}_o (\alpha_{h+1} \wedge \neg \dot{Q}_h(x) \wedge \neg \alpha_{h+2}(x)) \rightarrow \neg \text{Con}_{\alpha_{h+1}} ;$$

therefore :

$$\begin{aligned} \overline{|T}_{h+1} \alpha_{h+1}(x) &\rightarrow (\dot{Q}_h(x) \dot{\vee} \alpha_{h+2}(x)) \\ &\rightarrow \alpha_{h+2}(x) . \end{aligned}$$

By Prop. 4. , one concludes :

$$\overline{|T}_{h+1} \alpha_{h+1}(x) \rightarrow \alpha_{h+2}(x)$$

which is something more then required to show the theorem.

COROLLARY 14. $\overline{|T}_{n+1} \text{Con}_{\alpha_{n+1}} .$

PROOF. We have :

$$\begin{aligned} \overline{|T}_{n+1} x \approx x &\rightarrow \dot{Q}_n(\dot{x} \approx \dot{x}) \\ &\rightarrow \dot{Q}_{n+1}(\dot{x} \approx \dot{x}) \end{aligned}$$

and from there on, the proof is quite similar to that of Prop. 4.

PROPOSITION 15. For each $\psi \in \Sigma_{n+1}F \cup \Pi_{n+1}F$, with x free (ahronov), one has :

- (i) $\overline{|T}_{n+1} \dot{Q}_{n+1}(\overline{\neg \psi}(\dot{x})) \rightarrow \neg \dot{Q}_{n+1}(\psi(\dot{x})) ;$
- (ii) $\overline{|T}_n \neg \dot{Q}_{n+1}(\overline{\psi}(\dot{x})) \rightarrow \dot{Q}_{n+1}(\overline{\neg \psi}(\dot{x})) .$

PROOF. (i) follows from Cor. 14.

(ii) Let $\psi \in \Sigma_{n+1}F$; then

$$\overline{|T}_n \psi(x) \rightarrow \dot{Q}_n(\overline{\psi}(\dot{x}))$$

whence :

$$\overline{|T}_o \dot{T}_n((\dot{n})(\overline{\psi}(\dot{x})))$$

But, by prop. 13,

$$|\bar{T}_n \neg \dot{Q}_{n+1}(\bar{\psi}(\dot{x})) \rightarrow \neg \dot{Q}_n(\bar{\psi}(\dot{x}))|.$$

therefore (ii) holds. To get (ii) for $\psi \in \Sigma_{n+1}F$, apply it to $\neg \psi$.

PROPOSITION 16. The following are equivalent (with *ahronov* of formulas involved, when suitable):

- a) $|\bar{T}_n \forall x(\dot{Q}_n(\bar{\vartheta}(\dot{x}))) \rightarrow \forall x\vartheta(x)|$, for $\vartheta \in \Sigma_{n+1}F$;
- b) $|\bar{T}_n \forall x\vartheta(x)|$, for $\vartheta \in \Sigma_{n+1}F$ and such that $|\bar{T}_{n-1} \forall x\dot{Q}_n(\bar{\vartheta}(\dot{x}))|$;
- c) $|\bar{T}_n \dot{Q}_n(\bar{\vartheta}(\dot{x})) \rightarrow \vartheta(x)|$, for $\vartheta \in \Sigma_{n+1}F$;
- d) $|\bar{T}_n \xi(x) \rightarrow \dot{Q}_{n+1}(\bar{\xi}(\dot{x}))|$, for $\xi \in \Pi_{n+1}F$;
- e) $|\bar{T}_n \zeta(x) \rightarrow \dot{Q}_{n+1}(\bar{\zeta}(\dot{x}))|$, for $\zeta \in \Sigma_{n+2}F$;
- f) $|\bar{T}_n \Sigma_{n+1}F - \omega\text{-con}_{\alpha_n}|$;
- g) $|\bar{T}_n \Pi_{n+1}F - \omega\text{-con}_{\alpha_n}|$.

PROOF.

(a) \Rightarrow (b) : this is obvious.

(b) \Rightarrow (c) . To prove this, one employs an analogue of Lemma 2,18 of [2]; namely :

LEMMA. Let $\varphi \in Fm_{K_0}$, with x free (*ahronov*) ; put

$$\psi(x, y) = \text{Prf}_{\alpha_n}(\bar{\varphi}(\dot{x}), y) \rightarrow \varphi(x)$$

then one has :

$$|\bar{T}_{n-1} \forall x \forall y \dot{Q}_n(\bar{\psi}(\dot{x}, \dot{y}))|$$

The proof of the lemma is quite analogous to Feferman's : only observe that Prf_{α_n} is in $\Pi_{n+1}F$.

Having this lemma, one concludes just as in the proof of Th. 2.19 of [2].

(c) \Rightarrow (a) : This is obvious.

(e) \Rightarrow (d) . By hypothesis ,

$$|\bar{T}_n \xi(x) \rightarrow \neg \dot{Q}_n(\bar{\neg \xi}(\dot{x}))|;$$

moreover :

$$\overline{T}_n \neg \xi(x) \rightarrow \dot{Q}_n(\overline{\neg \xi(\dot{x})});$$

whence

$$\overline{T}_o \dot{T}_n(\overline{\neg \xi(\dot{x})} \rightarrow \dot{Q}_n(\overline{\neg \xi(\dot{x})})).$$

Therefore (by Lemma (1)) :

$$\overline{T}_n \xi(x) \rightarrow \dot{Q}_{n+1}(\overline{\xi(\dot{x})}).$$

(d) \Rightarrow (e) ; this is immediate, after lemma 10 (iii).

(e) \Rightarrow (c) is obvious, by B_n (ii).

(f) \Rightarrow (g) . Let $\psi(x, y) \in \Pi_{n+1} F$ (ahronov) ; one has :

$$\overline{T}_n \dot{Q}_n(\overline{\neg \forall x \psi(x, y)}) \rightarrow \dot{Q}_n(\overline{\neg \forall x \forall z \varphi(x, \dot{y}, z)})$$

(where $\psi = \forall z \psi(x, y, z)$, and $\varphi \in \Sigma_n F$)

$$\begin{aligned} &\rightarrow \neg \forall x \forall z \dot{Q}_n(\overline{\varphi(\dot{x}, \dot{y}, \dot{z})}) \\ &\rightarrow \exists x \neg \forall z \dot{Q}_n(\overline{\varphi(\dot{x}, \dot{y}, \dot{z})}) \\ &\rightarrow \exists x \neg \dot{Q}_n(\overline{\neg \forall z \varphi(\dot{x}, \dot{y}, z)}), \end{aligned}$$

where the last implication follows from Lemma 10(iv).

(g) \Rightarrow (f) : this follows from : $\Sigma_n F \subseteq \pi_{n+4} F$.

(c) \Rightarrow (g) . Let $\psi \in \Pi_{n+1} F$ ($\psi = \forall y \varphi(x, y, z)$, $\varphi \in \Sigma_n F$) ; then :

$$\begin{aligned} \overline{T}_n \dot{Q}_n(\overline{\neg \forall x \forall y \varphi(x, y, \dot{z})}) &\rightarrow \\ &\rightarrow \neg \forall x \forall y \varphi(x, y, z) \\ &\rightarrow \exists x \neg \dot{Q}_n(\overline{\forall x \varphi(\dot{x}, y, \dot{z})}) \end{aligned}$$

(this implication follows from B_n (i) and from Prop. 4)

$$\rightarrow \neg \forall x \dot{Q}_n(\overline{\psi(\dot{x}, \dot{z})})$$

(g) \Rightarrow (e). Let $\vartheta = \neg \forall x \psi(x, y)$, with $\psi \in \Sigma_n F$: then

$$\begin{aligned} \lfloor \bar{T}_n Q_n(\overline{\neg \forall x \psi}(x, \dot{y})) \rightarrow \neg \forall x \dot{Q}_n(\overline{\psi}(\dot{x}, \dot{y})) \\ \rightarrow \neg \forall x \psi(x, y) 0 ; \end{aligned}$$

(The last implication, by $B_n(i)$).

REMARKS 1. In the preceding proof, any implication having (e) or (f) as a consequent, would be obvious from the induction hypothesis; I have tried to use the latter the less possible; many of the implications follow simply from: $\lfloor \bar{T}_n \text{Con}_{\alpha_n}$; in particular, $B_n(ii)$ was only used in the proof of: (e) \Rightarrow (e).

2. This kind of analysis leads to the following:

COROLLARY 17. Let $(a'), \dots, (g')$ be obtained from $(a), \dots, (g)$ of Prop. 16, by substituting T_{n+1} to T_n ; let (h) be the following:

$$(h) \quad \lfloor \bar{T}_{n+1} \dot{Q}_{n+1}(\overline{\vartheta}(\dot{x})) \rightarrow \vartheta(x), \text{ for } \vartheta \in \Sigma_{n+1} F.$$

Then, under the only hypothesis that: $\lfloor \bar{T}_h \text{Con}_{\alpha_h}$, for $h \leq n+1$ (i.e. without using that $\Sigma F - \omega - \text{con}_{\alpha_{n+1}}$ is provable in T_{n+1}), one has:

$$(a') , (b') , (c') , (d') , (e') , (f') , (g') , (h)$$

are pairwise equivalent.

PROOF. For the most part, the proof of Prop. 16 works here also. The only implication that deserves attention is: (c') \Rightarrow (h).

One has:

$$\begin{aligned} \lfloor \bar{T}_{n+1} \dot{Q}_{n+1}(\overline{\vartheta}(\dot{x})) \rightarrow \neg \dot{Q}_{n+1}(\overline{\neg \vartheta}(\dot{x})) \\ \rightarrow \neg a_{n+1}(\overline{\neg \vartheta}(\dot{x})) \\ \rightarrow (\dot{T}_n(\overline{\vartheta}(\dot{x})) \rightarrow \dot{Q}_n(\overline{\vartheta}(\dot{x}))) \rightarrow \dot{Q}_n(\overline{\vartheta}(\dot{x})). \end{aligned}$$

Therefore:

$$\lfloor \bar{T}_{n+1} \dot{Q}_{n+1}(\overline{\vartheta}(\dot{x})) \longleftrightarrow \dot{Q}_n(\overline{\vartheta}(\dot{x})).$$

From proposition 16 there follows immediatly :

COROLLARY 18. B_{n+1} (i) holds.

PROPOSITION 19. The following are equivalent :

- (0) $|\bar{T}_{n+1} \dot{Q}_{n+1}(\bar{\vartheta}(\bar{x})) \rightarrow \vartheta(x), \text{ for } \vartheta \in \Sigma_{n+2}F \text{ (ahronov);}$
- (00) $|\bar{T}_{n+1} \Sigma_{n+1}F - \omega - \text{con}_{\alpha_{n+1}};$
- (000) $|\bar{T}_{n+1} \Pi_{n+2}F - \omega - \text{con}_{\alpha_{n+1}}.$

PROOF. (000) \Rightarrow (00) is immediate, and (00) \Rightarrow (000) is proved quite similarly to the proof of (f) \Rightarrow (g) in Prop. 16 ; (0) \Rightarrow (000) follows from Cor. 18 and 14 ; (00) \Rightarrow (0) follows Cor. 18.

COROLLARY 20. B_{n+1} (ii) holds.

THEOREM 21. There exist two sequences $(R_n)_{n \in \omega}$ and $(\alpha)_{\alpha \in \omega}$ which satisfy A_n, B_n, C_n, D_n and E_n .

Among the properties of these sequences, I list the following. Firstly, one can imitate the trick of Löb's in [4], to prove :

- THEOREM 22. (i) Let $g(x)$ be any formula such that, if $a \in St_{K_0}$ and $|\bar{Q}_n g(\bar{a})$, then $a \in Q_n$; then for any $p \in St_{K_0}$, if $|\bar{Q}_n \dot{Q}_n(\overline{g(\bar{p})}) \rightarrow g(\bar{p})$, then $p \in Q_n$.
- (ii) For $p \in St_{K_0}$, if $|\bar{Q}_n \dot{Q}_n(\bar{p}) \rightarrow p$, then $p \in Q_n$.

PROOF. By diagonalization, let $b \in St_{K_0}$ be such that :

$$|\bar{T}_0 (\dot{Q}_n(\bar{b}) \rightarrow g(\bar{p})) \leftrightarrow b.$$

Then :

$$|\bar{T}_0 \dot{Q}_n(\bar{b}) \rightarrow (\dot{Q}_n(\overline{\dot{Q}_n(\bar{b})}) \rightarrow \dot{Q}_n(\overline{g(\bar{p})})).$$

But :

$$|\bar{T}_{n-1} \dot{Q}_n(\bar{b}) \rightarrow \dot{Q}_n(\overline{\dot{Q}_n(\bar{b})}).$$

(This is true by Th. 5.4. of [1] for $n = 0$, and follows from B_n (i) for $n > 0$).

Then we get :

$$\vdash_{\bar{Q}_n} b ;$$

whence :

$$\vdash_{\bar{Q}_n} g(\bar{p})$$

and finally : $p \in Q_n$. ii) is proved in the same way.

THEOREM 23. Q_{n+1} does not belong to Σ_{n+1} .

PROOF. By the proof of Prop. 7, if a set S is numerable in Q_n , it is numerable in Q_n by a formula of Σ_{n+1}^F . Now let us suppose, by absurd, that some formula $g_n \in \Sigma_{n+1}^F$ numerates Q_{n+1} in Q_n . Let p be any sentence ; by considering the sentence b which is equivalent (in T_0) to : $\dot{Q}_n(\bar{b}) \rightarrow g_n(\bar{p})$, one would get, as before,

$$\vdash_{\bar{Q}_n} \dot{Q}_n(\bar{b}) \rightarrow \dot{Q}_n(\overline{g_n(\bar{p})})$$

but, by B_n (ii) :

$$\vdash_{\bar{Q}_{n+1}} \dot{Q}_n(\overline{g_n(\bar{p})}) \rightarrow g_n(\bar{p})$$

whence one could get :

$$\vdash_{\bar{Q}_{n+1}} b$$

whence :

$$\vdash_{\bar{Q}_{n+1}} g_n(\bar{p})$$

and hence :

$$\vdash_{\bar{Q}_n} g_n(\bar{p})$$

and finally :

$$p \in Q_{n+1}$$

Therefore each sentence would be in Q_{n+1} , which is absurd.

THEOREM 24. (i) Q_n is not complete, for any n ; in particular Con_{α_n} is undecidable in Q_n ; $\text{Con}_{\alpha_n} \in Q_{n+1}$, and also

$\text{Not-Comp}_{\alpha_n} = \exists x(\neg \dot{Q}_n(x) \wedge \neg \dot{Q}_n(\neg x) \wedge \text{St}_{K_0}(x))$
is in Q_{n+1} ; and lastly: $\vdash_{Q_{n+1}} \neg \dot{Q}_n(\overline{\text{Con}_{\alpha_n}})$.

(ii) $\bigcup_{n \in \omega} (Q_n \cap \text{St}_{K_0}) = V$.

(iii) $T_n \ Q_n \not\subseteq$; $T_{n+1} \not\subseteq T_n$.

(iv) (Hilbert-Bernays; Kuenecov; Trahtenbrot-see [5], Ch. XII). $\{V\} \in \Pi_2^0$.

PROOF. (i) and (ii) are immediate.

(iii) $\text{Con}_{\alpha_n} \in T_n$ but $\notin Q_n$; $\text{Con}_{\alpha_{n+1}} \in T_{n+1}$, but $\notin T_n$,

(otherwise it would $\in Q_{n+1}$).

(iv) follows from (ii).

Finally, it would be easy to prove (cfr. e.g. [7]) that the set V_0 of [3] is exactly $Q_1 \cap \text{St}_{K_0}$.

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—Oh me dolente! Come mi riscossi
quando mi prese dicendomi: ‘Forse
tu non pensavi ch’io loico fossi’!—

Dante, Inf. XXVII, 120-123.

REFERENCES

- [1] S. FEFERMAN, *Arithmetization of Metamathematics in a general setting*, Fund. Mat. **49** (1960) pp. 35-92.
- [2] S. FEFERMAN, *Transfinite recursive progressions of axiomatic theories*, Journ., of Symb. Logic, **27** (1962) pp. 259-316.
- [3] R. MAGARI, *Significato e verità nell'aritmetica peaniana*, Ann. di Mat. Pura e Appl., **4** (103) 1975, pp. 343-368.
- [4] M. H. LÖB, *Solution of a problem of Leon Henkin*, Journ. of Symb. Logic, **20** (1955) pp. 115-118.
- [5] H. ROGERS, Jr. *Theory of recursive functions and effective computability*, Mc Graw Hill; New York, 1967.
- [6] C. SMORYNSKI, *Consistency and related metamathematical properties*, Amsterdam Mathematisch Instituut, Rp. 75-02.
- [7] A. URSINI, *On the set of « meaningful » sentences of arithmetic*, to appear in Studia Logica.

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