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in normed spaces**

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## **$\theta$ -Mappings and quasiconformal mappings in normed spaces**

GIOVANNI PORRU (\*)

### **Introduction.**

The  $\theta$ -mappings introduced by Gehring in [2], and some definitions of quasiconformal mappings in  $R^n$ , are meaningful in every normed space. But such mappings have been investigated, till now, almost exclusively in  $R^n$ . Indeed, most of the procedures used in  $R^n$  (as Lebesgue integral) are meaningless in an infinite dimensional space  $H$ .

In this paper we consider  $\theta$ -mappings in a real normed space, and we prove for them a compactness theorem. Further, we show some properties of sequences of  $\theta$ -mappings. Next we consider the metric definition of quasiconformal mappings in a real normed space, and we give a relation between the  $\theta$ -mappings and the quasiconformal mappings.

1.  $\theta$ -MAPPINGS. Let  $H$  be a real normed space and  $\Omega, \Omega'$  domains of  $H$ . Following Gehring ([2], pag. 14) we give the

**DEFINITION 1.** *A homeomorphism  $T_X$  of  $\Omega$  onto  $\Omega'$  is said to be a  $\theta$ -mapping if there exists a function  $\theta(t)$ , which is continuous and increasing in  $0 \leq t < 1$  with  $\theta(0) = 0$ , such that the following are true.*

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(i) If  $X_0 \in \Omega$  and  $\|X - X_0\| < \inf_{Y \in \partial\Omega} \|X_0 - Y\| \equiv \varrho(X_0, \partial\Omega)$ , then

$$\frac{\|TX - TX_0\|}{\varrho(TX_0, \partial\Omega')} \leq \theta \left( \frac{\|X_0 - X\|}{\varrho(X_0, \partial\Omega)} \right)$$

(ii) The restriction of  $TX$  to any subdomain  $\Delta \subseteq \Omega$ , satisfies (i).

For the  $\theta$ -mappings we now prove some properties.

**THEOREM 1.** Suppose that  $\{T_n X\}$  is a sequence of  $\theta$ -mappings (with the same distortion function  $\theta(t)$ ) of  $\Omega$ , that

$$\lim_{n \rightarrow \infty} T_n X = TX$$

uniformly on each bounded domain  $\Delta \subset \Omega$  such that  $\varrho(\Delta, \partial\Omega) > 0$ , and that  $TX$  is a homeomorphism. Then  $TX$  is a  $\theta$ -mapping.

**PROOF.** Let  $\Delta \subset \Omega$  be a bounded domain such that  $\varrho(\Delta, \partial\Omega) > 0$ , and set  $\Delta'_n = T_n(\Delta)$ ,  $\Delta' = T(\Delta)$ . Since  $T_n X$  are  $\theta$ -mappings we have, for  $X_0 \in \Delta$ :

$$(1) \quad \frac{\|T_n X - T_n X_0\|}{\varrho(T_n X_0, \partial\Delta'_n)} \leq \theta \left( \frac{\|X - X_0\|}{\varrho(X_0, \partial\Delta)} \right), \quad n = 1, 2, \dots$$

for  $\|X - X_0\| < \varrho(X_0, \partial\Delta)$ . We have to show that

$$(2) \quad \frac{\|TX - TX_0\|}{\varrho(TX_0, \partial\Delta')} \leq \theta \left( \frac{\|X - X_0\|}{\varrho(X_0, \partial\Delta)} \right)$$

for  $\|X - X_0\| < \varrho(X_0, \partial\Delta)$ . First we prove that

$$(3) \quad \liminf_{n \rightarrow \infty} \varrho(T_n X_0, \partial\Delta'_n) \leq \varrho(TX_0, \partial\Delta').$$

We may suppose  $\varrho(TX_0, \partial\Delta') < \infty$ . If (3) is not true there exists a subsequence  $\{T_{n_k} X\}$  and a positive real number  $\sigma$  such that

$$(4) \quad \lim_{k \rightarrow \infty} \varrho(T_{n_k} X_0, \partial\Delta'_{n_k}) \geq \varrho(TX_0, \partial\Delta') + \sigma.$$

From (4) it follows that there exists a finite  $\nu$  such that

$$(5) \quad \varrho(TX_0, \partial\Delta'_{n_k}) > \varrho(TX_0, \partial\Delta') + \sigma/2$$

for  $k > \nu$ . By (5) we may find at least one point  $X' \in \partial\Delta'$  and a ball  $B(X', r')$  centered in  $X'$ , radius  $r'$ , and contained in  $\Delta'_{n_k}$ , for  $k > \nu$ . Further we may take a second ball  $B(Z', r) \subset B(X', r')$  with  $B(Z', r) \cap \Delta' = \emptyset$ . Let  $Z_k$  be the points of  $\Delta$  such that  $T_{n_k}Z_k = Z'$ . Since  $\{T_{n_k}X\}$  converges uniformly to  $TX$  on  $\Delta$ , we have

$$\|TZ_k - Z'\| = \|TZ_k - T_{n_k}Z_k\| < r$$

for  $k > \nu_1$ , where  $\nu_1 \geq \nu$  is a suitable finite number. The latter inequality shows that  $TZ_k$  belongs to  $B(Z', r)$ . Hence, we obtain a contradiction and inequality (3) does hold.

Since

$$\lim_{n \rightarrow \infty} \|T_n X - T_n X_0\| = \|TX - TX_0\|,$$

from (1) and (3), we derive (2).

Let now  $\Delta$  be a subdomain of  $\Omega$ . If  $X_0 \in \Delta$  and  $\varrho(X_0, \partial\Delta) = \infty$ , let  $\Delta_m \equiv B(X_0, m)$ . For the previous result we have, for  $\|X - X_0\| < m$ :

$$\frac{\|TX - TX_0\|}{\varrho(TX_0, \partial\Delta'_m)} \leq \theta\left(\frac{\|X - X_0\|}{\varrho(X_0, \partial\Delta_m)}\right) = \theta\left(\frac{\|X - X_0\|}{m}\right),$$

where  $\Delta'_m = T(\Delta_m)$ . Since  $\lim_{m \rightarrow \infty} \theta(\|X - X_0\|/m) = 0$  it must be

$$\lim_{m \rightarrow \infty} \varrho(TX_0, \partial\Delta'_m) = \infty. \text{ As } \varrho(TX_0, \partial\Delta') \geq \varrho(TX_0, \partial\Delta'_m)$$

it must be again  $\varrho(TX_0, \partial\Delta') = \infty$ , and inequality (2) holds for such  $\Delta$ .

Finally suppose  $\Delta \subseteq \Omega$ ,  $X_0 \in \Delta$  and such that  $\varrho(X_0, \partial\Delta) = c < \infty$ . Let  $\Delta_m = B(X_0, c - 1/m)$  with  $m > 1/c$ . Inequality (2) holds for  $\Delta_m$ . Further we have  $\Delta' = T(\Delta) \supset T(\Delta_m) = \Delta'_m$ , hence

$$\varrho(TX_0, \partial\Delta') \geq \varrho(TX_0, \partial\Delta'_m), \quad \varrho(X_0, \partial\Delta_m) = \varrho(X_0, \partial\Delta) - 1/m$$

and

$$\frac{\|TX - TX_0\|}{\varrho(TX_0, \partial\Delta')} \leq \frac{\|TX - TX_0\|}{\varrho(TX_0, \partial\Delta'_m)} \leq \theta \left( \frac{\|X - X_0\|}{\varrho(X_0, \partial\Delta_m)} \right) = \theta \left( \frac{\|X - X_0\|}{\varrho(X_0, \partial\Delta)^{-\frac{1}{m}}} \right).$$

For  $m \rightarrow \infty$ , because  $\theta(t)$  is continuous, we obtain inequality (2) for such  $\Delta$ .

**THEOREM 2.** *Suppose that  $\{T_n X\}$  is a sequence of  $\theta$ -mappings of  $\Omega$  and that*

$$\lim_{n \rightarrow \infty} T_n X = TX, \quad \|TX\| < \infty$$

*in  $\Omega$ . Then  $TX$  is either a constant or a continuous and injective mapping.*

In order to prove the Theorem we give some lemmas whose proofs are similar to those done by Gehring for  $H = R^n$  ([2]).

**LEMMA 1.** *Suppose that  $\{T_n X\}$  is a sequence of  $\theta$ -mappings of  $\Omega$  which are locally uniformly bounded on  $\Omega$ . Then  $T_n X$  are locally equicontinuous in  $\Omega$ .*

**PROOF.** Let  $Z \in \Omega$  and let  $B(Z, r)$  be a ball centered in  $Z$ , radius  $r < \varrho(Z, \partial\Omega)$  and such that  $T_n X$  are uniformly bounded in  $B(Z, r)$ . Choose  $X_0 \in \Omega - \bar{B}(Z, r)$  and let  $\Delta = \Omega - \{X_0\}$ . By hypothesis there exists a finite constant  $A$  (depending on  $B(Z, r)$  and  $X_0$ ) such that

$$\|T_n X - T_n X_0\| \leq \|T_n X\| + \|T_n X_0\| \leq A$$

for  $X \in B(Z, r)$  and all  $n$ . Hence

$$(6) \quad \varrho(T_n X, \partial\Delta'_n) \leq \|T_n X - T_n X_0\| \leq A$$

for  $X \in B(Z, r)$  and all  $n$ , where  $\Delta'_n = T_n(\Delta)$ . Since  $r < \varrho(Z, \partial\Omega)$  and  $X_0 \notin \bar{B}(Z, r)$ ,

$$(7) \quad \varrho(X, \partial\Delta) \geq a > 0$$

for  $X \in B(Z, r)$ . Now fix  $X \in B(Z, r)$  and choose  $Y$  so that  $\|Y -$

$\|X\| < a$ . Then

$$\frac{\|T_n Y - T_n X\|}{\varrho(T_n X, \partial \Delta'_n)} \leq \theta \left( \frac{\|Y - X\|}{\varrho(X, \partial \Delta)} \right).$$

and, for (6) – (7),

$$\|T_n Y - T_n X\| \leq A \theta \left( \frac{\|Y - X\|}{a} \right).$$

Since  $\lim_{t \rightarrow 0^+} \theta(t) = 0$ , this implies the desired equicontinuity of  $T_n X$  in  $B(Z, r)$ .

LEMMA 2. Suppose that  $\{T_n X\}$  is a sequence of  $\theta$ -mappings of  $\Omega$  onto  $\Omega'_n$ , that

$$\sup_n \|T_n X_0\| < \infty$$

for some  $X_0 \in \Omega$ , and that

$$\sup_n \varrho(0, \partial \Omega'_n) < \infty$$

where 0 denotes the null element of  $H$ . Then  $T_n X$  are locally equicontinuous in  $\Omega$ .

PROOF. Fix  $a$  so that  $0 < a < 1$ . Then, if we choose  $Y \in \Omega$  and  $X$  so that  $\|X - Y\| < a\varrho(Y, \partial \Omega)$  we have

$$\|T_n X - T_n Y\| \leq \theta(a) \varrho(T_n Y, \partial \Omega'_n)$$

for all  $n$ . Since

$$\varrho(T_n Y, \partial \Omega'_n) \leq \|T_n Y\| + \varrho(0, \partial \Omega'_n)$$

we thus obtain

$$(8) \quad \|T_n X\| \leq M \|T_n Y\| + N$$

where

$$M = 1 + \theta(a), \quad N = \theta(a) \sup_n \varrho(0, \partial \Omega'_n)$$

In particular we conclude that each point  $Y \in \Omega$  has a neighborhood  $U = U(Y) \subset \Omega$  such that (8) holds for all  $X \in U$ . Next, if we choose  $X \in \Omega$  and  $Y$  so that  $\|Y - X\| < \frac{a}{2} \varrho(X, \partial\Omega)$  we have

$$\|Y - X\| < \frac{a}{2} (\varrho(Y, \partial\Omega) + \|Y - X\|)$$

and thus

$$\|Y - X\| < \frac{a}{2 - a} \varrho(Y, \partial\Omega) < a\varrho(Y, \partial\Omega).$$

Hence we see that each point  $X \in \Omega$  has a neighborhood  $V = V(X) \subset \Omega$  such that (8) holds for all  $Y \in V$ . Now let  $G$  denote the set of points  $X \in \Omega$  for which

$$\sup_n \|T_n X\| < \infty.$$

If  $Y \in G$  and if  $U$  is the neighborhood described above, then (8) implies that

$$\sup_n \|T_n X\| \leq M \sup_n \|T_n Y\| + N$$

for all  $X \in U$ . Hence  $U \subset G$  and  $G$  is open. Similarly, if  $X \in \Omega - G$  and if  $V$  is the neighborhood described above, the same argument shows that  $V \subset \Omega - G$ , and hence that  $\Omega - G$  is open. Since  $\Omega$  is connected and  $X_0 \in G$ , we conclude that  $G = \Omega$ . We have shown that for  $Y \in G$ ,  $\sup_n \|T_n Y\| < \infty$ . Further, by (8), follows that  $T_n X$  are uniformly bounded in a suitable neighborhood  $U(Y)$ . The locally equicontinuity is now a consequence of Lemma 1.

**LEMMA 3.** *Suppose that  $\{T_n X\}$  is a sequence of  $\theta$ -mappings of  $\Omega$ , and that*

$$\sup_n \|T_n X_0\| < \infty, \quad \sup_n \|T_n X_1\| < \infty$$

*for a pair of distinct fixed points  $X_0, X_1 \in \Omega$ . Then  $T_n X$  are locally equicontinuous in  $\Omega$ .*

PROOF. Let  $\Delta = \Omega - \{X_1\}$  and  $\Delta'_n = T_n(\Delta)$ . Then  $T_n X_1 \in \partial \Delta'_n$  and hence

$$\sup_n \varrho(0, \partial \Delta'_n) \leq \sup_n \|T_n X_1\| < \infty.$$

Lemma 2 implies now the locally equicontinuity of  $T_n X$  in  $\Delta$ . Interchanging the roles of  $X_0$  and  $X_1$  we obtain the desired conclusion.

LEMMA 4. *Suppose that  $\{T_n X\}$  is a sequence of  $\theta$ -mappings of  $\Omega$ , that*

$$\lim_{n \rightarrow \infty} T_n X = TX, \quad \|TX\| < \infty$$

*in  $\Omega$ , that  $T_n X \neq Y'_n$  in  $\Omega$ , and that*

$$\lim_{n \rightarrow \infty} Y'_n = Y'.$$

*Then either  $TX \neq Y'$  in  $\Omega$  or  $TX \equiv Y'$  in  $\Omega$ .*

PROOF. Let  $G$  be the set of points  $X \in \Omega$  for which  $TX = Y'$ . Lemma 3 implies that  $T_n X$  are locally equicontinuous in  $\Omega$ . Hence  $TX$  is continuous and  $G$  is closed in  $\Omega$ . Now suppose  $X_0 \in G$  and let  $U$  be the set of points  $X$  for which  $\|X - X_0\| < a\varrho(X_0, \partial\Omega)$ , where  $a$  is some fixed constant,  $0 < a < 1$ . Then we have

$$\|T_n X - T_n X_0\| \leq \theta(a)\varrho(T_n X_0, \partial\Omega'_n)$$

for all  $X \in U$ , where  $\Omega'_n = T_n(\Omega)$ . Since  $Y'_n \notin \Omega'_n$ , we see that

$$\varrho(T_n X_0, \partial\Omega'_n) \leq \|T_n X_0 - Y'_n\|$$

and hence

$$\|TX - TX_0\| = \lim_{n \rightarrow \infty} \|T_n X_0 - T_n X\| \leq \theta(a) \lim_{n \rightarrow \infty} \|T_n X_0 - Y'_n\| = 0$$

for all  $X \in U$ . Hence  $U \subset G$  and  $G$  is open. Since  $\Omega$  is connected, we conclude, as desired, that either  $G = \Omega$  or  $TX \equiv Y'$  in  $\Omega$ .

PROOF OF THEOREM 2. Lemma 3 implies that  $T_n X$  are locally equicontinuous in  $\Omega$ , hence  $TX$  is continuous. If  $TX$  is not one to

one, we can find a pair of distinct points  $X_0, X_1 \in \Omega$ , such that  $TX_0 = TX_1 = Y'$ . Let  $\Delta = \Omega - \{X_1\}$  and  $T_n X_1 = Y'_n$ . Then  $T_n X \neq Y'_n$  in  $\Delta$  and

$$\lim_{n \rightarrow +\infty} Y'_n = Y'.$$

Since  $X_0 \in \Delta$  and  $TX_0 = Y'$ , Lemma 4 implies that  $TX = Y'$  in  $\Delta$  and hence that  $TX$  is constant in  $\Omega$ .

2. QUASICONFORMAL MAPPINGS. Let  $H$  be a real normed space,  $\Omega, \Omega'$  domains of  $H$  and  $TX$  a homeomorphism of  $\Omega$  onto  $\Omega'$ . For  $X_0 \in \Omega$  and  $0 < r < \varrho(X_0, \partial\Omega)$  we set

$$L(X_0, r) = \sup_{\|X - X_0\| = r} \|TX - TX_0\|$$

$$l(X_0, r) = \inf_{\|X - X_0\| = r} \|TX - TX_0\|$$

$$\delta(X_0) = \limsup_{r \rightarrow 0^+} \frac{L(X_0, r)}{l(X_0, r)}$$

According to Gehring metric definition of quasiconformality for  $H = R^n$ , we give the following

**DEFINITION 2.** A homeomorphism  $TX$  of a domain  $\Omega$  onto a domain  $\Omega'$  is said to be  $K$ -quasiconformal if  $\delta(X_0) \leq K$  for every point  $X_0$  of  $\Omega$ .  $TX$  is quasiconformal if it is  $K$ -quasiconformal for some  $K$  (finite).

If  $H = R^n$ , a homeomorphism is quasiconformal if and only if it is a  $\theta$ -mapping ([2]). Now we show the following

**THEOREM 3.** A homeomorphism  $TX$  of  $\Omega$  onto  $\Omega'$  is quasiconformal if  $T^{-1}X$  is a  $\theta$ -mapping.

**PROOF.** Let  $X_0 \in \Omega$  and  $c$ ,  $0 < c < \varrho(X_0, \partial\Omega)$ , so that

$$(9) \quad \|TX - TX_0\| < \varrho(TX_0, \partial\Omega')$$

for  $\|X - X_0\| < c$ . Next, for each  $r$ ,  $0 < r < c$  and  $\varepsilon > 0$ , choose  $X_1$  and  $X_2$  so that  $\|X_1 - X_0\| = \|X_2 - X_0\| = r$  and so that

$$(10) \quad \begin{cases} L(X_0, r) = \sup_{\|X - X_0\|=r} \|TX - TX_0\| < \|TX_1 - TX_0\| + \varepsilon \\ 1(X_0, r) = \inf_{\|X - X_0\|=r} \|TX - TX_0\| > \|TX_2 - TX_0\| - \varepsilon \end{cases}$$

If for some  $r$  and for each  $\varepsilon$  is

$$(11) \quad 1(X_0, r) + \varepsilon > L(X_0, r) - \varepsilon$$

then we have  $1(X_0, r) = L(X_0, r)$  for such  $r$ . If  $r$  is such that (11) is not true we have, for  $0 < \varepsilon < \varepsilon_0$ ,  $\varepsilon_0$  suitable,

$$(12) \quad 1(X_0, r) + \varepsilon < L(X_0, r) - \varepsilon.$$

Let  $\Delta = \Omega - \{X_1\}$  and  $\Delta' = T(\Delta)$ . Then  $\Delta' = \Omega' - \{TX_1\}$ . From (9), (10) and (12) we obtain

$$\|TX_2 - TX_0\| < 1(X_0, r) + \varepsilon < L(X_0, r) - \varepsilon < \|TX_1 - TX_0\| = \varrho(TX_0, \partial\Delta').$$

Since  $T^{-1}X$  is a  $\theta$ -mapping we have

$$1 = \frac{\|T^{-1}(TX_2) - T^{-1}(TX_0)\|}{\varrho(T^{-1}(TX_0), \partial\Delta)} \leq \theta \left( \frac{\|TX_2 - TX_0\|}{\varrho(Tx_0, \partial\Delta')} \right) < \theta \left( \frac{1(X_0, r) + \varepsilon}{L(X_0, r) - \varepsilon} \right)$$

and, as  $\theta(t)$  is increasing,

$$\frac{L(X_0, r) - \varepsilon}{1(X_0, r) + \varepsilon} < \frac{1}{\theta^{-1}(1)}$$

For  $\varepsilon \rightarrow 0^+$  we obtain

$$\frac{L(X_0, r)}{1(X_0, r)} \leq \frac{1}{\theta^{-1}(1)}$$

and also

$$\limsup_{r \rightarrow 0^+} \frac{L(X_0, r)}{1(X_0, r)} \leq \frac{1}{\theta^{-1}(1)} = K$$

As the latter inequality holds for each point  $X_0$  of  $\Omega$ , the theorem is proved.

Theorem 3 can be improved if  $TX$  is a diffeomorphism, that is a homeomorphism of  $\Omega$  onto  $\Omega'$ , such that  $TX$  and  $T^{-1}X$  are differentiable. For this we have:

LEMMA 5. *Suppose that  $TX$  is a quasiconformal diffeomorphism of  $\Omega$  onto  $\Omega'$ . Then the diffeomorphism  $T^{-1}X$  of  $\Omega'$  onto  $\Omega$  is quasiconformal.*

PROOF. Let  $X_0 \in \Omega$ . If  $A$  denotes the Fréchet derivative of  $TX$  in  $X_0$ , we have ([3]), for some constant  $K$ ,

$$(13) \quad 0 < \sup_{Z \neq 0} \frac{\|AZ\|}{\|Z\|} \leq K \inf_{Z \neq 0} \frac{\|AZ\|}{\|Z\|}.$$

Since the derivative of  $T^{-1}X$  in  $TX_0$  is  $A^{-1}$ , (13) implies

$$(14) \quad 0 < \sup_{Y \neq 0} \frac{\|A^{-1}Y\|}{\|Y\|} \leq K \inf_{Y \neq 0} \frac{\|A^{-1}Y\|}{\|Y\|}.$$

If  $X_0$  ranges onto  $\Omega$ , then  $TX_0$  ranges onto  $\Omega'$ , and (14) does hold in every point of  $\Omega'$ . Thus ([3])  $T^{-1}X$  is quasiconformal.

Theorem 3 and Lemma 5 give the following.

COROLLARY. *A diffeomorphism  $TX$  of  $\Omega$  onto  $\Omega'$  is quasiconformal if  $TX$  is a  $\theta$ -mapping.*

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