RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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θ-mappings and quasiconformal mappings in normed spaces

Rendiconti del Seminario Matematico della Università di Padova, tome 57 (1977), p. 173-182

<http://www.numdam.org/item?id=RSMUP_1977__57__173_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 57 (1977)

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Introduction.

The θ-mappings introduced by Gehring in [2], and some definitions of quasiconformal mappings in R^n, are meaningfull in every normed space. But such mappings have investigated, till now, almost exclusively in R^n. Indeed, most of the procedures used in R^n (as Lebesgue integral) are meaningless in a infinite dimensional space H.

In this paper we consider θ-mappings in a real normed space, and we prove for them a compactness theorem. Further, we show some properties of sequences of θ-mappings. Next we consider the metric definition of quasiconformal mappings in a real normed space, and we give a relation between the θ-mappings and the quasiconformal mappings.

1. θ-MAPPINGS. Let H be a real normed space and Ω, Ω' domains of H. Following Gehring ([2], pag. 14) we give the

DEFINITION 1. *A homeomorphism TX of Ω onto Ω' is said to be a θ-mapping if there exists a function $\theta(t)$, which is continuous and increasing in $\theta \leq t < 1$ with $\theta(0) = 0$, such that the following are true.*

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Lavoro eseguito nell'ambito del GNAFA del CNR.

(i) *If* $X_0 \in \Omega$ *and* $\|X - X_0\| < \inf_{Y \in \partial\Omega} \|X_0 - Y\| \equiv \varrho(X_0, \partial\Omega)$, *then*

$$\frac{\|TX - TX_0\|}{\varrho(TX_0, \partial\Omega')} \leq \theta\left(\frac{\|X_0 - X\|}{\varrho(X_0, \partial\Omega)}\right),$$

(ii) *The restriction of* TX *to any subdomain* $\Delta \subseteq \Omega$, *satisfies* (i).

For the θ-mappings we now prove some properties.

THEOREM 1. *Suppose that* $\{T_n X\}$ *is a sequence of* θ*-mappings (with the same distortion function* $\theta(t)$*) of* Ω, *that*

$$\lim_{n\to\infty} T_n X = TX$$

uniformly on each bounded domain $\Delta \subset \Omega$ *such that* $\varrho(\Delta, \partial\Omega) > 0$, *and that* TX *is a homeomorphism. Then* TX *is a* θ*-mapping.*

PROOF. Let $\Delta \subset \Omega$ be a bounded domain such that $\varrho(\Delta, \partial\Omega) > 0$, and set $\Delta'_n = T_n(\Delta)$, $\Delta' = T(\Delta)$. Since $T_n X$ are θ-mappings we have, for $X_0 \in \Delta$:

$$\frac{\|T_n X - T_n X_0\|}{\varrho(T_n X_0, \partial\Delta'_n)} \leq \theta\left(\frac{\|X - X_0\|}{\varrho(X_0, \partial\Delta)}\right), \quad n = 1, 2, \ldots \tag{1}$$

for $\|X - X_0\| < \varrho(X_0, \partial\Delta)$. We have to show that

$$\frac{\|TX - TX_0\|}{\varrho(TX_0, \partial\Delta')} \leq \theta\left(\frac{\|X - X_0\|}{\varrho(X_0, \partial\Delta)}\right) \tag{2}$$

for $\|X - X_0\| < \varrho(X_0, \partial\Delta)$. First we prove that

$$\liminf_{n\to\infty} \varrho(T_n X_0, \partial\Delta'_n) \leq \varrho(TX_0, \partial\Delta'). \tag{3}$$

We may suppose $\varrho(TX_0, \partial\Delta') < \infty$. If (3) is not true there exists a subsequence $\{T_{nk} X\}$ and a positive real number σ such that

$$\lim_{k\to\infty} \varrho(T_{n_k} X_0, \partial\Delta'_{n_k}) \geq \varrho(TX_0, \partial\Delta') + \sigma. \tag{4}$$

From (4) it follows that there exists a finitive ν such that

$$(5)\qquad \varrho(TX_0\,,\,\partial\Delta'_{n_k}) > \varrho(TX_0\,,\,\partial\Delta') + \sigma/2$$

for $k > \nu$. By (5) we may find at least one point $X' \in \partial\Delta'$ and a ball $B(X', r')$ centered in X', radius r', and contained in Δ'_{n_k}, for $k > \nu$. Further we may take a second ball $B(Z', r) \subset B(X', r')$ with $B(Z', r) \cap \Delta' = \varnothing$. Let Z_k be the points of Δ such that $T_{n_k}Z_k = Z'$. Since $\{T_{n_k}X\}$ converges uniformly to TX on Δ, we have

$$\|TZ_k - Z'\| = \|TZ_k - T_{n_k}Z_k\| < r$$

for $k > \nu_1$, where $\nu_1 \geq \nu$ is a suitable finite number. The latter inequality shows that TZ_k belongs to $B(Z', r)$. Hence, we obtain a contradiction and inequality (3) does hold.

Since

$$\lim_{n\to\infty} \|T_nX - T_nX_0\| = \|TX - TX_0\|,$$

from (1) and (3), we derive (2).

Let now Δ be a subdomain of Ω. If $X_0 \in \Delta$ and $\varrho(X_0\,,\,\partial\Delta) = \infty$, let $\Delta_m \equiv B(X_0\,,\,m)$. For the previous result we have, for $\|X - X_0\| < m$:

$$\frac{\|TX - TX_0\|}{\varrho(TX_0\,,\,\partial\Delta'_m)} \leq \theta\left(\frac{\|X - X_0\|}{\varrho(X_0\,,\,\partial\Delta_m)}\right) = \theta\left(\frac{\|X - X_0\|}{m}\right),$$

where $\Delta'_m = T(\Delta_m)$. Since $\lim\limits_{m\to\infty} \theta(\|X - X_0\|/m) = 0$ it must be

$$\lim_{m\to\infty} \varrho(TX_0\,,\,\partial\Delta'_m) = \infty\,.\ \text{As}\ \varrho(TX_0\,,\,\partial\Delta') \geq \varrho(TX_0\,,\,\partial\Delta'_m)$$

it must be again $\varrho(TX_0\,,\,\partial\Delta') = \infty$, and inequality (2) holds for such Δ.

Finally suppose $\Delta \subseteq \Omega$, $X_0 \in \Delta$ and such that $\varrho(X_0\,,\,\partial\Delta) = c < \infty$. Let $\Delta_m = B(X_0\,,\,c - 1/m)$ with $m > 1/c$. Inequality (2) holds for Δ_m. Further we have $\Delta' = T(\Delta) \supset T(\Delta_m) = \Delta'_m$, hence

$$\varrho(TX_0\,,\,\partial\Delta') \geq \varrho(TX_0\,,\,\partial\Delta'_m)\,,\ \varrho(X_0\,,\,\partial\Delta_m) = \varrho(X_0\,,\,\partial\Delta) - 1/m$$

and

$$\frac{\|TX - TX_0\|}{\varrho(TX_0, \partial\Delta')} \le \frac{\|TX - TX_0\|}{\varrho(TX_0, \partial\Delta'_m)} \le \theta\left(\frac{\|X - X_0\|}{\varrho(X_0, \partial\Delta_m)}\right) = \theta\left(\frac{\|X - X_0\|}{\varrho(X_0, \partial\Delta) - \frac{1}{m}}\right).$$

For $m \to \infty$, because $\theta(t)$ is continuous, we obtain inequality (2) for such Δ.

THEOREM 2. *Suppose that* $\{T_n X\}$ *is a sequence of* θ*-mappings of* Ω *and that*

$$\lim_{n\to\infty} T_n X = TX, \; \|TX\| < \infty$$

in Ω. *Then* TX *is either a constant or a continuous and injective mapping.*

In order to prove the Theorem we give some lemmas whose proofs are similar to those done by Gehring for $H = R^n$ ([2]).

LEMMA 1. *Suppose that* $\{T_n X\}$ *is a sequence of* θ*-mappings of* Ω *which are locally uniformly bounded on* Ω. *Then* $T_n X$ *are locally equicontinuous in* Ω.

PROOF. Let $Z \in \Omega$ and let $B(Z, r)$ be a ball centered in Z, radius $r < \varrho(Z, \partial\Omega)$ and such that $T_n X$ are uniformly bounded in $B(Z, r)$. Choose $X_0 \in \Omega - \bar{B}(Z, r)$ and let $\Delta = \Omega - \{X_0\}$. By hypothesis there exists a finite constant A (depending on $B(Z, r)$ and X_0) such that

$$\|T_n X - T_n X_0\| \le \|T_n X\| + \|T_n X_0\| \le A$$

for $X \in B(Z, r)$ and all n. Hence

$$\varrho(T_n X, \partial\Delta'_n) \le \|T_n X - T_n X_0\| \le A \tag{6}$$

for $X \in B(Z, r)$ and all n, where $\Delta'_n = T_n(\Delta)$. Since $r < \varrho(Z, \partial\Omega)$ and $X_0 \notin \bar{B}(Z, r)$,

$$\varrho(X, \partial\Delta) \ge a > 0 \tag{7}$$

for $X \in B(Z, r)$. Now fix $X \in B(Z, r)$ and choose Y so that $\|Y -$

$- X\| < a$. Then

$$\frac{\|T_n Y - T_n X\|}{\varrho(T_n X, \partial \Delta_n')} \le \theta\left(\frac{\|Y - X\|}{\varrho(X, \partial \Delta)}\right).$$

and, for (6) — (7) ,

$$\|T_n Y - T_n X\| \le A\, \theta\left(\frac{\|Y - X\|}{a}\right).$$

Since $\lim_{t \to 0+} \theta(t) = 0$, this implies the desired equicontinuity of $T_n X$ in $B(Z, r)$.

Lemma 2. *Suppose that $\{T_n X\}$ is a sequence of θ-mappings of Ω onto Ω_n', that*

$$\sup_n \|T_n X_0\| < \infty$$

for some $X_0 \in \Omega$, and that

$$\sup_n \varrho(0\, ,\, \partial \Omega_n') < \infty$$

where 0 denotes the null element of H. Then $T_n X$ are locally equicontinuous in Ω.

Proof. Fix a so that $0 < a < 1$. Then, if we choose $Y \in \Omega$ and X so that $\|X - Y\| < a\varrho(Y, \partial\Omega)$ we have

$$\|T_n X - T_n Y\| \le \theta(a)\, \varrho(T_n Y, \partial \Omega_n')$$

for all n. Since

$$\varrho(T_n Y, \partial \Omega_n') \le \|T_n Y\| + \varrho(0, \partial \Omega_n')$$

we thus obtain

$$\|T_n X\| \le M\, \|T_n Y\| + N \tag{8}$$

where

$$M = 1 + \theta(a)\, , \quad N = \theta(a) \sup_n \varrho(0, \partial \Omega_n')$$

In particular we conclude that each point $Y \in \Omega$ has a neighborhood $U = U(Y) \subset \Omega$ such that (8) holds for all $X \in U$. Next, if we choose $X \in \Omega$ and Y so that $\|Y - X\| < \frac{a}{2} \varrho(X, \partial\Omega)$ we have

$$\|Y - X\| < \frac{a}{2} \left(\varrho (Y, \partial\Omega) + \|Y - X\|\right)$$

and thus

$$\|Y - X\| < \frac{a}{2 - a} \varrho(Y, \partial\Omega) < a\varrho(Y, \partial\Omega) .$$

Hence we see that each point $X \in \Omega$ has a neighborhood $V = V(X) \subset \Omega$ such that (8) holds for all $Y \in V$. Now let G denote the set of points $X \in \Omega$ for which

$$\sup_n \|T_n X\| < \infty .$$

If $Y \in G$ and if U is the neighborhood described above, then (8) implies that

$$\sup_n \|T_n X\| \leq M \sup_n \|T_n Y\| + N$$

for all $X \in U$. Hence $U \subset G$ and G is open. Similary, if $X \in \Omega - G$ and if V is the neighborhood described above, the same argument shows that $V \subset \Omega - G$, and hence that $\Omega - G$ is open. Since Ω is connected and $X_0 \in G$, we conclude that $G = \Omega$. We have shown that for $Y \in G$, $\sup \|T_n Y\| < \infty$. Further, by (8), follows that $T_n X$ are uniformly bounded in a suitable neighborhood $U(Y)$. The locally equicontinuity is now a consequence of Lemma 1.

LEMMA 3. *Suppose that* $\{T_n X\}$ *is a sequence of* θ*-mappings of* Ω, *and that*

$$\sup_n \|T_n X_0\| < \infty , \quad \sup_n \|T_n X_1\| < \infty$$

for a pair of distintic fixed points $X_0, X_1 \in \Omega$. *Then* $T_n X$ *are locally equicontinuous in* Ω.

PROOF. Let $\Delta = \Omega - \{X_1\}$ and $\Delta'_n = T_n(\Delta)$. Then $T_n X_1 \in \partial \Delta'_n$ and hence

$$\sup_n \varrho(0, \partial \Delta'_n) \leq \sup_n \|T_n X_1\| < \infty .$$

Lemma 2 implies now the locally equicontinuity of $T_n X$ in Δ. Interchanging the roles of X_0 and X_1 we obtain the desired conclusion.

LEMMA 4. *Suppose that $\{T_n X\}$ is a sequence of θ-mappings of Ω, that*

$$\lim_{n \to \infty} T_n X = TX , \quad \|TX\| < \infty$$

in Ω, that $T_n X \neq Y'_n$ in Ω, and that

$$\lim_{n \to \infty} Y'_n = Y'.$$

Then either $TX \neq Y'$ in Ω or $TX \equiv Y'$ in Ω.

PROOF. Let G be the set of points $X \in \Omega$ for which $TX = Y'$. Lemma 3 implies that $T_n X$ are locally equicontinuous in Ω. Hence TX is continuous and G is closed in Ω. Now suppose $X_0 \in G$ and let U be the set of points X for which $\|X - X_0\| < a\varrho(X_0, \partial \Omega)$, where a is some fixed constant, $0 < a < 1$. Then we have

$$\|T_n X - T_n X_0\| \leq \theta(a) \varrho(T_n X_0 , \partial \Omega'_n)$$

for all $X \in U$, where $\Omega'_n = T_n(\Omega)$. Since $Y'_n \notin \Omega'_n$, we see that

$$\varrho(T_n X_0 , \partial \Omega'_n) \leq \|T_n X_0 - Y'_n\|$$

and hence

$$\|TX - TX_0\| = \lim_{n \to \infty} \|T_n X_0 - T_n X\| \leq \theta(a) \lim_{n \to \infty} \|T_n X_0 - Y'_n\| = 0$$

for all $X \in U$. Hence $U \subset G$ and G is open. Since Ω is connected, we conclude, as desired, that either $G = \Omega$ or $TX \equiv Y'$ in Ω.

PROOF OF THEOREM 2. Lemma 3 implies that $T_n X$ are locally equicontinuous in Ω, hence TX is continuous. If TX is not one to one, we can find a pair of distintic points X_0, $X_1 \in \Omega$, such that $TX_0 = TX_1 = Y'$. Let $\Delta = \Omega - \{X_1\}$ and $T_n X_1 = Y'_n$. Then $T_n X \neq$ $\neq Y'_n$ in Δ and

$$\lim_{n \to +\infty} Y'_n = Y'.$$

Since $X_0 \in \Delta$ and $TX_0 = Y'$, Lemma 4 implies that $TX = Y'$ in Δ and hence that TX is constant in Ω.

2. Quasiconformal mappings. Let H be a real normed space, Ω, Ω' domains of H and TX a homeomorphism of Ω onto Ω'. For $X_0 \in \Omega$ and $0 < r < \varrho(X_0, \partial\Omega)$ we set

$$L(X_0, r) = \sup_{\|X - X_0\| = r} \|TX - TX_0\|$$

$$1(X_0, r) = \inf_{\|X - X_0\| = r} \|TX - TX_0\|$$

$$\delta(X_0) = \limsup_{r \to 0+} \frac{L(X_0, r)}{1(X_0, r)}$$

According to Gehring metric definition of quasiconformality for $H = R^n$, we give the following

Definition 2. *A homeomorphism TX of a domain Ω onto a domain Ω' is said to be K-quasiconformal if $\delta(X_0) \leq K$ for every point X_0 of Ω. TX is quasiconformal if it is K-quasiconformal for some K (finite).*

If $H = R^n$, a homeomorphism is quasiconformal if and only if it is a θ-mapping ([2]). Now we show the following

Theorem 3. *A homeomorphism TX of Ω onto Ω' is quasiconformal if $T^{-1}X$ is a θ-mapping.*

Proof. Let $X_0 \in \Omega$ and c, $0 < c < \varrho(X_0, \partial\Omega)$, so that

$$\|TX\text{-}TX_0\| < \varrho(TX_0, \partial\Omega') \tag{9}$$

for $\|X-X_0\| < c$. Next, for each r, $0 < r < c$ and $\varepsilon > 0$, choose X_1 and X_2 so that $\|X_1-X_0\| = \|X_2-X_0\| = r$ and so that

$$(10) \qquad \begin{cases} L(X_0\,, r) = \sup\limits_{\|X-X_0\|=r} \|TX-TX_0\| < \|TX_1-TX_0\| + \varepsilon \\ 1\,(X_0\,, r) = \inf\limits_{\|X-X_0\|=r} \|TX-TX_0\| > \|TX_2-TX_0\| - \varepsilon \end{cases}$$

If for some r and for each ε is

$$(11) \qquad 1\,(X_0\,, r) + \varepsilon > L(X_0\,, r) - \varepsilon$$

then we have $1\,(X_0\,, r) = L(X_0\,, r)$ for such r. If r is such that (11) is not true we have, for $0 < \varepsilon < \varepsilon_0$, ε_0 suitable,

$$(12) \qquad 1\,(X_0\,, r) + \varepsilon < L\,(X_0\,, r) - \varepsilon\,.$$

Let $\Delta = \Omega - \{X_1\}$ and $\Delta' = T(\Delta)$. Then $\Delta' = \Omega' - \{TX_1\}$. From (9), (10) and (12) we obtain

$$\|TX_2-TX_0\| < 1(X_0\,, r) + \varepsilon < L(X_0\,, r) - \varepsilon < \|TX_1-TX_0\| = \varrho(TX_0\,, \partial\Delta')\,.$$

Since $T^{-1}\,X$ is a θ-mapping we have

$$1 = \frac{\|T^{-1}(TX_2) - T^{-1}(TX_0)\|}{\varrho(T^{-1}(TX_0)\,, \partial\Delta)} \leq \theta\left(\frac{\|TX_2 - TX_0\|}{\varrho(Tx_0\,, \partial\Delta')}\right) < \theta\left(\frac{1(X_0\,, r)+\varepsilon}{L(X_0\,, r)-\varepsilon}\right)$$

and, as $\theta\,(t)$ is increasing,

$$\frac{L(X_0\,, r)-\varepsilon}{1(X_0\,, r)+\varepsilon} < \frac{1}{\theta^{-1}(1)}$$

For $\varepsilon \to 0^+$ we obtain

$$\frac{L(X_0\,, r)}{1(X_0\,, r)} \leq \frac{1}{\theta^{-1}(1)}$$

and also

$$\limsup_{r\to 0+} \frac{L(X_0\,, r)}{1(X_0\,, r)} \leq \frac{1}{\theta^{-1}(1)} = K$$

As the latter inequality holds for each point X_0 of Ω, the theorem is proved.

Theorem 3 can be improved if TX is a diffeomorphism, that is a homeomorphism of Ω onto Ω', such that TX and $T^{-1}X$ are differentiable. For this we have:

LEMMA 5. *Suppose that TX is a quasiconformal diffeomorphism of Ω onto Ω'. Then the diffeomorphism $T^{-1}X$ of Ω' onto Ω is quasiconformal.*

PROOF. Let $X_0 \in \Omega$. If A denotes the Frèchet derivative of TX in X_0, we have ([3]), for some constant K,

$$0 < \sup_{Z \neq 0} \frac{\|AZ\|}{\|Z\|} \leq K \inf_{Z \neq 0} \frac{\|AZ\|}{\|Z\|}. \tag{13}$$

Since the derivative of $T^{-1}X$ in TX_0 is A^{-1}, (13) implies

$$0 < \sup_{Y \neq 0} \frac{\|A^{-1}Y\|}{\|Y\|} \leq K \inf_{Y \neq 0} \frac{\|A^{-1}Y\|}{\|Y\|}. \tag{14}$$

If X_0 ranges onto Ω, then TX_0 ranges onto Ω', and (14) does hold in every point of Ω'. Thus ([3]) $T^{-1}X$ is quasiconformal.

Theorem 3 and Lemma 5 give the following.

COROLLARY. *A diffeomorphism TX of Ω onto Ω' is quasiconformal if TX is a θ-mapping.*

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Manoscritto pervenuto in redazione il 28 marzo 1977.