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The Weak Cauchy Problem
for Abstract Differential Equations.

S. Zaidman (*)

Introduction.

We consider the weak Cauchy problem in arbitrary Banach space for equations \((d/dt - A)u = 0\), as were defined by Kato-Tanabe. After proving some elementary relationships, we obtain a result which shows how uniqueness of Cauchy problem for strong solutions in the second dual space implies uniqueness of the weak Cauchy problem.

A simple result by Barbu-Zaidman (Notices A.M.S., April 1973, 73T-B120) gets then a new proof, and an uniqueness result for weakened solutions by Liubic-Krein gets a partial extension.

The last result is a certain extension of Barbu-Zaidman result to non-reflexive \(B\)-spaces, using as a main tool in the proof Phillips's theorem on dual semi-groups in \(B\)-spaces.

§ 1. - Let \(X\) be a given Banach space, and \(X^*\) be its dual space. If \(A\) is a linear closed operator with dense domain \(D(A) \subset X\), mapping \(D(A)\) into \(X\), then, the dual operator \(A^*\) is defined on the set \(D(A^*) = \{x^* \in X^*, \text{ s.t. } \exists y^* \in X^* \text{, satisfying} \}

\[
\langle x^*, Ax \rangle = \langle y^*, x \rangle \quad \forall x \in D(A)
\]

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By definition $A^* x^* = y^*$, and $A^*$ is a well-defined linear operator from $\mathcal{D}(A^*)$ into $\mathcal{X}^*$.

Furthermore, the domain $\mathcal{D}(A^*)$ is a total set in $\mathcal{X}^*$; this means that, given any element $x \in \mathcal{X}$, $x \neq \theta$, $\exists x^* \in \mathcal{D}(A^*)$, such that $x^*(x) \neq 0$; hence, if $x^*(x) = 0$, $\forall x^* \in \mathcal{D}(A^*)$, then $x = \theta$.

Let now be given a finite interval $-\infty < a < b < +\infty$ on the real axis; a class of «test-functions» associated to the operator $\mathcal{D}(A^*)$ and to the given interval, denoted here by $K_{A^*}[a, b]$, consists of continuously differentiable functions $\phi^*(t)$, $a < t < b \to \mathcal{X}^*$, which are $= \theta$ near $b$ (that is $\phi^*(t) = \theta$ for $b - \delta < t < b$, where $\delta$ depends on $\phi^*$); furthermore, $\phi^*(t)$ belong to $\mathcal{D}(A^*)$, $\forall t \in [a, b]$, and $(A^* \phi^*)(t)$ is $\mathcal{X}^*$-continuous on $[a, b]$.

Obviously, if $\varphi(t)$ is scalar-valued, $C^1[a, b]$-function, and $\varphi = 0$ near $b$, and if $\phi^*$ is any element in $\mathcal{D}(A^*)$, then $\varphi(t) \phi^*$ belongs to $K_{A^*}[a, b]$.

Let us consider now the Bochner space $L^p_{loc}([a, b]; \mathcal{X})$, where $p$ is any real $> 1$, consisting of strongly measurable $\mathcal{X}$-valued functions $f$ defined on $[a, b)$, such that $\int_a^b \|f(t)\|_\mathcal{X}^p \, dt < \infty$ for any $c < b$.

The weak forward Cauchy problem is here defined as follows: given any element $u_a \in \mathcal{X}$ and any function $f(t) \in L^p_{loc}([a, b]; \mathcal{X})$, find a function $u(t) \in L^p_{loc}([a, b]; \mathcal{X})$ verifying

$$
(1.2) \quad -\langle \phi^*(a), u_a \rangle - \int_a^b \langle \frac{d\phi^*}{dt}, u(t) \rangle \, dt = \int_a^b \langle (A^* \phi^*)(t), u(t) \rangle \, dt + \\
+ \int_a^b \langle \phi^*(t), f(t) \rangle \, dt, \quad \forall \phi^* \in K_{A^*}[a, b].
$$

In similar way define a weak backward Cauchy problem: The class $K_{A^*}[a, b]$ is defined like $K_{A^*}[a, b]$, with the only difference that the test-functions must be null near $a$, instead of being null near $b$.

There is also the space $L^p_{loc}((a, b]; \mathcal{X})$ of $\mathcal{X}$-mesurable functions such that $\int_a^b \|f\|_\mathcal{X}^p \, dt < \infty$, $\forall c > a$, $c < b$.

Then given any element $u_b \in \mathcal{X}$, and again $f \in L^p_{loc}((a, b]; \mathcal{X})$, find
u(t) ∈ L^p_{loc}((a, b]; \mathcal{X})$, satisfying

\begin{equation}
-\langle \phi^*(b), u_0 \rangle - \int_a^b \left\langle \frac{d\phi^*}{dt}, u(t) \right\rangle dt = \int_a^b \langle (A^*\phi^*)(t), u(t) \rangle dt + \int_a^b \langle \phi^*(t), f(t) \rangle dt, \quad \forall \phi^* \in K_{A^*}(a, b].
\end{equation}

**REMARK.** This definitions are slightly more general than the weakened Cauchy problem as defined for example in S. G. Krein [3]; extending his definition from the interval \([0, T]\) to an arbitrary interval \([a, b]\), we say that \(u(t), a < t < b \rightarrow \mathcal{X}\) is a weakened solution of

\begin{equation}
\dot{u}(t) = Au(t) + f(t), \quad u(a) = u_0 \in \mathcal{X}.
\end{equation}

where \(f(t), a < t < b \rightarrow \mathcal{X}\) is \(\mathcal{X}\)-continuous, if: \(u(t)\) is \(\mathcal{X}\)-continuous on the closed interval \([a, b]\); \(u(t)\) is \(\mathcal{X}\)-differentiable with continuous derivative on the half-open interval \((a, b]\); \(u(t)\) is \(\mathcal{D}(A)\) on same \((a, b]\);

\[ u'(t) = Au(t) + f(t), \quad a < t < b, \quad u(a) = u_0. \]

The following result holds:

**PROPOSITION 1.1.** If \(u(t)\) is a weakened solution of (1.4), then (1.2) is also verified.

Consider the equality \(u'(t) - Au(t) = f(t)\), valid on the half-open interval \(a < t < b\). Then take any test-function \(\phi^*(t) \in K_{A^*}[a, b]\) (\(\phi^*\) is null near \(b\)). We get obviously

\begin{equation}
\langle \phi^*(t), u'(t) \rangle - \langle \phi^*(t), Au(t) \rangle = \langle \phi^*(t), f(t) \rangle, \quad a < t < b.
\end{equation}

Also we see that

\begin{equation}
\frac{d}{dt} \langle \phi^*(t), u(t) \rangle = \left\langle \frac{d\phi^*}{dt}, u(t) \right\rangle + \left\langle \phi^*(t), \frac{du}{dt} \right\rangle, \quad a < t < b.
\end{equation}
If we integrate (1.6) between $a + \varepsilon$ and $b$, $\forall \varepsilon > 0$, we obtain

$$-\langle \phi^*(a + \varepsilon), u(a + \varepsilon) \rangle = \int_{a + \varepsilon}^{b} \{\langle \phi^*(t), u(t) \rangle + \langle \phi^*(t), \dot{u}(t) \rangle\} dt.$$  

Let us integrate now (1.5) between $a$ and $b$, and remark also that $\langle \phi^*(t), Au(t) \rangle = \langle A^* \phi^*(t), u(t) \rangle$, $a < t < b$. We get

$$\int_{a}^{b} \langle \phi^*(t), u'(t) \rangle dt = -\langle \phi^*(a + \varepsilon), u(a + \varepsilon) \rangle - \int_{a + \varepsilon}^{b} \langle \phi^*(t), u(t) \rangle dt =$$

$$= \int_{a + \varepsilon}^{b} \langle A^* \phi^*(t), u(t) \rangle dt + \int_{a + \varepsilon}^{b} \langle \phi^*(t), f(t) \rangle dt.$$  

By continuity of all functions here involved, one obtains, when $\varepsilon \to 0$

$$-\langle \phi^*(a), u(a) \rangle - \int_{a}^{b} \langle \phi^*(t), u(t) \rangle dt - \int_{a}^{b} \langle A^* \phi^*(t), u(t) \rangle dt = \int_{a}^{b} \langle \phi^*(t), f(t) \rangle dt.$$  

A converse result is also given in the following.

**Proposition 1.2.** Let us assume: $f(t), a < t < b \to \mathcal{X}$, be strongly continuous; $u_a \in \mathcal{X}$ be arbitrarily given. Then $u(t), a < t < b \to \mathcal{X}$ be a $\mathcal{X}$-continuous function, which is continuously differentiable for $a < t < b$, and belongs to $\mathcal{D}(A)$ for $t \in (a, b)$. Let also (1.2) be satisfied. Then it follows that $u' - Au = f$ on $a < t < b$, and also $u(a) = u_a$.

In order to prove this simple fact, we shall first introduce in (1.2) test-functions of the special form $\phi^*(t) = v(t)x^*$ where $x^* \in \mathcal{D}(A^*)$ and $v(t)$ is scalar-valued continuously differentiable function which $= 0$ near $a$ and near $b$. It results then, if $[a_1, b_1] \subset (a, b)$ contains $\text{supp} \, \phi^*$

$$-\int_{a_1}^{b_1} \langle \phi^*(t), u(t) \rangle dt = \int_{a_1}^{b_1} \langle (A^* \phi^*)(t), u(t) \rangle dt + \int_{a_1}^{b_1} \langle \phi^*(t), f(t) \rangle dt.$$  

As $u(t)$ is continuously differentiable on $[a_1, b_1]$, and $\phi^*(a_1) = \phi^*(b_1) = 0$ (intervals $(a, a_1), (b_1, b)$ are in the null set of $\phi^*$: hence, by continuity,
\( \phi^*(a_1) = \phi^*(b_1) = \theta \) also), it results
\[
- \int_{a_1}^{b_1} \langle \dot{x}^*(t), u(t) \rangle \, dt = \int_{a_1}^{b_1} \langle \dot{x}^*(t), \dot{u}(t) \rangle \, dt.
\]
Also because \( u(t) \in \mathcal{D}(A) \) for \( t \in [a_1, b_1] \), it is \( \langle \dot{x}^*(t), Au(t) \rangle = \langle A^* \dot{x}^*(t), u(t) \rangle \). Hence, relation (1.7) becomes
\[
(1.8) \quad \int_{a_1}^{b_1} \langle \dot{x}^*(t), u(t) \rangle \, dt = \int_{a_1}^{b_1} \langle \dot{x}^*(t), (Au)(t) \rangle \, dt + \int_{a_1}^{b_1} \langle \dot{x}^*(t), f(t) \rangle \, dt
\]
or, as \( \dot{x}^*(t) \) is here \( = \nu(t)x^* \),
\[
(1.9) \quad \int_{a_1}^{b_1} \langle x^*, u'(t) \rangle \nu(t) \, dt = \int_{a_1}^{b_1} \langle x^*, (Au)(t) \rangle \nu(t) \, dt + \int_{a_1}^{b_1} \langle x^*, f(t) \rangle \nu(t) \, dt
\]
or
\[
(1.10) \quad \int_{a_1}^{b_1} \langle x^*, u'(t) - Au(t) - f(t) \rangle \nu(t) \, dt = 0.
\]
By continuity of the scalar function \( \langle x^*, u'(t) - Au(t) - f(t) \rangle \) in \( [a_1, b_1] \), letting \( \nu(t) \) to vary, we get \( \langle x^*, u'(t) - Au(t) - f(t) \rangle = 0 \) in \( [a_1, b_1] \), \( \forall x^* \in \mathcal{D}(A^*) \).
\[
\left\{ \begin{array}{l}
\text{If } \int_\alpha^\beta \phi(t) \nu(t) \, dt = 0, \forall \nu \in C^1_0(\alpha, \beta), \phi \in C[\alpha, \beta] \Rightarrow \phi = 0 \text{ on } (\alpha, \beta) \text{; if not, } \exists \xi \in (\alpha, \beta), \phi(\xi) > 0 \text{ say; in } (\xi - \delta, \xi + \delta), \phi > 0 \text{; take } 0 < \nu, \nu = 1 \text{ on } (\xi - \delta/2, \xi + \delta/2), = 0 \text{ outside } (\xi - \delta, \xi + \delta), \in C^1; \text{ then }
\int_\alpha^\xi \phi \nu \, dt = \int_{\xi - \delta/2}^{\xi + \delta/2} \phi \, dt > 0, \text{ absurde.}
\end{array} \right.
\]
If \( \phi = 0 \text{ on } (\alpha, \beta), \Rightarrow \phi = 0 \text{ on } [\alpha, \beta] \).

Now, if we fix \( t \in [a_1, b_1] \), and vary \( x^* \) over the total set \( \mathcal{D}(A^*) \), we get \( u'(t) = Au(t) + f(t) \).
This is true for any \( t \in [a_1, b_1] \), hence for any \( t \in (a, b) \) too. But \( u'(t), f(t) \), hence \( Au(t) \) are continuous on \( t = b \); so we obtain \( u'(b) = Au(b) + f(b) \) also to be valid.

We still must prove that \( u(a) = u_a \).

Consider again the relation (1.2), for general test-functions \( \phi^*(t) \in \mathcal{K}_A[a, b) \). Take an arbitrary small \( \varepsilon > 0 \), and get

\[
- \langle \phi^*(a), u_a \rangle - \int_a^{a + \varepsilon} \langle \phi^*(t), u(t) \rangle \, dt - \int_{a + \varepsilon}^b \langle \phi^*(t), u(t) \rangle \, dt =
\]

\[
= \int_a^{a + \varepsilon} \langle (A^* \phi^*)(t), u(t) \rangle \, dt + \int_{a + \varepsilon}^b \langle (A^* \phi^*)(t), u(t) \rangle \, dt + \int_{a + \varepsilon}^b \langle \phi^*(t), f(t) \rangle \, dt
\]

we have also,

\[
\int_{a + \varepsilon}^b \langle \phi^*(t), u(t) \rangle \, dt = \int_{a + \varepsilon}^b \frac{d}{dt} \langle \phi^*(t), u(t) \rangle \, dt - \int_{a + \varepsilon}^b \langle \phi^*(t), u(t) \rangle \, dt =
\]

\[
= - \langle \phi^*(a + \varepsilon), u(a + \varepsilon) \rangle - \int_{a + \varepsilon}^b \langle \phi^*(t), u(t) \rangle \, dt ,
\]

and

\[
\int_{a + \varepsilon}^b \langle (A^* \phi^*)(t), u(t) \rangle \, dt = \int_{a + \varepsilon}^b \langle \phi^*(t), Au(t) \rangle \, dt ;
\]

so, we get

\[
- \langle \phi^*(a), u_a \rangle - \int_a^{a + \varepsilon} \langle \phi^*(t), u(t) \rangle \, dt + \langle \phi^*(a + \varepsilon), u(a + \varepsilon) \rangle +
\]

\[
+ \int_{a + \varepsilon}^b \langle \phi^*(t), u(t) \rangle \, dt = \int_{a + \varepsilon}^b \langle \phi^*(t), Au(t) \rangle \, dt + \int_{a + \varepsilon}^b \langle \phi^*(t), f(t) \rangle \, dt + \int_{a}^{a + \varepsilon} \langle A^* \phi^*, u \rangle \, dt .
\]

But \( \dot{u}(t) = Au(t) + f(t) \) on \( a + \varepsilon < t < b \), as was proved above. Hence,
it remains
\[- \langle \phi^*(a), u_a \rangle - \int_a^{a+\varepsilon} \langle \phi^*(t), u(t) \rangle \, dt + \langle \phi^*(a + \varepsilon), u(a + \varepsilon) \rangle = \]
\[= \int_a^{a+\varepsilon} \langle A^* \phi^*(t), u(t) \rangle \, dt + \int_a^{a+\varepsilon} \langle \phi^*(t), f(t) \rangle \, dt.\]

If now let \( \varepsilon \to 0 \), it remains only, using continuity of \( u \) on \([a, b]\), that
\[\langle \phi^*(a), u(a) - u_a \rangle = 0, \quad \forall \phi^* \in K_A[a, b].\]

We can now take \( \phi^*(t) = v_0(t) x^* \), where \( v_0(t) \in C^1[a, b] \), equals 1 near \( a \), and = 0 near \( b \), and \( x^* \in \mathcal{D}(A^*) \). Hence
\[\langle x^*, u(a) - u_a \rangle = 0 \quad \forall x^* \in \mathcal{D}(A^*)\]
which is a total set in \( \mathcal{X}^* \), and again, it will be \( u(a) = u_a \). Q.E.D.

§ 2. - In this section we shall prove that uniqueness of Cauchy problem for strong solutions on an interval \([a, b]\) in the second dual space \( \mathcal{X}^{**} \), implies uniqueness of the weak Cauchy problem in the same interval, in the original space \( \mathcal{X} \).

If \( A \) is linear, closed operator with dense domain in the \( B \)-space \( \mathcal{X} \), we saw that the dual operator \( A^* \) is linear, defined on a total set in \( \mathcal{X}^* \); also \( A^* \) is closed on this set; in fact, let \( x_n^* \in \mathcal{D}(A^*) \), \( x_n^* \to x^*_0 \in \mathcal{X}^* \), \( A^* x_n^* \to y_0^* \in \mathcal{X}^* \). From relations \( \langle x^*_n, A x \rangle = \langle A^* x^*_n, x \rangle \), \( \forall x \in \mathcal{D}(A) \), we get, as \( n \to \infty \), \( \langle x^*_0, A x \rangle = \langle y_0^*, x \rangle \), \( \forall x \in \mathcal{D}(A) \). Hence, by definition of \( A^* \), it is \( x^*_0 \in \mathcal{D}(A^*) \), \( A^* x^*_0 = y_0^* \), so \( A^* \) is closed.

Let us assume from now on the supplementary.

HYPOTHESIS. \( A^* \) is an operator with dense domain in \( \mathcal{X}^* \).

(REMARK. This holds allways when \( \mathcal{X} \) is a reflexive \( B \)-space; the proof is similar to a classical one in Hilbert spaces).

Then, the second dual operator \( A^{**} = (A^*)^* \) will be a well defined operator on a total set \( \mathcal{D}(A^{**}) \subset \mathcal{X}^{**} \) the second dual space of \( \mathcal{X} \). More precisely \( \mathcal{D}(A^{**}) = \{ \psi^{**} \in \mathcal{X}^{**}, \text{ such that } \exists \phi^{**} \in \mathcal{X}^{**}, \text{ satisfying relation } \langle \psi^{**}, A^* \phi^{**} \rangle = \langle \phi^{**}, \phi^* \rangle, \forall \phi^* \in \mathcal{D}(A^*) \} \) and if \( \psi^{**} \in \mathcal{D}(A^{**}) \).
\[ A^{**} y = x^{**}. \] We also know the existence of a canonical map \( J: \mathcal{X} \rightarrow \mathcal{X}^{**}, \) which is linear and isometric; precisely, any element \( x \in \mathcal{X} \) defines a linear continuous functional \( f^{**} \) on \( \mathcal{X}^{*} \), by: \( \langle f^{**}, x^{*} \rangle = \langle x^{*}, x \rangle \), \( \forall x^{*} \in \mathcal{X}^{*} \). Then put \( Jx = f^{**} \), so that \( \langle x^{*}, x \rangle = \langle Jx, x^{*} \rangle \), \( \forall x^{*} \in \mathcal{D}(A^{*}) \).

Let now \( u(t), a < t < b, \) be a \( C^1[a, b; \mathcal{X}] \) function such that \( \forall t \in [a, b] \) and \( u'(t) = \Delta u(t) \) on \([a, b].\) This is a strong solution on \([a, b],\) and \( u(a) \) belongs necessarily to \( \mathcal{D}(A). \) Then \( (Ju)(t) \) is a \( C^1[a, b; \mathcal{X}^{**}] \) function, as easily seen, and \( (d/dt)(Ju) = J(du/dt). \) We prove now following

**Theorem 2.1.** Let us assume that for any function \( u(t) \in C^1([a, b], \mathcal{X}) \) such that

i) \( (Ju)(t) \in \mathcal{D}(A^{**}), a < t < b, \)

ii) \( (d/dt)(Ju) - A^{**}(Ju) = 0 \) on \([a, b],\)

iii) \( (Ju)(a) = \theta, \)

it is \( (Ju)(t) = \theta, \forall t \in [a, b]. \) Then, there is unicity of the forward weak Cauchy problem on \([a, b].\)

**Proof.** What we must prove is the following: \( v(t) \in L^p_{\text{loc}}([a, b); \mathcal{X}), \)

\[
(2.1) \quad -\int_a^b \langle \phi^*(t), v(t) \rangle \ dt = \int_a^b \langle (A^* \phi^*)'(t), v(t) \rangle \ dt , \quad \forall \phi^* \in K_{\mathcal{X}}[a, b],
\]

implies \( v(t) = \theta \) almost everywhere on \([a, b].\)

Now, using a suggestion by professor S. Agmon (in Pisa, Italy), we start by extending \( v(t) \) to \((-\infty, b),\) as follows: \( \tilde{v}(t) = v(t) \) for \( a < t < b, \) \( \tilde{v}(t) = \theta \) for \(-\infty < t < a. \) It holds now the following

**Lemma 2.1.** The extended function \( \tilde{v}(t) \) verifies the integral identity

\[
(2.2) \quad \int_{-\infty}^b \langle \psi^*(t) + (A^* \psi^*)'(t), \tilde{v}(t) \rangle \ dt = 0
\]

for any function \( \psi^*(t), -\infty < t < b \rightarrow \mathcal{X}^{*}, \) continuously differentiable there, such that \( \psi^*(t) \in \mathcal{D}(A^{*}), \forall t \in (-\infty, b], \) \( A^* \psi^*(t) \) is \( \mathcal{X}^*-\)continuous; support \( \psi^* \) is compact in \((-\infty, b) \) (i.e. \( \psi^* = \theta \) near \( b \) and near \(-\infty). \)
In fact, (2.2) is the same as

\begin{equation}
\int_{a}^{b} \langle \psi^*(t) + (A^* \psi^*)(t), \nu(t) \rangle \, dt = 0.
\end{equation}

But the restriction to \([a, b]\) of the above considered test function \(\psi^*(t)\) is obviously in the class \(K_x[a, b]\), (because it was null near \(b\), and had all regularity required properties).

Hence, by (2.1), the lemma is proved.

A second, needed result (already announced in our paper [6]) is as follows:

Take any scalar function \(\alpha(t) \in C^1(-\infty, \infty)\), which = 0 for \(|t| > \varepsilon\); for any \(w(t) \in L^p_{\text{loc}}(-\infty, b; \mathcal{X})\) (\(\mathcal{X}\)-mesurable on \((-\infty, b)\), such that \(\int_a^b \|w\|^p_{\mathcal{X}} \cdot dt < \infty\), \(\forall \alpha > -\infty, \beta > \alpha, \beta < b\), we can consider the mollified function

\[ (w \ast \alpha)(t) = \int_{t-\varepsilon}^{t+\varepsilon} w(\tau) \alpha(t - \tau) \, d\tau \]

which is well-defined for \(-\infty < t < b - \varepsilon\), is strongly continuously differentiable, and

\[ \frac{d}{dt} (w \ast \alpha) = \int_{t-\varepsilon}^{t+\varepsilon} w(\tau) \alpha(t - \tau) \, d\tau, \quad -\infty < t < b - \varepsilon. \]

We have

**Lemma 2.2.** If \(w(t) \in L^p_{\text{loc}}(-\infty, b; \mathcal{X})\) verifies the integral identity:

\begin{equation}
\int_{-\infty}^{b} \langle \psi^*(t) + (A^* \psi^*)(t), \nu(t) \rangle \, dt = 0
\end{equation}

\(\forall \psi^*\) as in Lemma 2.1, then, it is \(J(w \ast \alpha) \in D(A^{**})\), and \((d/dt)J(w \ast \alpha)(t) = A^{**}(J(w \ast \alpha)(t))\) holds, \(\forall t \in (-\infty, b - \varepsilon)\) where \(J\) is the canonical map of \(\mathcal{X}\) in \(\mathcal{X}^{**}\).
Take in fact any fixed $t_0 \in (-\infty, b - \varepsilon)$, and consider then the functions $\psi^{*, \varepsilon}(t) = \alpha^{\varepsilon}(t_0 - t)x^*$, where $x^* \in \mathcal{D}(A^*)$. These are good test functions because $\alpha^{\varepsilon}(t_0 - t) = 0$ for

$$|t - t_0| \gg \varepsilon,$$

hence in any case, $\alpha^{\varepsilon}(t_0 - t) = 0$ near $b$ and near $-\infty$.

There is also $(d/dt)\psi^{*, \varepsilon} = -\alpha^{\varepsilon}(t_0 - t)x^*$. Writing now (2.4), we get

$$\int_{-\infty}^{b} \left< \alpha^{\varepsilon}(t_0 - t)x^*, w(t) \right> dt = \int_{-\infty}^{b} \alpha^{\varepsilon}(t_0 - t) \langle A^*x^*, w(t) \rangle dt$$

or also

$$\left< A^*x^*, \int_{-\infty}^{b} \alpha^{\varepsilon}(t_0 - t)w(t) dt \right> = \left< x^*, \int_{-\infty}^{b} \alpha^{\varepsilon}(t_0 - t)w(t) dt \right>, \quad \forall x^* \in \mathcal{D}(A^*),$$

or

$$\left< A^*x^*, (w^*\alpha^\varepsilon)(t_0) \right> = \left< x^*, (w^*\alpha^\varepsilon)'(t_0) \right>, \quad \forall x^* \in \mathcal{D}(A^*).$$

Here, if we introduce the canonical imbedding operator $J$, we have:

$$\langle J(w^*\alpha^\varepsilon)(t_0), A^*x^* \rangle = \langle J(w^*\alpha^\varepsilon)'(t_0), x^* \rangle, \quad \forall x^* \in \mathcal{D}(A^*).$$

Now if we use definition of $\mathcal{D}(A^{**})$ and of $A^{**}$, we see that $J(w^*\alpha^\varepsilon) \cdot (t_0) \in \mathcal{D}(A^{**})$, and

$$A^{**}(J(w^*\alpha^\varepsilon)(t_0)) = J(w^*\alpha^\varepsilon)'(t_0) = \frac{d}{dt} J(w^*\alpha^\varepsilon)(t_0)$$

which is the desired Lemma 2.2.

We pass now to the final steps of the proof.

Take $w(t) = \tilde{v}(t)$ the function used in Lemma 2.1; as $v(t) \in L^p_{\text{loc}} \cdot ([a, b); \mathfrak{X})$ and $\tilde{v} = \theta$ for $t < a$, it is obvious that

$$\tilde{v}(t) = w(t) \in L^p_{\text{loc}}[(-\infty, b); \mathfrak{X}], \quad \left( \int_{a}^{\beta} \|w(t)\|^p dt = \int_{a}^{\beta} \|v(t)\|^p dt < \infty \text{ for } \beta < b, \alpha < a \right).$$
Let us apply Lemma 2.2 to \( \hat{v}(t) \). We obtain that \((\hat{v} \ast \alpha_\varepsilon)(t)\) is well-defined on \(-\infty < t < b - \varepsilon\) where is continuously differentiable; also \(J(\hat{v} \ast \alpha_\varepsilon) \in \mathcal{D}(A^{**})\) and

\[
(2.5) \quad \frac{d}{dt} (J(\hat{v} \ast \alpha_\varepsilon)) = A^{**}(J(\hat{v} \ast \alpha_\varepsilon)) \quad \text{holds on } -\infty < t < b - \varepsilon.
\]

Remark also that \((\hat{v} \ast \alpha_\varepsilon)(t) = \theta\) for \(t < a - \varepsilon\), because it is

\[
(\hat{v} \ast \alpha_\varepsilon)(t) = \int_{t - \varepsilon}^{t + \varepsilon} \hat{v}(\tau) \alpha_\varepsilon(t - \tau) d\tau
\]

and \(\hat{v}(\tau) = \theta\) for \(a - \varepsilon < \tau < a\).

Hence, (2.5) holds on \(a - \varepsilon < t < b - \varepsilon\), and also \((\hat{v} \ast \alpha_\varepsilon)(a - \varepsilon) = \theta\) so \(J(\hat{v} \ast \alpha_\varepsilon)(a - \varepsilon) = \theta\).

Now, if \((\hat{v} \ast \alpha_\varepsilon)(t) = Z(t)\), we see that, in the space \(\mathcal{X}^{**}\), it is:

\[
(JZ)'(t) = A^{**}JZ(t) \quad \text{on } [a - \varepsilon, b - \varepsilon], \quad \text{and } JZ(a - \varepsilon) = \theta.
\]

Put then \(t = \sigma - \varepsilon\) and \(Z(t) = Z(\sigma - \varepsilon) = u(\sigma)\); when \(a - \varepsilon < t < b - \varepsilon\), we get \(a - \varepsilon < \sigma - \varepsilon < b - \varepsilon\), or \(a < \sigma < b\); also \(Ju'(\sigma) = JZ'(t)\), so that

\[
(Ju)'(\sigma) = A^{**}(Ju)(\sigma) \quad \text{in } \mathcal{X}^{**}, \quad a < \sigma < b, \quad \text{and } Ju(a) = (JZ)(a - \varepsilon) = \theta.
\]

Applying the hypothesis of the theorem, it follows that \(u(t) = \theta\) on \([a, b]\), hence, \(Z(t) = \theta\) on \([a - \varepsilon, b - \varepsilon]\), that is

\[
(\hat{v} \ast \alpha_\varepsilon)(t) = \theta \quad \text{on } [a - \varepsilon, b - \varepsilon].
\]

Now, take a sequence of functions \(\alpha_n(t)\) which are non-negative, \(= 0\) for \(|t| > 1/n\), continuously differentiable, such that

\[
\int_{-1/n}^{1/n} \alpha_n(\sigma) d\sigma = 1.
\]

We obtain then, in the usual way, as for scalar-valued functions, the relation:

\[
\lim_{n \to \infty} \int_{a_1}^{b_1} \|v(t) - (\hat{v} \ast \alpha_n)(t)\|^p dt = 0, \quad \forall b_1 < b, \quad a_1 > a.
\]
§ 3. - We shall give now some applications of Theorem 2.1. To start with, we give a proof of the following result (see [2]). Let \( \mathcal{X} \) be a reflexive \( B \)-space; \( A \) be the infinitesimal generator of a strongly continuous semi-group of class \( C_0 \); \( A^* \) be the dual operator to \( A \). Let \( u(t), 0 < t < T \to \mathcal{X} \) be a strongly continuous function, verifying the integral identity

\[
\int_0^T \langle \varphi^*(t) + (A^* \varphi^*) (t), u(t) \rangle \, dt = 0
\]

for any function \( \varphi^*(t), 0 < t < T \to \mathcal{X}^* \), which is continuously differentiable in \( \mathcal{X}^* \), belongs to \( \mathcal{D}(A^*) \), \( \forall t \in [0, T] \), \( (A^* \varphi^*) \) is \( \mathcal{X}^* \)-continuous, \( 0 < t < T \), and \( \varphi^*(t) \) is null near 0 and near \( T \). Let also be \( u(0) = \theta \); then \( u(t) = \theta \), \( 0 < t < T \).

Let us remark first that \( A \) is linear closed with dense domain in \( \mathcal{X} \) as any generator of a \( C_0 \) semi-group. By reflexivity of \( \mathcal{X} \) (which means, as usual, that \( J(\mathcal{X}) = \mathcal{X}^{**} \)), it follows that \( \mathcal{D}(A^*) \) is dense in \( \mathcal{X}^* \), and that \( A^{**} (J x) = J(A x), \forall x \in \mathcal{D}(A) \), and \( J(\mathcal{D}(A)) = \mathcal{D}(A^{**}) \), (see [9]).

We shall see now that hypothesis i)-ii)-iii) of Theorem 2.1 are verified.

Take hence \( u(t) \in C^1([0, T]; \mathcal{X}) \); assuming that \( Ju \in \mathcal{D}(A^{**}) = J(\mathcal{D}(A)) \) means: \( \forall t \in [0, T], J v(t) \in \mathcal{D}(A) \), such that \( J v(t) = Ju(t) \); as \( J^{-1} \) exists, \( \Rightarrow v(t) = u(t) \); hence \( u(t) \in \mathcal{D}(A), 0 < t < T \). Also, \( A^{**}. \cdot (Ju(t)) = J(Au(t)) \); We assumed in ii) that \( (d/dt)Ju - A^{**}(Ju) = 0 \) on \([0, T] \). But \( (d/dt)Ju = J(du/dt), \) as \( u \in C^1([0, T]; \mathcal{X}) \). Hence ii) becomes \( J(du/dt) - J(Au) = 0 \) on \([0, T] \) which implies \( u' - Au = 0 \) on \([0, T] \).

Furthermore iii) implies obviously that \( u(0) = \theta \), again because \( J^{-1} \) exists \( (\mathcal{X}^{**} \to \mathcal{X}) \).

Now, the well-known unicity result for strong solutions of \( (d/dt - A)w = 0 \) when \( A \) is generator of a \( C_0 \)-semi-group (see for example [7], theorem 2.2.2) implies that \( u(t) = \theta \) on \([0, T] \), so \( Ju(t) = \theta \) on \([0, T] \) too. Hence, all conditions of theorem 2.1 are fulfilled, and by now we can conclude that:
If the relation
\[
\int_0^T \langle \phi^*(t) + (A^* \phi^*)(t), u(t) \rangle \, dt = 0
\]
holds \( \forall \phi \in K_\alpha[0, T] \), then \( u = \theta \) on \( [0, T] \) (in fact, \( u \)-continuous is in \( L^p_{\text{loc}} \), and \( u = \theta \) a.e. on \( [0, T] \) \( \Rightarrow u = \theta \) everywhere on \( [0, T] \)). Hence, it remains to check precisely that
\[
(3.2) \quad \int_0^T \langle \phi^*(t) + (A^* \phi^*)(t), u(t) \rangle \, dt = 0 \quad \forall \phi^* \in K_\alpha[0, T].
\]
Remember that our hypothesis here is slightly different: we assume in fact that it is
\[
(3.3) \quad \int_0^T \langle \phi^*(t) + (A^* \phi^*)(t), u(t) \rangle \, dt = 0
\]
for test-functions regular as those in \( K_\alpha[0, T] \) but null near 0 as well as near \( T \), which forms a subclass of \( K_\alpha[0, T] \) (denoted usually as \( K_\alpha'(0, T) \)). We added however the condition \( u(0) = \theta \). So, it remains to prove that (3.2) holds.

Take henceforth an arbitrary \( \phi^*(t) \in K_\alpha[0, T] \). Then consider, for any \( \varepsilon > 0 \), a scalar-valued function \( \nu_\varepsilon(t) \in C'[0, T] \), which = 0 for \( 0 < t < \varepsilon \), and = 1 for \( 2\varepsilon < t < T \), satisfying also an estimate \( |\dot{\nu}_\varepsilon(t)| < c/\varepsilon \), \( 0 < t < T \).

Then the product \( \nu_\varepsilon(t) \phi^*(t) \) is also = \( \theta \) near \( t = 0 \), so it is in the subclass of admissible here test-functions. We get from (3.3) the following equality
\[
(3.4) \quad \int_0^T \langle \dot{\nu}_\varepsilon \phi^* + \nu_\varepsilon \dot{\phi}^* + \nu_\varepsilon A^* \phi^*, u \rangle \, dt = 0, \quad \forall \varepsilon > 0, \ \phi^* \in K_\alpha[0, T].
\]
Obviously (3.4) reduces to the following
\[
\int_\varepsilon^{2\varepsilon} \langle \dot{\nu}_\varepsilon, u \rangle \, dt + \int_\varepsilon^{2\varepsilon} \langle \nu_\varepsilon \dot{\phi}, u \rangle \, dt + \int_{2\varepsilon}^T \langle \dot{\phi}, u \rangle \, dt + \int_\varepsilon^{2\varepsilon} \langle \nu_\varepsilon A^* \phi^*, u \rangle \, dt + \int_{2\varepsilon}^T \langle A^* \phi, u \rangle \, dt = 0, \quad \forall \varepsilon > 0, \ \forall \phi^* \in K_\alpha[0, T].
\]
Now, for $\varepsilon \to 0$, the first integral is estimated as

$$\lim_{\varepsilon \to 0} \frac{\int_{\varepsilon}^{2\varepsilon} \langle \phi^*, u \rangle \, dt}{\varepsilon} \leq \sup_{\varepsilon \leq t \leq 2\varepsilon} |\langle \phi^*, u \rangle| \cdot \varepsilon;$$

as $u(0) = 0$, $u(t) \to 0$ when $t \to 0$, hence $\sup_{\varepsilon \leq t \leq 2\varepsilon} |\langle \phi^*, u \rangle| \to 0$ with $\varepsilon$. The other integrals containing $\varepsilon$ are easily handled so that we obtain

$$\int_{0}^{T} \langle \phi^*, u \rangle \, dt + \int_{0}^{T} \langle A^* \phi^*, u \rangle \, dt = 0, \quad \forall \phi^* \in K_A^*[0, T)$$

which finishes our proof.

REMARK. The original proof of [2] was given using the adjoint semi group theory in reflexive spaces in a very natural way. We shall see later on a similar proof for the non-reflexive case (§5).

§ 4. — We shall deal here with the following unicity result for weakened solutions (see [3], Theorem 3.1, p. 81):

«Let be $A$ a linear operator in the $B$-space $X$, such that $R(A) = \{ \lambda - A \}^{-1} \in L(X, X)$ for $\lambda$ real $> \lambda_0$, and

$$\lim_{\lambda \to +\infty} \frac{\ln \| R(\lambda) \|}{\lambda} = h_A < \infty.$$"

Let $u(t)$ be a weakened solution of $u' - Au = 0$ on the interval $0 < t < T$, such that $u(0) = 0$, and assume also that $h_A < T$. Then $u(t) = 0$ in $0 < t < T - h_A$.

A slight generalization is possible, replacing $[0, T]$ by an arbitrary real interval $[a, b]$.

THEOREM 4.1. Under the same hypothesis on $A$, and if $h_A < b - a$, any weakened solution $u(t)$ of $u' - Au = 0$ on $a < t < b$, such that $u(a) = 0$, is $\theta$ on $a < t < b - h_A$.

We can in fact take $T = b - a$ in the above theorem; so if $u(0) = 0$, we get $u(t) = \theta$ on $[0, b - a - h_A]$.
To prove theorem 4.1, let us put \( u(t + a) = u_a(t) \); it maps the interval \( 0 < t < b - a \) into \( \mathcal{X} \). Also it is \( \dot{u}_a(t) = u'(t + a) = A u(t + a) = A u_a(t) \) for \( 0 < t < b - a \).

Hence \( u_a(t) \) is weakened solution on \( 0 < t < b - a \), and \( u_a(0) = \sigma = u(a) = 0 \); so, \( u_a(t) = 0 \) on \( 0 < t < b - a - h_a \) that is \( u(t + a) = 0 \) for \( 0 < t < b - a - h_a \), hence for \( a < t + a < b - h_a \), which gives \( u(t) = 0 \) for \( a < t < b - h_a \).

Now we shall see a partial extension of Theorem 4.1 in general \( B \)-spaces, taking weak solutions instead of weakened. Precisely, we propose ourselves to prove the following

**Theorem 4.2.** Let \( A \) be a linear operator in the \( B \)-space \( \mathcal{X} \), such that \( (\lambda - A)^{-1} \in \mathcal{L}(\mathcal{X}, \mathcal{X}) \) for \( \lambda \) real > \( \lambda_0 \) and assume also that

\[
\lim_{\lambda \to \infty} \frac{\ln \| R(\lambda; A) \|}{\lambda} = h_4 < \infty.
\]

Let also be \( \mathcal{D}(A^*) \) a dense subset of \( \mathcal{X}^* \), and \( \mathcal{D}(A) \) be dense in \( \mathcal{X} \) (*). Assume finally that

\[
\int_a^b \langle \phi^* + A^* \phi^*, u \rangle \, dt = 0
\]

\( \forall \phi^* \in K^*([a, b]), \) where \( u \in L^p_{\text{loc}}([a, b]; \mathcal{X}) \). Then, \( u = 0 \) a.e. on \( a < t < b - h_4 \), provided \( h_4 < b - a \).

Let us start the proof by remembering Phillips's fundamental results (see [4], [8]) concerning resolvents of dual operators.

Let \( T \) be linear closed operator with dense domain \( \mathcal{D}(T) \subset \mathcal{X} \), and \( T^* \) be its dual operator (acting on a total set in \( \mathcal{X}^* \), \( \mathcal{D}(T^*) \)). Then the resolvent sets \( \varrho(T) \) and \( \varrho(T^*) \) coincide; also, for any \( \lambda \in \varrho(T) \), it is \( (R(\lambda; T^*))^* = R(\lambda; T) \).

Apply this result to our operator \( A \) which is linear closed in \( \mathcal{X} \), because we assume that \( R(\lambda; A) \) exists \( \in \mathcal{L}(\mathcal{X}, \mathcal{X}) \) for \( \lambda \geq \lambda_0 \), \( \lambda \) real, and \( \mathcal{D}(A) \) is dense by hypothesis. We obtain that for \( \lambda \) real > \( \lambda_0 \),

(*) The existence of \( (\lambda - A)^{-1} \in \mathcal{L}(\mathcal{X}, \mathcal{X}) \) does not implies in general, that \( \mathcal{D}(\lambda - A) = \mathcal{D}(A) \) is dense in \( \mathcal{X} \).

It suffices to consider \( \mathcal{X} = C[0, 1]; \) \( A = \frac{d^2}{dx^2} \) defined on functions in \( C^2[0, 1] \) which vanish for \( x = 0 \) and \( x = 1 \). Considering the equation \( u'' = f \), \( \forall f \in C[0, 1] \), we find a unique solution \( u \in \mathcal{D}(A) \), depending continuously on \( f \). However, \( \mathcal{D}(A) \) is not dense in \( \mathcal{X} \).
$R(\lambda, A^*)$ also $\in \mathcal{L}(\mathcal{X}^*, \mathcal{X}^*)$, and $R(\lambda; A^*) = \left[ R(\lambda; A) \right]^*$. We know also that $\| [R(\lambda; A)]^* \| = \| R(\lambda; A) \|$ hence $\| R(\lambda; A^*) \| = \| R(\lambda; A) \|$ and consequently

$$\lim_{\lambda \to \infty} \frac{\ln \| R(\lambda; A^*) \|}{\lambda} = h_A \text{ too }.$$

Now, $\mathcal{D}(A^*)$ is also dense in $\mathcal{X}^*$, and $A^*$ is closed. It follows that $R(\lambda; A^{**}) \in \mathcal{L}(\mathcal{X}^{**}, \mathcal{X}^{**})$, $\forall \lambda$ real $\geq \lambda_0$, and for these $\lambda$, $\| R(\lambda; A^{**}) \| = \| R(\lambda; A) \|$ so,

$$\lim_{\lambda \to \infty} \frac{\ln \| R(\lambda; A^{**}) \|}{\lambda} = h_A < \infty \text{ too }.$$

Now we shall apply theorem 2.1 on the interval $a \leq t \leq b - h_A$. Let us consider consequently a function $u(t) \in C^1[a, b - h_A; \mathcal{X}]$, such that $Ju \in \mathcal{D}(A^{**})$, $a \leq t \leq b - h_A$, $(d/dt)(Ju) - A^{**}(Ju) = 0$ on $a \leq t \leq b - h_A$, and $(Ju)(a) = \theta$.

Let us apply now theorem 4.1 taking $A^{**}$ instead of $A$ which is possible by the above (remarking also that here the solutions are strong which is better than weakened). It follows that $Ju(t) = \theta$ on $a \leq t \leq b - h_A$. Hence theorem 2.1 is applicable on $[a, b - h_A]$ and we get uniqueness of weak solutions, as desired.

§ 5. — In this section we present a variant of the unicity result considered in § 3, which is valid in more general, non-reflexive $B$-spaces.

Let us start by remembering Phillips's theorem on dual semi-groups (see [4], [5], [8]).

Consider in the $B$-space $\mathcal{X}$, a linear closed operator $A$ with domain $\mathcal{D}(A)$ dense in $\mathcal{X}$, and assume that $A$ generates a semi-group of class $\mathcal{C}_0$ of linear continuous operators $T(t)$, $0 \leq t < \infty \to \mathcal{L}(\mathcal{X}, \mathcal{X})$.

Now, as previously, the dual operator $A^*$ of $A$ is a closed linear transformation on $\mathcal{D}(A^*) \subset \mathcal{X}^*$ to $\mathcal{X}^*$. We know that $\mathcal{D}(A^*)$ is a total set in $\mathcal{X}^*$, but in general $\mathcal{D}(A^*)$ is not dense in $\mathcal{X}^*$ so that $A^*$ is not necessarily the infinitesimal generator of a strongly continuous semi-group in $\mathcal{X}^*$.

Therefore it is convenient to consider the so called $\mathfrak{C}$-dual space $\mathcal{X}^\mathfrak{C}$ of $\mathcal{X}$, defined by $\mathcal{X}^\mathfrak{C} = \overline{\mathcal{D}(A^*)}$ (closure in $\mathcal{X}^*$). In the case of reflexive $\mathcal{X}$, we have $\mathcal{X}^\mathfrak{C} = \mathcal{X}^*$, else $\mathcal{X}^\mathfrak{C}$ may be a proper subset of $\mathcal{X}^*$.

Let us define now the operator $A^\mathfrak{C}$ to be the restriction of the
dual operator $A^*$ to the domain

\begin{equation}
\mathcal{D}(A^o) = \{x^* \in \mathcal{X}^*, x^* \in \mathcal{D}(A^*)\} \text{ such that } A^* x^* \in \mathcal{X}^o. \tag{5.1}
\end{equation}

Furthermore, let $T^*(t)$ be, for any $t > 0$, the dual operator of $T(t)$, and then $T^o(t)$ be the restriction of $T^*(t)$ to $\mathcal{X}^o$; then $T^o(t) \in \mathcal{L}(\mathcal{X}^o, \mathcal{X}^o)$, $t > 0$, and it is a semi-group of class $(C_0)$ having $A^o$ as infinitesimal generator.

Our aim is to prove the following

**Theorem 5.1.** Let $u(t)$ be a continuous function, $0 < t < T$ to $\mathcal{X}$, such that $u(0) = 0$, and satisfying relation

\begin{equation}
\int_0^T \langle \phi^o + A^o \phi^o, u(t) \rangle \, dt = 0 \tag{5.1}
\end{equation}

for any function $\phi^o(t)$, $0 < t < T \rightarrow \mathcal{D}(A^o)$, $\phi^o \in C^t[0, T; \mathcal{X}^o]$, $A^o \phi^o \in \mathcal{D}(A^o)$, $\phi^o(t) \in \mathcal{D}(A^o)$, $(A^o \phi^o)(t) \in C[0, T; \mathcal{X}^o]$, $\phi^o = 0$ near $0$ and near $T$. Then $u(t) = 0$ on $[0, T]$.

**Remark.** Before giving the proof, let us consider the particular case of reflexive space $\mathcal{X}$. Then $\mathcal{X} = \mathcal{X}^*$, $A^o = A^*$, so we find again the previously proved theorem in § 3.

**Proof of the theorem.** We have firstly

**Lemma 5.1.** The relation

\begin{equation}
\int_0^T \langle \phi^o + A^o \phi^o, u \rangle \, dt = 0 \tag{5.2}
\end{equation}

is verified for the more general class of test-function: $\phi^o(t) \in C^t[0, T; \mathcal{X}^o]$, $\phi^o(t) \in \mathcal{D}(A^o)$, $(A^o \phi^o)(t) \in C[0, T; \mathcal{X}^o]$, $\phi^o(T) = 0$.

Let us consider, $\forall \varepsilon > 0$, a scalar-valued function $v_\varepsilon(t)$, continuously differentiable on $0 < t < T$, $= 0$ for $0 < t < \varepsilon$, $T - \varepsilon < t < T$, $= 1$ for $2\varepsilon < t < T - 2\varepsilon$, such that $|v_\varepsilon'(t)| < c/\varepsilon$, $0 < t < T$, $|v_\varepsilon(t)| \leq 1$, $0 < t < T$; then $v_\varepsilon(t) \phi^o(t)$ is a test-function as required in theorem 5.1, because it vanishes near $t = 0$ and near $t = T$. We can write henceforth the relation (5.2) for $v_\varepsilon \phi^o$, and obtain the following:

\begin{equation}
\int_0^T \langle v_\varepsilon \phi^o + v_\varepsilon \phi^o, u \rangle \, dt = -\int_0^T v_\varepsilon \langle A^o \phi^o, u \rangle \, dt. \tag{5.2}
\end{equation}
The right-hand integral splits as
\[ -\int_{2s}^{T-2s} \langle A^\circ \phi^\circ, u \rangle \, dt - \int_s^{2s} \langle A^\circ \phi^\circ, u \rangle \, dt - \int_{T-2s}^{T-\varepsilon} \langle A^\circ \phi^\circ, u \rangle \, dt \]
and is readily seen that
\[ \lim_{\varepsilon \to 0} \int_0^T \langle A^\circ \phi^\circ, u \rangle \, dt = -\int_0^T \langle A^\circ \phi^\circ, u \rangle \, dt . \]

The left-hand side integral equals
\[ \int_0^T \dot{v}_e \langle \phi^\circ, u \rangle \, dt + \int_0^T v_e \langle \phi^\circ, u \rangle \, dt = I_1 + I_2 . \]

Actually it results
\[ I_1 = \int_{2s}^{T-\varepsilon} \langle \dot{v}_e \phi^\circ, u \rangle \, dt + \int_{T-2s}^{T} \langle v_e \phi^\circ, u \rangle \, dt = I_3 + I_4 . \]

Now, \( \lim_{\varepsilon \to 0} I_3 = 0 \), essentially because \( |\dot{v}_e| < c/\varepsilon \), and \( u(0) = \theta \). Also \( \lim_{\varepsilon \to 0} I_4 = 0 \), essentially because \( |\dot{v}_e| < c/\varepsilon \), and \( \phi^\circ(T) = \theta \). As for \( I_2 \), it is obviously seen to converge to \( \int_0^T \langle \phi^\circ, u \rangle \, dt \), as \( \varepsilon \to 0 \). Hence, altogether, for \( \varepsilon \to 0 \) we get
\[ \int_0^T \langle \phi^\circ, u \rangle \, dt + \int_0^T \langle A^\circ \phi^\circ, u \rangle \, dt = 0 , \]

and the Lemma is proved.

We can continue now the proof of our theorem.

Let us take an arbitrarily given function \( k^\circ(t) \in C^1[0, T; \mathfrak{I}^\circ] \). Then consider in the \( \bigcirc \)-dual space \( \mathfrak{I}^\circ \), the strong inhomogeneous Cauchy
Due to the fact that $A^\circ$ is the generator of a (Co)-semigroup $T^\circ(t)$ in $\mathcal{X}_\circ$, by a well-known result of Phillips ([7], Theorem 2.2.3), the problem (5.3) has a unique solution (given by the formula $\psi^\circ(t) = -\int_0^t T^\circ(t-\sigma) \cdot k^\circ(\sigma) \, d\sigma$, but this is not important here).

Consider now the function $\phi^\circ(t)$, defined for $0 < t < T$ through the relation $\phi^\circ(t) = \psi^\circ(T-t)$.

It is continuously differentiable in $\mathcal{X}_\circ$ on $0 < t < T$; it belongs to $\mathcal{D}(A^\circ)$, $\forall t \in [0, T]$, and $(A^\circ \phi^\circ)(t) = (A^\circ \psi^\circ)(T-t)$ is continuous, $0 < t < T \to \mathcal{X}_\circ$. Finally, $\phi^\circ(T) = \psi^\circ(0) = \theta$. Hence, $\phi^\circ(T)$ is an admissible test-function, and the relation $\int_0^T \langle \phi^\circ + A^\circ \phi^\circ, u \rangle \, dt = 0$ is verified.

Furthermore, $d\phi^\circ/dt = -\psi^\circ(T-t)$ and consequently we get:

$$\phi^\circ(t) + A^\circ \phi^\circ(t) = -\psi^\circ(T-t) + A^\circ \psi^\circ(T-t) = k^\circ(T-t),$$

in view of (5.3). Hence, we obtained the identity

$$\int_0^T \langle k^\circ(T-t), u(t) \rangle \, dt = 0,$$

for any $k^\circ \in C^1[0, T; \mathcal{X}_\circ]$, or, obviously, as $t \to T-t$ maps $C^1[0, T; \mathcal{X}_\circ]$ onto itself,

$$\int_0^T \langle h^\circ(t), u(t) \rangle \, dt = 0 \quad \forall h^\circ \in C^1[0, T; \mathcal{X}_\circ].$$

Take in particular $h^\circ(t) = v(t)x^*$, where $x^* \in \mathcal{X}_\circ$. Then

$$\int_0^T v(t) \langle x^*, u(t) \rangle \, dt = 0,$$

if $v(t) \in C^1[0, T]$.

As $\langle x^*, u \rangle$ is scalar-continuous on $[0, T]$, we obtain $\langle x^*, u(t) \rangle = 0,$
\(\forall t \in [0, T].\) But we can let \(x^*\) to vary in the total set \(\mathcal{D}(A^*) \subset X^0.\)

It follows that \(u(t) = \theta, \forall t \in [0, T].\)

This ends the proof of our theorem.

A simple corollary is the following.

**Theorem 5.2.** Let \(u(t) \in C([0, T]; X),\) such that \(u(0) = \theta\) and assume that

\[
\int_0^T \langle \phi^* + A^* \phi^*, u \rangle \, dt = 0,
\]

for any function \(\phi^*(t), 0 < t < T \rightarrow \mathcal{D}(A^*),\) belonging to \(C^1([0, T]; X^*),\) such that \(A^* \phi^* \in C([0, T]; X^*)\) and \(\phi^* = \theta\) near 0 and near \(T.\) Then \(u(t) = \theta\) on \([0, T].\)

In fact it suffices to remark that the class of test-functions considered here contains as a subset the class considered in the theorem 5.1, because \(A^0\) is a certain restriction of \(A^*\) to an (eventually) smaller domain. Hence, the relation (5.2) is verified and theorem 5.2 implies \(u = \theta\) on \([0, T].\)

We have also the following.

**Theorem 5.3.** Let \(A\) be the generator of a \((C_0)\) semi-group \(T(t)\) in the \(B\)-space \(X,\) and \(A^*; \mathcal{D}(A^*) \subset X^* \rightarrow X^*\) be its dual operator, defined on the total set \(\mathcal{D}(A^*).\)

Let \(u(t)\) a continuous function, \(0 < t < T \rightarrow X,\) such that \(u(0) = u_0\) given arbitrarily in \(X,\) and satisfying the relation

\[
\int_0^T \langle \phi^* + A^* \phi^*, u \rangle \, dt = 0, \quad \forall \phi^*(t) \in K_{A^*}(0, T) \quad (1).
\]

Then \(u(t)\) has the representation \(u(t) = T(t)u_0, 0 < t < T.\)

Let us consider in fact the strongly continuous function \(v(t), 0 < t < T \rightarrow X,\) given by \(v(t) = T(t)u_0.\) Then (5.5) is valid also for this function \(v.\)

In fact, let \((u_n)_T \subset \mathcal{D}(A)\) be a sequence convergent to \(u_0.\) Let also

\[(1)\] This is the class of test-functions considered in Theorem 5.2.
\( v_n(t) = T(t)u_n \), so that, as well-known, it is \( \dot{v}_n = Av_n \), \( 0 \leq t \leq T \). Now

\[
\int_0^T \langle \phi^*, v_n \rangle \, dt = -\int_0^T \langle \phi^*, \dot{v}_n \rangle \, dt,
\]
as obviously seen. Furthermore is \( \langle A^*\phi^*, v_n \rangle = \langle \phi^*, Av_n \rangle \), \( \forall t \in [0, T] \). It follows

\[
\int_0^T \langle \phi^* + A^*\phi^*, v_n \rangle \, dt = -\int_0^T \langle \phi^*, \dot{v}_n \rangle \, dt + \int_0^T \langle \phi^*, Av_n \rangle \, dt = 0.
\]

when \( n \to \infty, v_n(t) \to v(t) \) uniformly on \( [0, T] \), as \( \sup_{0 \leq t \leq T} \| T(t) \| = C_T < \infty \) so, it results:

\[
\int_0^T \langle \phi^* + A^*\phi^*, v \rangle \, dt = 0
\]
too. If we take now \( w(t) = u(t) - v(t) \), then (5.5) is verified for \( w(t) \), and \( w(0) = \theta \). By previous theorem, it follows \( u(t) = v(t) = T(t)u_0 \) on \( 0 \leq t \leq T \).

REFERENCES


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