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The weak Cauchy problem for abstract differential equations

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The Weak Cauchy Problem
for Abstract Differential Equations.

S. Zaidman (*)

Introduction.

We consider the weak Cauchy problem in arbitrary Banach space for equations \((d/dt - A)u = 0\), as were defined by Kato-Tanabe. After proving some elementary relationships, we obtain a result which shows how uniqueness of Cauchy problem for strong solutions in the second dual space implies uniqueness of the weak Cauchy problem.

A simple result by Barbu-Zaidman (Notices A.M.S., April 1973, 73T-B120) gets then a new proof, and an uniqueness result for weakened solutions by Liubic-Krein gets a partial extension.

The last result is a certain extension of Barbu-Zaidman result to non-reflexive \(B\)-spaces, using as a main tool in the proof Phillips's theorem on dual semi-groups in \(B\)-spaces.

§ 1. – Let \(\mathcal{X}\) be a given Banach space, and \(\mathcal{X}^*\) be its dual space. If \(A\) is a linear closed operator with dense domain \(\mathcal{D}(A) \subset \mathcal{X}\), mapping \(\mathcal{D}(A)\) into \(\mathcal{X}\), then, the dual operator \(A^*\) is defined on the set \(\mathcal{D}(A^*) = \{x^* \in \mathcal{X}^*, \text{ s.t. } \exists y^* \in \mathcal{X}^*, \text{ satisfying} \}

\[
(1.1) \quad \langle x^*, Ax \rangle = \langle y^*, x \rangle \quad \forall x \in \mathcal{D}(A) \}
\]

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By definition $A^* x^* = y^*$, and $A^*$ is a well-defined linear operator from $\mathcal{D}(A^*)$ into $\mathcal{X}^*$.

Furthermore, the domain $\mathcal{D}(A^*)$ is a total set in $\mathcal{X}^*$; this means that, given any element $x \in \mathcal{X}^*$, $x \neq \theta$, $\exists x^* \in \mathcal{D}(A^*)$, such that $x^*(x) \neq 0$; hence, if $x^*(x) = 0$, $\forall x^* \in \mathcal{D}(A^*)$, then $x = \theta$.

Let now be given a finite interval $-\infty < a < b < +\infty$ on the real axis; a class of « test-functions » associated to the operator $\mathcal{D}(A^*)$ and to the given interval, denoted here by $K_{A^*}[a, b]$, consists of continuously differentiable functions $\phi^*(t)$, $a < t < b \rightarrow \mathcal{X}^*$, which are $= 0$ near $b$ (that is $\phi^*(t) = \phi$ for $b - \delta < t < b$, where $\delta$ depends on $\phi^*$); furthermore, $\phi^*(t)$ belongs to $\mathcal{D}(A^*)$, $\forall t \in [a, b]$, and $(A^*\phi^*)(t)$ is $\mathcal{X}^*$-continuous on $[a, b]$.

Obviously, if $\varphi(t)$ is scalar-valued, $C^1[a, b]$-function, and $\varphi = 0$ near $b$, and if $\phi^*$ is any element in $\mathcal{D}(A^*)$, then $\varphi(t)\phi^*$ belongs to $K_{A^*}[a, b]$.

Let us consider now the Bochner space $L^p_{\text{loc}}([a, b]; \mathcal{X})$, where $p$ is any real $\geq 1$, consisting of strongly measurable $\mathcal{X}$-valued functions $f$ defined on $[a, b]$, such that $\int_a^b \|f(t)\|_{\mathcal{X}}^p \, dt < \infty$ for any $c < b$.

The weak forward Cauchy problem is here defined as follows: given any element $u_a \in \mathcal{X}$ and any function $f(t) \in L^p_{\text{loc}}([a, b]; \mathcal{X})$, find a function $u(t) \in L^p_{\text{loc}}([a, b]; \mathcal{X})$ verifying

$$
(1.2) \quad -\langle \varphi^*(a), u_a \rangle - \int_a^b \left( \frac{d\varphi^*}{dt}, u(t) \right) \, dt = \int_a^b \langle (A^*\varphi^*)(t), u(t) \rangle \, dt + \\
+ \int_a^b \langle \varphi^*(t), f(t) \rangle \, dt, \quad \forall \varphi^* \in K_{A^*}[a, b].
$$

In similar way define a weak backward Cauchy problem: The class $K_{A^*}(a, b]$ is defined like $K_{A^*}[a, b)$, with the only difference that the test-functions must be null near $a$, instead of being null near $b$.

There is also the space $L^p_{\text{loc}}((a, b]; \mathcal{X})$ of $\mathcal{X}$-mesurable functions such that $\int_c^b \|f\|_{\mathcal{X}}^p \, dt < \infty$, $\forall c > a$, $c < b$.

Then given any element $u_b \in \mathcal{X}$, and again $f \in L^p_{\text{loc}}((a, b]; \mathcal{X})$, find
u(t) \in L^p_{\text{loc}}((a, b]; \mathcal{X})$, satisfying

\begin{equation}
(1.3) \quad - \langle \varphi^*(b), u_0 \rangle - \int_a^b \langle \frac{d\varphi^*}{dt}, u(t) \rangle \, dt = \int_a^b \langle (A^* \varphi^*)(t), u(t) \rangle \, dt + \int_a^b \langle \varphi^*(t), f(t) \rangle \, dt, \quad \forall \varphi^* \in K_{A^*}(a, b).
\end{equation}

REMARK. This definitions are slightly more general than the weakened Cauchy problem as defined for example in S. G. Krein [3]; extending his definition from the interval \([0, T]\) to an arbitrary interval \([a, b]\), we say that \(u(t), a < t < b \rightarrow \mathcal{X}\) is a weakened solution of

\begin{equation}
(1.4) \quad \dot{u}(t) = Au(t) + f(t), \quad u(a) = u_0 \in \mathcal{X},
\end{equation}

where \(f(t), a < t < b \rightarrow \mathcal{X}\) is \(\mathcal{X}\)-continuous, if: \(u(t)\) is \(\mathcal{X}\)-continuous on the closed interval \([a, b]\); \(u(t)\) is \(\mathcal{X}\)-differentiable with continuous derivative on the half-open interval \((a, b]\); \(u(t) \in \mathcal{D}(A)\) on same \((a, b]\);

\[ u'(t) = Au(t) + f(t), \quad a < t < b, \quad u(a) = u_0. \]

The following result holds:

PROPOSITION 1.1. If \(u(t)\) is a weakened solution of (1.4), then (1.2) is also verified.

Consider the equality \(u'(t) - Au(t) = f(t)\), valid on the half-open interval \(a < t < b\). Then take any test-function \(\phi^*(t) \in K_{A^*}[a, b] \) (\(\phi^*\) is null near \(b\)). We get obviously

\begin{equation}
(1.5) \quad \langle \dot{\phi}^*(t), u'(t) \rangle - \langle \dot{\phi}^*(t), Au(t) \rangle = \langle \phi^*(t), f(t) \rangle, \quad a < t < b.
\end{equation}

Also we see that

\begin{equation}
(1.6) \quad \frac{d}{dt} \langle \phi^*(t), u(t) \rangle = \left\langle \frac{d\phi^*}{dt}(t), u(t) \right\rangle + \left\langle \phi^*(t), \frac{du}{dt}(t) \right\rangle, \quad a < t < b.
\end{equation}
If we integrate (1.6) between \(a + \varepsilon \) and \(b, \forall \varepsilon > 0\), we obtain

\[
- \langle \phi^*(a + \varepsilon), u(a + \varepsilon) \rangle = \int_{a+\varepsilon}^{b} \{ \langle \phi^*(t), u(t) \rangle + \langle \phi^*(t), \dot{u}(t) \rangle \} \, dt.
\]

Let us integrate now (1.5) between \(a + \varepsilon \) and \(b\), and remark also that \(\langle \phi^*(t), Au(t) \rangle = \langle A\phi^*(t), u(t) \rangle, a < t < b\). We get

\[
\int_{a+\varepsilon}^{b} \langle \phi^*(t), u'(t) \rangle \, dt = - \langle \phi^*(a + \varepsilon), u(a + \varepsilon) \rangle - \int_{a+\varepsilon}^{b} \langle \phi^*(t), u(t) \rangle \, dt = \int_{a+\varepsilon}^{b} \langle A\phi^*(t), u(t) \rangle \, dt + \int_{a+\varepsilon}^{b} \langle \phi^*(t), f(t) \rangle \, dt.
\]

By continuity of all functions here involved, one obtains, when \(\varepsilon \to 0\)

\[
- \langle \phi^*(a), u(a) \rangle - \int_{a}^{b} \langle \phi^*(t), u(t) \rangle \, dt - \int_{a}^{b} \langle A\phi^*(t), u(t) \rangle \, dt = \int_{a}^{b} \langle \phi^*(t), f(t) \rangle \, dt.
\]

A converse result is also given in the following.

**Proposition 1.2.** Let us assume: \(f(t), a < t < b \to \mathbb{X}\), be strongly continuous; \(u_a \in \mathbb{X}\) be arbitrarily given. Then \(u(t), a < t < b \to \mathbb{X}\) be a \(\mathbb{X}\)-continuous function, which is continuously differentiable for \(a < t < b\), and belongs to \(D(A)\) for \(t \in (a, b)\). Let also (1.2) be satisfied. Then it follows that \(u' - Au = f\) on \(a < t < b\), and also \(u(a) = u_a\).

In order to prove this simple fact, we shall first introduce in (1.2) test-functions of the special form \(\phi^*(t) = \nu(t)x^*\) where \(x^* \in D(A^*)\) and \(\nu(t)\) is scalar-valued continuously differentiable function which \(= 0\) near \(a\) and near \(b\). It results then, if \([a_1, b_1] \subset (a, b)\) contains \(\text{supp} \phi^*\)

\[
(1.7) - \int_{a_1}^{b_1} \langle \phi^*(t), u(t) \rangle \, dt = \int_{a_1}^{b_1} \langle A\phi^*(t), u(t) \rangle \, dt + \int_{a_1}^{b_1} \langle \phi^*(t), f(t) \rangle \, dt.
\]

As \(u(t)\) is continuously differentiable on \([a_1, b_1]\), and \(\phi^*(a_1) = \phi^*(b_1) = 0\) (intervals \((a, a_1), (b_1, b)\) are in the null set of \(\phi^*\): hence, by continuity,
\( \phi^*(a_1) = \phi^*(b_1) = \theta \) also, it results

\[
- \int_{a_1}^{b_1} \langle \phi^*(t), u(t) \rangle \, dt = \int_{a_1}^{b_1} \langle \phi^*(t), \dot{u}(t) \rangle \, dt.
\]

Also because \( u(t) \in \mathcal{D}(A) \) for \( t \in [a_1, b_1] \), it is \( \langle \phi^*(t), Au(t) \rangle = \langle A^* \phi^*(t), u(t) \rangle \). Hence, relation (1.7) becomes

\[
(1.8) \quad \int_{a_1}^{b_1} \langle \phi^*(t), \dot{u}(t) \rangle \, dt = \int_{a_1}^{b_1} \langle \phi^*(t), (Au)(t) \rangle \, dt + \int_{a_1}^{b_1} \langle \phi^*(t), f(t) \rangle \, dt
\]

or, as \( \phi^*(t) \) is here = \( v(t)x^* \),

\[
(1.9) \quad \int_{a_1}^{b_1} \langle x^*, u'(t)v(t) \rangle \, dt = \int_{a_1}^{b_1} \langle x^*, (Au)(t)v(t) \rangle \, dt + \int_{a_1}^{b_1} \langle x^*, f(t)v(t) \rangle \, dt
\]

or

\[
(1.10) \quad \int_{a_1}^{b_1} \langle x^*, u'(t) - Au(t) - f(t) \rangle v(t) \, dt = 0.
\]

By continuity of the scalar function \( \langle x^*, u'(t) - Au(t) - f(t) \rangle \) in \([a_1, b_1]\), letting \( v(t) \) to vary, we get \( \langle x^*, u'(t) - Au(t) - f(t) \rangle = 0 \) in \([a_1, b_1]\), \( \forall x^* \in \mathcal{D}(A^*) \).

\[
\text{If } \int_{a}^{b} \phi(t)v(t) \, dt = 0, \forall v \in C^1_0(\alpha, \beta), \phi \in C[\alpha, \beta] \Rightarrow \phi = 0 \text{ on } (\alpha, \beta) \; \text{if not, } \exists \xi \in (\alpha, \beta), \phi(\xi) > 0 \text{ say}; \text{ in } (\xi - \delta, \xi + \delta), \phi > 0; \text{ take } 0 < \nu, \nu = 1 \text{ on } (\xi - \delta/2, \xi + \delta/2), = 0 \text{ outside } (\xi - \delta, \xi + \delta), \in C^1; \text{ then}
\]

\[
\int_{a}^{b} \phi v \, dt = \int_{\xi - \delta}^{\xi + \delta/2} \phi(v) \, dt > \int_{\xi - \delta/2}^{\xi + \delta/2} \phi \, dt > 0, \quad \text{absurde}.
\]

If \( \phi = 0 \) on \( (\alpha, \beta) \), \( \Rightarrow \phi = 0 \) on \([\alpha, \beta]\).

Now, if we fix \( t \in [a_1, b_1] \), and vary \( x^* \) over the total set \( \mathcal{D}(A^*) \), we get \( u'(t) = Au(t) + f(t) \).
This is true for any \( t \in [a_1, b_1] \), hence for any \( t \in (a, b) \) too. But 
\( u'(t), f(t) \), hence \( Au(t) \) are continuous on \( t = b \); so we obtain \( u'(b) = Au(b) + f(b) \) also to be valid.

We still must prove that \( u(a) = u_a \).

Consider again the relation (1.2), for general test-functions \( \phi^*(t) \in K_\Lambda[a, b) \). Take an arbitrary small \( \varepsilon > 0 \), and get

\[
- \langle \phi^*(a), u_a \rangle - \int_a^{a+\varepsilon} \langle \phi^*(t), u(t) \rangle \, dt - \int_{a+\varepsilon}^{b} \langle \phi^*(t), u(t) \rangle \, dt =
\]

\[
= \int_a^{a+\varepsilon} \langle (A^*\phi^*)(t), u(t) \rangle \, dt + \int_{a+\varepsilon}^{b} \langle (A^*\phi^*)(t), u(t) \rangle \, dt + \int_{a}^{b} \langle \phi^*(t), f(t) \rangle \, dt
\]

we have also,

\[
\int_{a+\varepsilon}^{b} \langle \phi^*(t), u(t) \rangle \, dt = \int_{a+\varepsilon}^{b} \frac{d}{dt} \langle \phi^*(t), u(t) \rangle \, dt - \int_{a+\varepsilon}^{b} \langle \phi^*(t), \dot{u}(t) \rangle \, dt =
\]

\[
= - \langle \phi^*(a + \varepsilon), u(a + \varepsilon) \rangle - \int_{a+\varepsilon}^{b} \langle \phi^*(t), \dot{u}(t) \rangle \, dt,
\]

and

\[
\int_{a+\varepsilon}^{b} \langle (A^*\phi^*)(t), u(t) \rangle \, dt = \int_{a+\varepsilon}^{b} \langle \phi^*(t), Au(t) \rangle \, dt;
\]

so, we get

\[
- \langle \phi^*(a), u_a \rangle - \int_{a}^{a+\varepsilon} \langle \phi^*(t), u(t) \rangle \, dt + \langle \phi^*(a + \varepsilon), u(a + \varepsilon) \rangle +
\]

\[
+ \int_{a+\varepsilon}^{b} \langle \phi^*(t), \dot{u}(t) \rangle \, dt = \int_{a+\varepsilon}^{b} \langle \phi^*(t), Au(t) \rangle \, dt + \int_{a}^{b} \langle \phi^*(t), f(t) \rangle \, dt + \int_{a}^{b} \langle A^*\phi^*, u \rangle \, dt.
\]

But \( \dot{u}(t) = Au(t) + f(t) \) on \( a + \varepsilon \leq t < b \), as was proved above. Hence,
it remains
\[
- \langle \phi^*(a), u_a \rangle - \int_a^{a+\varepsilon} \langle \phi^*(t), u(t) \rangle \, dt + \langle \phi^*(a+\varepsilon), u(a+\varepsilon) \rangle =
\]
\[
= \int_a^{a+\varepsilon} \langle A^* \phi^*(t), u(t) \rangle \, dt + \int_a^{a+\varepsilon} \langle \phi^*(t), f(t) \rangle \, dt .
\]

If now let \( \varepsilon \to 0 \), it remains only, using continuity of \( u \) on \([a, b]\), that
\[
\langle \phi^*(a), u(a) - u_a \rangle = 0 , \quad \forall \phi^* \in K_{A^*}[a, b] .
\]

We can now take \( \phi^*(t) = \nu_0(t)x^* \), where \( \nu_0(t) \in C'[a, b] \), equals 1 near \( a \), and equals 0 near \( b \), and \( x^* \in \mathcal{D}(A^*) \). Hence
\[
\langle x^*, u(a) - u_a \rangle = 0 \quad \forall x^* \in \mathcal{D}(A^*)
\]
which is a total set in \( \mathcal{X}^* \), and again, it will be \( u(a) = u_a \). Q.E.D.

§ 2. – In this section we shall prove that uniqueness of Cauchy problem for strong solutions on an interval \([a, b]\) in the second dual space \( \mathcal{X}^{**} \), implies uniqueness of the weak Cauchy problem in the same interval, in the original space \( \mathcal{X} \).

If \( A \) is linear, closed operator with dense domain in the B-space \( \mathcal{X} \), we saw that the dual operator \( A^* \) is linear, defined on a total set in \( \mathcal{X}^* \); also \( A^* \) is closed on this set; in fact, let \( x_n^* \in \mathcal{D}(A^*) \), \( x_n^* \to x^* \in \mathcal{X}^* \), \( A^*x_n^* \to y_0^* \in \mathcal{X}^* \). From relations \( \langle x_n^*, Ax \rangle = \langle A^*x_n^*, x \rangle \), \( \forall x \in \mathcal{D}(A) \), we get, as \( n \to \infty \), \( \langle x_0^*, Ax \rangle = \langle y_0^*, x \rangle \), \( \forall x \in \mathcal{D}(A) \). Hence, by definition of \( A^* \), it is \( x_0^* \in \mathcal{D}(A^*) \), \( A^*x_0^* = y_0^* \), so \( A^* \) is closed.

Let us assume from now on the supplementary.

HYPOTHESIS. \( A^* \) is an operator with dense domain in \( \mathcal{X}^* \).

(REMARK. This holds always when \( \mathcal{X} \) is a reflexive B-space; the proof is similar to a classical one in Hilbert spaces).

Then, the second dual operator \( A^{**} = (A^*)^* \) will be a well defined operator on a total set \( \mathcal{D}(A^{**}) \subset \mathcal{X}^{**} \) the second dual space of \( \mathcal{X} \). More precisely \( \mathcal{D}(A^{**}) = \{ \psi^{**} \in \mathcal{X}^{**} \text{ such that } \exists x^{**} \in \mathcal{X}^{**}, \text{ satisfying relation } \langle \psi^{**}, A^*\phi^* \rangle = \langle x^{**}, \phi^* \rangle, \forall \phi^* \in \mathcal{D}(A^*) \} \) and if \( \psi^{**} \in \mathcal{D}(A^{**}) \),
A** y~ = x**. We also know the existence of a canonical map J: X → X**, which is linear and isometric; precisely, any element x ∈ X defines a linear continuous functional f** on X*, by: \langle f**, x* \rangle = \langle x*, x \rangle, \forall x* ∈ X*. Then put Jx = f**, so that \langle x*, x \rangle = \langle Jx, x* \rangle, \forall x* ∈ D(A*).

Let now u(t), a ≤ t ≤ b, be a C¹[a, b; X] function such that u(t) ∈ D(A), \forall t ∈ [a, b] and u'(t) = Au(t) on [a, b]. This is a strong solution on [a, b], and u(a) belongs necessarily to D(A). Then (Ju)(t) is a C¹[a, b; X**] function, as easily seen, and \( (d/dt)(Ju) = J(du/dt) \). We prove now following

**Theorem 2.1.** Let us assume that for any function u(t) ∈ C¹([a, b], X)] such that

i) \((Ju)(t) ∈ D(A**), a < t < b,

ii) \((d/dt)(Ju) - A**(Ju) = 0 \) on [a, b],

iii) \((Ju)(a) = 0, \)

it is \((Ju)(t) = 0, \forall t ∈ [a, b]. \) Then, there is unicity of the forward weak Cauchy problem on [a, b].

**Proof.** What we must prove is the following: \( v(t) \in L^p_{loc}([a, b]; X), \) and

\[
\int_a^b \langle \phi^*(t), v(t) \rangle \, dt = \int_a^b \langle (A\ast \phi^*)(t), v(t) \rangle \, dt, \quad \forall \phi^* ∈ K_{A}[a, b],
\]

implies \( v(t) = 0 \) almost everywhere on [a, b].

Now, using a suggestion by professor S. Agmon (in Pisa, Italy), we start by extending \( v(t) \) to \((−∞, b), \) as follows: \( \tilde{v}(t) = v(t) \) for \( a < t < b, \tilde{v}(t) = 0 \) for \( −∞ < t < a. \) It holds now the following

**Lemma 2.1.** The extended function \( \tilde{v}(t) \) verifies the integral identity

\[
\int_{-∞}^b \langle \psi^*(t) + (A\ast \psi^*)(t), \tilde{v}(t) \rangle \, dt = 0
\]

for any function \( \psi^*(t), −∞ < t < b → X*, \) continuously differentiable there, such that \( \psi^*(t) ∈ D(A*), \forall t ∈ (−∞, b], \) \( A\ast \psi^*(t) \) is \( X*-\)continuous; support \( \psi^* \) is compact in \((−∞, b) \) (i.e. \( \psi^* = 0 \) near \( b \) and near \( −∞). \)
In fact, (2.2) is the same as

\[(2.3) \quad \int_{a}^{b} \langle \psi^{*}(t) + (A^{*} \psi^{*})(t), \varphi(t) \rangle \, dt = 0. \]

But the restriction to \([a, b]\) of the above considered test function \(\psi^{*}(t)\) is obviously in the class \(K_{a}^{*}\), (because it was null near \(b\), and had all regularity required properties).

Hence, by (2.1), the lemma is proved.

A second, needed result (already announced in our paper [6]) is as follows:

Take any scalar function \(\alpha_{\epsilon}(t) \in C^{1}(-\infty, \infty)\), which = 0 for \(|t| > \epsilon\); for any \(w(t) \in L^{p}_{loc}(-\infty, b; \mathcal{X})\) (\(\mathcal{X}\)-mesurable on \((-\infty, b)\), such that \(\int_{a}^{b} \|w\|_{L^{p}\mathcal{X}} \cdot dt < \infty, \forall \alpha > -\infty, \beta > \alpha, \beta < b\)), we can consider the mollified function

\[(w * \alpha_{\epsilon})(t) = \int_{t-\epsilon}^{t+\epsilon} w(\tau) \alpha_{\epsilon}(t - \tau) \, d\tau\]

which is well-defined for \(-\infty < t < b - \epsilon\), is strongly continuously differentiable, and

\[\frac{d}{dt} (w * \alpha_{\epsilon}) = \int_{t-\epsilon}^{t+\epsilon} w(\tau) \alpha_{\epsilon}'(t - \tau) \, d\tau, \quad -\infty < t < b - \epsilon.\]

We have

**Lemma 2.2.** If \(w(t) \in L^{p}_{loc}(-\infty, b; \mathcal{X})\) verifies the integral identity:

\[(2.4) \quad \int_{-\infty}^{b} \langle \psi^{*}(t) + (A^{*} \psi^{*})(t), w(t) \rangle \, dt = 0\]

\(\forall \psi^{*} \text{ as in Lemma 2.1, then, it is } J(w * \alpha_{\epsilon}) \in \mathcal{D}(A^{**}), \text{ and } (d/dt)J(w * \alpha_{\epsilon}) = A^{**}(J(w * \alpha_{\epsilon})) \text{ holds, } \forall t \in (-\infty, b - \epsilon) \text{ where } J \text{ is the canonical map of } \mathcal{X} \text{ in } \mathcal{X}^{**}.\)
Take in fact any fixed \( t_0 \in (-\infty, b - \varepsilon) \), and consider then the functions \( \psi_{t_0,t}^*(t) = \alpha_\varepsilon(t_0 - t)x^* \), where \( x^* \in \mathcal{D}(A^*) \). These are good test functions because \( \alpha_\varepsilon(t_0 - t) = 0 \) for \( |t - t_0| \gg \varepsilon \), hence in any case, \( \alpha_\varepsilon(t_0 - t) = 0 \) near \( b \) and near \(-\infty\).

There is also \( (d/dt)\psi_{t_0,t}^* = -\alpha_\varepsilon(t_0 - t)x^* \). Writing now (2.4), we get

\[
\int_{-\infty}^{b} \langle \alpha_\varepsilon(t_0 - t)x^*, w(t) \rangle dt = \int_{-\infty}^{b} \alpha_\varepsilon(t_0 - t) \langle A^*x^*, w(t) \rangle dt
\]

or also

\[
\int_{-\infty}^{b} \langle A^*x^*, \int_{-\infty}^{t} \alpha_\varepsilon(t_0 - t) w(t) dt \rangle = \int_{-\infty}^{b} \alpha_\varepsilon(t_0 - t) \langle A^*x^*, w(t) \rangle dt
\]

or

\[
\langle A^*x^*, (w^\ast \alpha_\varepsilon)(t_0) \rangle = \langle x^*, (w^\ast \alpha_\varepsilon)'(t_0) \rangle, \quad \forall x^* \in \mathcal{D}(A^*),
\]

Here, if we introduce the canonical imbedding operator \( J \), we have:

\[
\langle J(w^\ast \alpha_\varepsilon)(t_0), A^*x^* \rangle = \langle J(w^\ast \alpha_\varepsilon)'(t_0), x^* \rangle, \quad \forall x^* \in \mathcal{D}(A^*).
\]

Now if we use definition of \( \mathcal{D}(A^{**}) \) and of \( A^{**} \), we see that \( J(w^\ast \alpha_\varepsilon) \cdot (t_0) \in \mathcal{D}(A^{**}) \), and

\[
A^{**}(J(w^\ast \alpha_\varepsilon)(t_0)) = J(w^\ast \alpha_\varepsilon)'(t_0) = \frac{d}{dt} J(w^\ast \alpha_\varepsilon)(t_0)
\]

which is the desired Lemma 2.2.

*We pass now to the final steps of the proof.*

Take \( w(t) = \tilde{v}(t) \) the function used in Lemma 2.1; as \( v(t) \in L^p_{loc} \cdot ([a, b); \mathcal{X}) \) and \( \tilde{v} = \theta \) for \( t < a \), it is obvious that

\[
\tilde{v}(t) = w(t) \in L^p_{loc}([-\infty, b); \mathcal{X}].
\]

\[
\cdot \left( \int_{a}^{b} \| w(t) \|^p dt = \int_{a}^{b} \| v(t) \|^p dt < \infty \text{ for } \beta < b, \alpha < a \right).
\]
Let us apply Lemma 2.2 to $\tilde{v}(t)$. We obtain that $(\tilde{v} \ast \alpha_\varepsilon)(t)$ is well-defined on $-\infty < t < b - \varepsilon$ where is continuously differentiable; also $J(\tilde{v} \ast \alpha_\varepsilon) \in \mathcal{D}(A^{**})$ and

$$
\frac{d}{dt} (J(\tilde{v} \ast \alpha_\varepsilon)) = A^{**}(J(\tilde{v} \ast \alpha_\varepsilon)) \quad \text{holds on} \quad -\infty < t < b - \varepsilon.
$$

Remark also that $(\tilde{v} \ast \alpha_\varepsilon)(t) = \theta$ for $t \leq a - \varepsilon$, because it is

$$
(\tilde{v} \ast \alpha_\varepsilon)(t) = \int_{t-\varepsilon}^{t+\varepsilon} \tilde{v}(\tau) \alpha_\varepsilon(t - \tau) d\tau
$$

and $\tilde{v}(\tau) = \theta$ for $a - \varepsilon \leq \tau \leq a$.

Hence, (2.5) holds on $a - \varepsilon \leq t < b - \varepsilon$, and also $(\tilde{v} \ast \alpha_\varepsilon)(a - \varepsilon) = 0$ so $J(\tilde{v} \ast \alpha_\varepsilon)(a - \varepsilon - \varepsilon) = \theta$.

Now, if $(\tilde{v} \ast \alpha_\varepsilon)(t) = Z(t)$, we see that, in the space $\mathcal{X}^{**}$, it is:

$$(JZ)'(t) = A^{**}JZ(t) \quad \text{on} \quad [a - \varepsilon, b - \varepsilon], \quad \text{and} \quad JZ(a - \varepsilon - \varepsilon) = \theta.$$ 

Put then $t = \sigma - \varepsilon$ and $Z(t) = Z(\sigma - \varepsilon) = u(\sigma)$; when $a - \varepsilon \leq t < b - \varepsilon$, we get $a - \varepsilon \leq \sigma - \varepsilon < b - \varepsilon$, or $a \leq \sigma < b$; also $Jw'(\sigma) = JZ'(t)$, so that $(Jw)'(\sigma) = A^{**}(Jw)(\sigma)$ in $\mathcal{X}^{**}$, $a \leq \sigma < b$, and $Jw(a) = (JZ)(a - \varepsilon - \varepsilon) = \theta$.

Applying the hypothesis of the theorem, it follows that $u(t) = \theta$ on $[a, b]$, hence, $Z(t) = \theta$ on $[a - \varepsilon, b - \varepsilon]$, that is

$$(\tilde{v} \ast \alpha_\varepsilon)(t) = \theta \quad \text{on} \quad [a - \varepsilon, b - \varepsilon].$$

Now, take a sequence of functions $\alpha_n(t)$ which are non-negative, $= 0$ for $|t| > 1/n$, continuously differentiable, such that

$$
\frac{1}{n} \int_{-1/n}^{1/n} \alpha_n(\sigma) d\sigma = 1.
$$

We obtain then, in the usual way, as for scalar-valued functions, the relation:

$$
\lim_{n \to \infty} \int_{a_1}^{b_1} \|v(t) - (\tilde{v} \ast \alpha_n)(t)\|^p \, dt = 0, \quad \forall b_1 < b, \; a_1 > a.
$$
§ 3. We shall give now some applications of Theorem 2.1. To start with, we give a proof of the following result (see [2]). Let \( X \) be a reflexive B-space; \( A \) be the infinitesimal generator of a strongly continuous semi-group of class \( C_0 \); \( A^* \) be the dual operator to \( A \). Let \( u(t) \), \( 0 < t < T \rightarrow X \), be a strongly continuous function, verifying the integral identity

\[
\int_0^T \langle \phi^*(t) + (A^* \phi^*)(t), u(t) \rangle \, dt = 0
\]

for any function \( \phi^*(t), 0 < t < T \rightarrow X^* \), which is continuously differentiable in \( X^* \), belongs to \( \mathcal{D}(A^*) \), \( \forall t \in [0, T] \), \( (A^* \phi^*) \) is \( X^* \)-continuous, \( 0 < t < T \), and \( \phi^*(t) \) is null near 0 and near \( T \). Let also be \( u(0) = 0 \); then \( u(t) = 0, 0 < t < T \).

Let us remark first that \( A \) is linear closed with dense domain in \( X \) as any generator of a \( C_0 \) semi-group. By reflexivity of \( X \) (which means, as usual, that \( J(X) = X^{**} \)), it follows that \( \mathcal{D}(A^*) \) is dense in \( X^* \), and that \( A^{**}(Jx) = J(Ax), \forall x \in \mathcal{D}(A), \) and \( J(D(A)) = D(A^{**}), \) (see [9]).

We shall see now that hypothesis i)-ii)-iii) of Theorem 2.1 are verified.

Take hence \( u(t) \in C^1([0, T]; X) \); assuming that \( J(t) \in D(A^{**}) = J(D(A)) \) means: \( \forall t \in [0, T], \exists v(t) \in \mathcal{D}(A), \) such that \( Jv(t) = Ju(t); \) as \( J^{-1} \) exists, \( \Rightarrow v(t) = u(t); \) hence \( u(t) \in \mathcal{D}(A), 0 < t < T. \) Also, \( A^{**}. \)

\[
\cdot (Ju(t)) = J(Au(t)); \quad \text{We assumed in ii) that (d/dt)Ju - A^{**}(Ju) = 0 on [0, T]. But (d/dt)Ju = J(du/dt), as } u \in C^1([0, T]; X). \quad \text{Hence ii) becomes } J(du/dt) - J(Au) = 0 \text{ on [0, T] which implies } u' - Au = 0 \text{ on [0, T].}
\]

Furthermore iii) implies obviously that \( u(0) = 0, \) again because \( J^{-1} \) exists (\( X^{**} \rightarrow X \)).

Now, the well-known unicity result for strong solutions of \( (d/dt - A)v = 0 \) when \( A \) is generator of a \( C_0 \)-semi-group (see for example [7], theorem 2.2.2) implies that \( u(t) = 0 \) on [0, T), so \( Ju(t) = 0 \) on [0, T] too. Hence, all conditions of theorem 2.1 are fulfilled, and by now we can conclude that:
If the relation
\[ \int_0^T \langle \phi^*(t) + (A^*\phi^*)(t), u(t) \rangle \, dt = 0 \]
holds \( \forall \phi \in K_{A^*}[0, T) \), then \( u = \theta \) on \([0, T]\) (in fact, \( u \)-continuous is in \( L_{\text{loc}}^p \), and \( u = \theta \) a.e. on \([0, T] \Rightarrow u = \theta \) everywhere on \([0, T]\)). Hence, it remains to check precisely that
\[ (3.2) \quad \int_0^T \langle \phi^*(t) + (A^*\phi^*)(t), u(t) \rangle \, dt = 0 \quad \forall \phi^* \in K_{A^*}[0, T). \]

Remember that our hypothesis here is slightly different: we assume in fact that it is
\[ (3.3) \quad \int_0^T \langle \phi^*(t) + (A^*\phi^*)(t), u(t) \rangle \, dt = 0 \]
for test-functions regular as those in \( K_{A^*}[0, T) \) but null near 0 as well as near \( T \), which forms a subclass of \( K_{A^*}[0, T) \) (denoted usually as \( K_{A^*}[0, T) \)). We added however the condition \( u(0) = \theta \). So, it remains to prove that (3.2) holds.

Take henceforth an arbitrary \( \phi^*(t) \in K_{A^*}[0, T) \). Then consider, for any \( \epsilon > 0 \), a scalar-valued function \( v_\epsilon(t) \in C[0, T] \), which = 0 for \( 0 < t < \epsilon \), and = 1 for \( 2\epsilon < t < T \), satisfying also an estimate \( |\dot{v}_\epsilon(t)| < c/\epsilon \), \( 0 < t < T \).

Then the product \( v_\epsilon(t)\phi^*(t) \) is also = \( \theta \) near \( t = 0 \), so it is in the subclass of admissible here test-functions. We get from (3.3) the following equality
\[ (3.4) \quad \int_0^T \langle \dot{v}_\epsilon \phi^* + v_\epsilon \dot{\phi}^* + v_\epsilon A^*\phi^*, u \rangle \, dt = 0 , \quad \forall \epsilon > 0 , \ \phi^* \in K_{A^*}[0, T). \]

Obviously (3.4) reduces to the following
\[
\int_0^{2\epsilon} \langle \dot{v}_\epsilon, u \rangle \, dt + \int_0^{2\epsilon} \langle v_\epsilon \phi^*, u \rangle \, dt + \int_0^T \langle \phi^*, u \rangle \, dt + \\
+ \int_0^{2\epsilon} \langle v_\epsilon A^*\phi^*, u \rangle \, dt + \int_0^T \langle A^*\phi^*, u \rangle \, dt = 0 , \quad \forall \epsilon > 0 , \ \forall \phi^* \in K_{A^*}[0, T). \]
Now, for $\varepsilon \to 0$, the first integral is estimated as

$$\left| \int_{\varepsilon}^{2\varepsilon} \hat{v}_\varepsilon \langle \phi^*, u \rangle \, dt \right| \leq \frac{\varepsilon}{\varepsilon} \sup_{\varepsilon \leq t \leq 2\varepsilon} |\langle \phi^*, u \rangle| \cdot \varepsilon;$$

as $u(0) = 0$, $u(t) \to 0$ when $t \to 0$, hence $\sup_{\varepsilon \leq t \leq 2\varepsilon} |\langle \phi^*, u \rangle| \leq K \sup_{\varepsilon \leq t \leq 2\varepsilon} \|u(t)\| \to 0$

with $\varepsilon$. The other integrals containing $\varepsilon$ are easily handled so that we obtain

$$\int_{0}^{T} \langle \phi^*, u \rangle \, dt + \int_{0}^{T} \langle A^* \phi^*, u \rangle \, dt = 0, \quad \forall \phi^* \in K_A(0, T)$$

which finishes our proof.

Remark. The original proof of [2] was given using the adjoint semi group theory in reflexive spaces in a very natural way. We shall see later on a similar proof for the non-reflexive case (§5).

§ 4. — We shall deal here with the following unicity result for weakened solutions (see [3], Theorem 3.1, p. 81):

"Let be $A$ a linear operator in the $B$-space $\mathcal{X}$, such that $R(\lambda; A) = (\lambda - A)^{-1} \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ for $\lambda$ real $\geq \lambda_0$, and

$$\lim_{\lambda \to +\infty} \frac{\ln \|R(\lambda)\|}{\lambda} = h_A < \infty.$$"

Let $u(t)$ be a weakened solution of $u' - Au = 0$ on the interval $0 < t < T$, such that $u(0) = \theta$, and assume also that $h_A < T$. Then $u(t) = \theta$ in $0 < t < T - h_A$.

A slight generalization is possible, replacing $[0, T]$ by an arbitrary real interval $[a, b]$.

Theorem 4.1. Under the same hypothesis on $A$, and if $h_A < b - a$, any weakened solution $u(t)$ of $u' - Au = 0$ on $a < t < b$, such that $u(a) = \theta$, is $\theta$ on $a < t < b - h_A$.

We can in fact take $T = b - a$ in the above theorem; so if $u(0) = \theta$, we get $u(t) = \theta$ on $[0, b - a - h_A]$. 
To prove theorem 4.1, let us put \( u(t + a) = u_a(t) \); it maps the interval \( 0 < t < b - a \) into \( \mathcal{X} \). Also it is \( \dot{u}_a(t) = u'(t + a) = Au(t + a) = Au_a(t) \) for \( 0 < t < b - a \).

Hence \( u_a(t) \) is weakened solution on \( 0 < t < b - a \), and \( u_a(0) = u(a) = 0 \); so, \( u_a(t) = 0 \) on \( 0 < t < b - a - h_A \) that is \( u(t + a) = 0 \) for \( 0 < t < b - a - h_A \), hence for \( a < t + a < b - h_A \), which gives \( u(t) = 0 \) for \( a < t < b - h_A \).

Now we shall see a partial extension of Theorem 4.1 in general \( B \)-spaces, taking weak solutions instead of weakened. Precisely, we propose ourselves to prove the following

**Theorem 4.2.** Let \( A \) be a linear operator in the \( B \)-space \( \mathcal{X} \), such that \( (\lambda - A)^{-1} \in \mathcal{L}(\mathcal{X}, \mathcal{X}) \) for \( \lambda \) real \( > \lambda_0 \) and assume also that

\[
\lim_{\lambda \to \infty} \frac{-\ln \| R(\lambda; A) \|}{\lambda} = h_A < \infty.
\]

Let also be \( \mathcal{D}(A^*) \) a dense subset of \( \mathcal{X}^* \), and \( \mathcal{D}(A) \) be dense in \( \mathcal{X} \) (*). Assume finally that

\[
\int_a^b \langle \phi^* + A^* \phi^*, u \rangle \, dt = 0
\]

\( \forall \phi^* \in K_a^*[a, b) \), where \( u \in L^p_{\text{loc}}([a, b); \mathcal{X}) \). Then, \( u = 0 \) a.e. on \( a < t < b - h_A \), provided \( h_A < b - a \).

Let us start the proof by remembering Phillips's fundamental results (see [4], [8]) concerning resolvents of dual operators.

*Let \( T \) be linear closed operator with dense domain \( \mathcal{D}(T) \subset \mathcal{X} \), and \( T^* \) be its dual operator (acting on a total set in \( \mathcal{X}^* \), \( \mathcal{D}(T^*) \)). Then the resolvent sets \( \rho(T) \) and \( \rho(T^*) \) coincide; also, for any \( \lambda \in \rho(T) \), it is \( (R(\lambda; T))^* = R(\lambda; T^*) \).*

Apply this result to our operator \( A \) which is linear closed in \( \mathcal{X} \), because we assume that \( R(\lambda; A) \) exists \( \in \mathcal{L}(\mathcal{X}, \mathcal{X}) \) for \( \lambda > \lambda_0 \), \( \lambda \) real, and \( \mathcal{D}(A) \) is dense by hypothesis. We obtain that for \( \lambda \) real \( > \lambda_0 \),

\( (*) \) The existence of \( (\lambda - A)^{-1} \in \mathcal{L}(\mathcal{X}, \mathcal{X}) \) does not implies in general, that \( \mathcal{D}(\lambda - A) = \mathcal{D}(A) \) is dense in \( \mathcal{X} \).

It suffices to consider \( \mathcal{X} = C[0, 1]; A = d^2/dx^2 \) defined on functions in \( C^2[0, 1] \) which vanish for \( x = 0 \) and \( x = 1 \). Considering the equation \( u'' = f \), \( \forall f \in C[0, 1] \), we find a unique solution \( u \in \mathcal{D}(A) \), depending continuously on \( f \). However, \( \mathcal{D}(A) \) is not dense in \( \mathcal{X} \).
\(R(\lambda, A^*)\) also \(\in \mathcal{L}(\mathfrak{X}^*, \mathfrak{X}^*)\), and \(R(\lambda; A^*) = [R(\lambda; A)]^*\). We know also that \(\| [R(\lambda; A)]^* \| = \| [R(\lambda; A)] \|\) hence \(\| R(\lambda; A^*) \| = \| R(\lambda; A) \|\) and consequently

\[
\lim_{\lambda \to \infty} \frac{\ln \| R(\lambda; A^*) \|}{\lambda} = h_A \text{ too.}
\]

Now, \(\mathcal{D}(A^*)\) is also dense in \(\mathfrak{X}^*\), and \(A^*\) is closed. It follows that \(R(\lambda; A^{**}) \in \mathcal{L}(\mathfrak{X}^{**}, \mathfrak{X}^{**})\), \(\forall \lambda \) real \(\geq \lambda_0\), and for these \(\lambda\), \(\| R(\lambda; A^{**}) \| = \| R(\lambda; A) \|\) so,

\[
\lim_{\lambda \to \infty} \frac{\ln \| R(\lambda; A^{**}) \|}{\lambda} = h_A < \infty \text{ too.}
\]

Now we shall apply theorem 2.1 on the interval \(a < t < b - h_A\). Let us consider consequently a function \(u(t) \in \mathcal{C}^1[a, b - h_A; \mathfrak{X}]\), such that \(Ju \in \mathcal{D}(A^{**})\), \(a < t < b - h_A\), \((d/dt)(Ju) - A^{**}(Ju) = 0\) on \(a < t < b - h_A\), and \((Ju)(a) = \theta\).

Let us apply now theorem 4.1 taking \(A^{**}\) instead of \(A\) which is possible by the above (remarking also that here the solutions are strong which is better than weakened). It follows that \(Ju(t) = \theta\) on \(a < t < b - h_A\). Hence theorem 2.1 is applicable on \([a, b - h_A]\) and we get uniqueness of weak solutions, as desired.

§ 5. - In this section we present a variant of the unicity result considered in § 3, which is valid in more general, non-reflexive \(B\)-spaces.

Let us start by remembering Phillips's theorem on dual semi-groups (see [4], [5], [8]).

Consider in the \(B\)-space \(\mathfrak{X}\), a linear closed operator \(A\) with domain \(\mathcal{D}(A)\) dense in \(\mathfrak{X}\), and assume that \(A\) generates a semi-group of class \((C_0)\) of linear continuous operators \(T(t), 0 < t < \infty \to \mathcal{L}(\mathfrak{X}, \mathfrak{X})\).

Now, as previously, the dual operator \(A^*\) of \(A\) is a closed linear transformation on \(\mathcal{D}(A^*) \subset \mathfrak{X}^*\) to \(\mathfrak{X}^*\). We know that \(\mathcal{D}(A^*)\) is a total set in \(\mathfrak{X}^*\), but in general \(\mathcal{D}(A^*)\) is not dense in \(\mathfrak{X}^*\) so that \(A^*\) is not necessarily the infinitesimal generator of a strongly continuous semi-group in \(\mathfrak{X}^*\).

Therefore it is convenient to consider the so called \(\ominus\)-dual space \(\mathfrak{X}^\ominus\) of \(\mathfrak{X}\), defined by \(\mathfrak{X}^\ominus = \overline{\mathcal{D}(A^*)}\) (closure in \(\mathfrak{X}^*\)). In the case of reflexive \(\mathfrak{X}\), we have \(\mathfrak{X}^\ominus = \mathfrak{X}^*\), else \(\mathfrak{X}^\ominus\) may be a proper subset of \(\mathfrak{X}^*\).

Let us define now the operator \(A^\ominus\) to be the restriction of the
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Dual operator $A^*$ to the domain

(5.1) $\mathcal{D}(A^\diamond) = [x^* \in \mathfrak{X}^*, x^* \in \mathcal{D}(A^*) \text{ such that } A^*x^* \in \mathfrak{X}^\diamond].$

Furthermore, let $T^*(t)$ be, for any $t > 0$, the dual operator of $T(t)$, and then $T^\diamond(t)$ be the restriction of $T^*(t)$ to $\mathfrak{X}^\diamond$; then $T^\diamond(t) \in \mathcal{L}(\mathfrak{X}^\diamond, \mathfrak{X}^\diamond)$, $t > 0$, and it is a semi-group of class $(C^0)$ having $A^\diamond$ as infinitesimal generator.

Our aim is to prove the following

**Theorem 5.1.** Let $u(t)$ be a continuous function, $0 < t \leq T$ to $\mathfrak{X}$, such that $u(0) = 0$, and satisfying relation

\[ \int_0^T \langle \phi^\diamond + A^\diamond \phi^\diamond, u(t) \rangle \, dt = 0 \]

for any function $\phi^\diamond(t), 0 < t < T \in \mathcal{D}(A^\diamond), \phi^\diamond \in C^1[0, T; \mathfrak{X}^\diamond], A^\diamond \phi^\diamond \in C[0, T; \mathfrak{X}^\diamond], \phi^\diamond = 0$ near $0$ and near $T$. Then $u(t) = 0$ on $[0, T]$.

**Remark.** Before giving the proof, let us consider the particular case of reflexive space $\mathfrak{X}$. Then $\mathfrak{X}^\diamond = \mathfrak{X}^*$, $A^\diamond = A^*$, so we find again the previously proved theorem in § 3.

**Proof of the theorem.** We have firstly

**Lemma 5.1.** The relation

(5.2) \[ \int_0^T \langle \phi^\diamond + A^\diamond \phi^\diamond, u \rangle \, dt = 0 \]

is verified for the more general class of test-function: $\phi^\diamond(t) \in C^1[0, T; \mathfrak{X}^\diamond], \phi^\diamond(t) \in \mathcal{D}(A^\diamond), (A^\diamond \phi^\diamond)(t) \in C[0, T; \mathfrak{X}^\diamond], \phi^\diamond(T) = 0$. Let us consider, $\forall \varepsilon > 0$, a scalar-valued function $v_\varepsilon(t)$, continuously differentiable on $0 < t < T$, $= 0$ for $0 < t < \varepsilon$, $T - \varepsilon < t < T$, $= 1$ for $2\varepsilon < t < T - 2\varepsilon$, such that $|v_\varepsilon'(t)| < c/\varepsilon, 0 < t < T, |v_\varepsilon(t)| < 1, 0 < t < T$; then $v_\varepsilon(t)\phi^\diamond(t)$ is a test-function as required in theorem 5.1, because it vanishes near $t = 0$ and near $t = T$. We can write henceforth the relation (5.2) for $v_\varepsilon\phi^\diamond$, and obtain the following:

\[ \int_0^T \langle \dot{v}_\varepsilon \phi^\diamond + v_\varepsilon \dot{\phi}^\diamond, u \rangle \, dt = - \int_0^T v_\varepsilon \langle A^\diamond \phi^\diamond, u \rangle \, dt. \]
The right-hand integral splits as
\[
-\int_{2s}^{T-2s} v_0 \langle \mathcal{A}^\otimes \phi^\otimes, u \rangle \, dt - \int_{2s}^{2s} \langle \mathcal{A}^\otimes \phi^\otimes, u \rangle \, dt - \int_{T-2s}^{T-2s} \langle \mathcal{A}^\otimes \phi^\otimes, u \rangle \, dt
\]
and is readily seen that
\[
\lim_{\varepsilon \to 0} -\int_0^T v_0 \langle \mathcal{A}^\otimes \phi^\otimes, u \rangle \, dt = -\int_0^T \langle \mathcal{A}^\otimes \phi^\otimes, u \rangle \, dt.
\]

The left-hand side integral equals
\[
\int_0^T \dot{v}_0 \langle \phi^\otimes, u \rangle \, dt + \int_0^T v_0 \langle \dot{\phi}^\otimes, u \rangle \, dt = I_1 + I_2.
\]

Actually it results
\[
I_1 = \int_{2s}^{T-2s} \dot{v}_0 \langle \phi^\otimes, u \rangle \, dt + \int_{T-2s}^{T-2s} \dot{v}_0 \langle \phi^\otimes, u \rangle \, dt = I_3 + I_4.
\]

Now, \( \lim_{\varepsilon \to 0} I_3 = 0 \), essentially because \( |\dot{v}| < \varepsilon/2 \), and \( u(0) = \theta \). Also \( \lim_{\varepsilon \to 0} I_4 = 0 \), essentially because \( |\dot{v}| < \varepsilon/3 \), and \( \phi^\otimes(T) = \theta \). As for \( I_2 \), it is obviously seen to converge to \( \int_0^T \langle \phi^\otimes, u \rangle \, dt \), as \( \varepsilon \to 0 \). Hence, altogether, for \( \varepsilon \to 0 \) we get
\[
\int_0^T \langle \phi^\otimes, u \rangle \, dt + \int_0^T \langle \mathcal{A}^\otimes \phi^\otimes, u \rangle \, dt = 0,
\]
and the Lemma is proved.

We can continue now the proof of our theorem.

Let us take an arbitrarily given function \( k^\otimes(t) \in C^1[0, T; \mathcal{X}^*] \). Then consider in the \( \mathcal{O} \)-dual space \( \mathcal{X}^\otimes \), the strong inhomogeneous Cauchy
problem

\[ \frac{d\psi^\circ}{dt} - A^\circ \psi^\circ = - k^\circ, \quad \psi^\circ(0) = \theta . \]

Due to the fact that \( A^\circ \) is the generator of a \((C_0)\)-semigroup \( T^\circ(t) \) in \( X^\circ \), by a well-known result of Phillips ([7], Theorem 2.2.3), the problem (5.3) has a unique solution (given by the formula \( \psi^\circ(t) = - \int_0^t T^\circ(t - \sigma) \cdot k^\circ(\sigma) \, d\sigma \), but this is not important here).

Consider now the function \( \phi^\circ(t) \), defined for \( 0 < t < T \) through the relation \( \phi^\circ(t) = \psi^\circ(T - t) \).

It is continuously differentiable in \( X^\circ \) on \( 0 < t < T \); it belongs to \( D(A^\circ) \), \( \forall t \in [0, T] \), and \( (A^\circ \phi^\circ)(t) = (A^\circ \psi^\circ)(T - t) \) is continuous, \( 0 < t < T \rightarrow X^\circ \). Finally, \( \phi^\circ(T) = \psi^\circ(0) = \theta \). Hence, \( \phi^\circ(T) \) is an admissible test-function, and the relation \( \int_0^T \langle \phi^\circ + A^\circ \phi^\circ, u \rangle \, dt = 0 \) is verified.

Furthermore, \( d\phi^\circ / dt = - \psi^\circ(T - t) \) and consequently we get:

\[ \phi^\circ(t) + A^\circ \phi^\circ(t) = - \psi^\circ(T - t) + A^\circ \psi^\circ(T - t) = k^\circ(T - t), \]

in view of (5.3). Hence, we obtained the identity

\[ \int_0^T \langle k^\circ(T - t), u(t) \rangle \, dt = 0 , \]

for any \( k^\circ \in C^1[0, T; X^\circ] \), or, obviously, as \( t \rightarrow T - t \) maps \( C^1[0, T; X^\circ] \) onto itself,

\[ \int_0^T \langle k^\circ(t), u(t) \rangle \, dt = 0 \quad \forall k^\circ \in C^1[0, T; X^\circ] . \]

Take in particular \( h^\circ(t) = v(t)x^* \), where \( x^* \in X^\circ \). Then

\[ \int_0^T \langle v(t)x^*, u(t) \rangle \, dt = 0 , \quad \text{if } v(t) \in C^1[0, T] . \]

As \( \langle x^*, u \rangle \) is scalar-continuous on \([0, T]\), we obtain \( \langle x^*, u(t) \rangle = 0 \),
\( \forall t \in [0, T] \). But we can let \( x^* \) to vary in the total set \( \mathcal{D}(A^*) \subset X^\circ \). It follows that \( u(t) = \theta \), \( \forall t \in [0, T] \).

This ends the proof of our theorem.

A simple corollary is the following

**Theorem 5.2.** - Let \( u(t) \in C([0, T]; X) \), such that \( u(0) = 0 \) and assume that

\[
\begin{align*}
\int_0^T \langle \phi^* + A^* \phi^*, u \rangle \, dt &= 0 ,
\end{align*}
\]

for any function \( \phi^*(t) \), \( 0 < t < T \rightarrow \mathcal{D}(A^*) \), belonging to \( C^1([0, T]; X^*) \), such that \( A^*\phi^* \in C([0, T]; X^*) \) and \( \phi^* = \theta \) near 0 and near T. Then \( u(t) = \theta \) on \([0, T]\).

In fact it suffices to remark that the class of test-functions considered here contains as a subset the class considered in the theorem 5.1, because \( A^\circ \) is a certain restriction of \( A^* \) to an (eventually) smaller domain. Hence, the relation (5.2) is verified and theorem 5.2 implies \( u = \theta \) on \([0, T]\).

We have also the following

**Theorem 5.3.** Let \( A \) be the generator of a \( (C_0) \) semi-group \( T(t) \) in the B-space \( X \), and \( A^*; \mathcal{D}(A^*) \subset X^* \rightarrow X^* \) be its dual operator, defined on the total set \( \mathcal{D}(A^*) \).

Let \( u(t) \) a continuous function. \( 0 < t < T \rightarrow X \), such that \( u(0) = u_0 \) given arbitrarily in \( X \), and satisfying the relation

\[
\begin{align*}
\int_0^T \langle \phi^* + A^* \phi^*, u \rangle \, dt &= 0 , \quad \forall \phi^*(t) \in K_{A^*}(0, T) \ (1) .
\end{align*}
\]

Then \( u(t) \) has the representation \( u(t) = T(t)u_0 \), \( 0 < t < T \).

Let us consider in fact the strongly continuous function \( v(t) \), \( 0 < t < T \rightarrow X \), given by \( v(t) = T(t)u_0 \). Then (5.5) is valid also for this function \( v \).

In fact, let \( (u_n)_T^\circ \subset \mathcal{D}(A) \) be a sequence convergent to \( u_0 \). Let also

\[(1) \text{ This is the class of test-functions considered in Theorem 5.2.}\]
\( v_n(t) = T(t)u_n \), so that, as well-known, it is \( \dot{v}_n = Av_n \), \( 0 < t < T \). Now

\[
\int_0^T \langle \phi^*, v_n \rangle \, dt = -\int_0^T \langle \phi^*, \dot{v}_n \rangle \, dt,
\]
as obviously seen. Furthermore is \( \langle A^*\phi^*, v_n \rangle = \langle \phi^*, Av_n \rangle \), \( \forall t \in [0, T] \).

It follows

\[
\int_0^T \langle \phi^* + A^*\phi^*, v_n \rangle \, dt = -\int_0^T \langle \phi^*, \dot{v}_n \rangle \, dt + \int_0^T \langle \phi^*, Av_n \rangle \, dt = 0.
\]

when \( n \to \infty, v_n(t) \to v(t) \) uniformly on \([0, T]\), as \( \sup_{0 \leq t \leq T} \| T(t) \| = C_T < \infty \), so, it results:

\[
\int_0^T \langle \phi^* + A^*\phi^*, v \rangle \, dt = 0
\]

too. If we take now \( w(t) = u(t) - v(t) \), then (5.5) is verified for \( w(t) \), and \( w(0) = \theta \). By previous theorem, it follows \( u(t) = v(t) = T(t)u_0 \) on \( 0 < t < T \).

**REFERENCES**


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