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On the Algebraic Compactness of Some Complete Modules.

S. BAZZONI (*)

Introduction.

Let R be a commutative ring with unit.

An R -module M is algebraically compact if every finitely soluble family of linear equations over R in M has a simultaneous solution.

If R is a noetherian ring and Ω is the set of the maximal ideals of R , we can define over any R -module M the Ω -adic topology, by taking as a base of neighborhoods of 0 the submodules IM , where I is a finite intersection of powers of the maximal ideals.

If R is any ring and M is any R -module, we can define on M the R -topology, by taking as a base of neighborhoods of 0 the submodules rM with $0 \neq r \in R$.

Warfield [W₁] has proved that any algebraically compact R -module is complete in the Ω -adic topology, if R is a noetherian ring, and in the R -topology if R is any ring.

Moreover, Warfield has raised the problem to see if any complete Hausdorff module over a noetherian ring is necessarily algebraically compact.

In this work we answer in the affirmative to the question posed by Warfield and we characterize the noetherian rings R such that

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any R -module which is complete and Hausdorff in the R -topology is algebraically compact.

1. Complete modules in the Ω -adic topology.

Let R be a noetherian commutative ring with unit, M a topological R -module equipped with the Ω -adic topology.

We denote by \widehat{M} the Ω -adic completion of M .

(« Complete module » means « Hausdorff complete module »).

The purpose of this section is to prove that, for any R -module M , \widehat{M} is an algebraic compact R -module.

First of all, we recall that M is topologically isomorphic to the product $\prod_{m \in \Omega} \widehat{M}_m$ where \widehat{M}_m denotes the m -adic completion of the localization of M at m , so since the class of algebraically compact modules is closed under direct products, we shall have to settle the problem only with respect to the m -adic completion of a module.

By a suitable definition of pure submodule, Warfield has proved that the class of algebraically compact modules, coincides with the class of pure-injective modules. Therefore we now recall the principal definitions concerning the concept of purity and pure-injectivity.

DEFINITION 1. *Let R be a ring, \mathcal{S} a class of R -modules.*

N is an \mathcal{S} -pure submodule of an R -module M , if every element of \mathcal{S} is projective for the exact sequence:

$$0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0 .$$

An equivalent definition to definition 1 is the following: ([W₂])

DEFINITION 1'. *Let \mathcal{S} be a class of R -modules.*

N is an \mathcal{S} -pure submodule of an R -module M if it is a direct summand of any module H such that: a) $N \leq H \leq M$, b) $H/N \in \mathcal{S}$.

Walker ([W₂]) has introduced the notion of \mathcal{S} -copure submodule by dualizing the definition 1' in the following way:

DEFINITION 2. *A submodule N of an R -module M is \mathcal{S} -copure in M , if for every submodule H of N such that $N/H \in \mathcal{S}$, N/H is a summand of M/H .*

We are interesting with a particular class of modules, namely we consider the class \mathcal{F} of all finitely presented modules, so that we give the following definition:

DEFINITION 3. *A submodule of a module M is pure (copure) in M if it is \mathcal{F} -pure (\mathcal{F} -copure).*

Moreover we say that a module is pure-injective if it is injective for any pure exact sequence.

REMARK. If R is a noetherian ring; the class \mathcal{F} is the class of all finitely generated modules.

LEMMA 1. *Let R be an artinian local ring. If N is a pure submodule of M , then it is also copure in M .*

PROOF. Let H be a submodule of N such that $N/H \in \mathcal{F}$.

We have to prove that N/H is a summand of M/H .

Since N is pure in M , N/H is pure in M/H ([W₂], Theor. 2.1); moreover, since the maximal ideal of R is nilpotent, the Ω -adic topology over any R -module is the discrete topology, so N/H is a finitely generated and a complete module in the Ω -adic topology.

Then, by Theor. 3 of [W₁], N/H is pure-injective and therefore it is a summand of M/H . //

LEMMA 2. *Let R be a ring satisfying the hypotheses of the preceding lemma. If N is a copure submodule of a module M , then it is a summand of M .*

PROOF. Let:

$$\mathcal{F}(N) = \{N_\alpha \leq N : N/N_\alpha \in \mathcal{F}\}$$

$$N^{\mathcal{F}} = \bigcap_{N_\alpha \in \mathcal{F}(N)} N_\alpha$$

For each $0 \neq x \in N$, let H_x be a submodule of N maximal with respect to the property of not containing x .

It is easy to verify that the submodule generated by $x + H_x$ is simple and essential in N/H_x .

Therefore the injective envelope $E(N/H_x)$ of N/H_x is isomorphic to $E(R/\mathfrak{m}) = E$. Now, by [M] Theor. 3.4 and 3.11, $E = \bigcup_k E_k$ where E_k is an increasing sequence of finitely generate submodules of E with $E_k = \{x \in E : \mathfrak{m}^k x = 0\}$. Therefore, since $\mathfrak{m}^h = 0$, for a convenient integer h , we have $E = E_h$; then N/H_x is finitely generated since it is

a submodule of the noetherian module E and then we have:

$$0 = \bigcap_{0 \neq \alpha \in N} H_\alpha \supseteq \bigcap_\alpha N_\alpha = N^{\mathcal{F}}.$$

Now, ([W₂], Corollary 2.9') the group $\text{Copext}(L, N)$ of the copure extensions of N by a generical module L , is the image of the homomorphism $f: \text{Ext}(L, N^{\mathcal{F}}) \rightarrow \text{Ext}(L, N)$ induced by the inclusion $N^{\mathcal{F}} \rightarrow N$. Then, since $N^{\mathcal{F}} = 0$, N is a summand of every module in which it is a copure submodule. //

THEOREM 1. *Let R be a noetherian ring, \mathfrak{m} an element of Ω and M an R -module. For every $k \in \mathbb{N}$, $M/\mathfrak{m}^k M$ is an algebraically compact R -module.*

PROOF. $M/\mathfrak{m}^k M$ is an R/\mathfrak{m}^k -module, then by lemmas 1 and 2, it is an algebraically compact R/\mathfrak{m}^k -module. Moreover we can easily deduce from the definition of algebraically compactness, that $M/\mathfrak{m}^k M$ is also an R -module algebraically compact //.

Let M be an R -module, we denote by $B(M)$ the Bohr compactification of M , that is:

$$B(M) = \text{Hom}_{\mathbb{Z}}(\text{Hom}_{\mathbb{Z}}(M, K), K)$$

where K is the circle group ([W₁], § 3).

Let ω_M be the natural homomorphism of M in $B(M)$; then $\omega_M(M) = \hat{M}$ is canonically isomorphic to M .

Warfield ([W₁], § 3), has proved that $B(M)$ is a topological compact R -module and that \hat{M} is a pure (and dense) submodule of $B(M)$.

Now we have the following:

THEOREM 2. *Let R be a noetherian ring, \mathfrak{m} a maximal ideal of R . The \mathfrak{m} -adic completion of any R -module M is an algebraically compact R -module.*

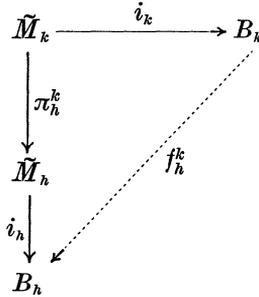
PROOF. Let $M_k = M/\mathfrak{m}^k M$ and π_h^k the natural homomorphisms $\pi_h^k: M_k \rightarrow M_h$ ($k > h$, $h, k \in \mathbb{N}$); then we have:

$$\hat{M} \approx \varprojlim_k \{M_k, \pi_h^k \quad k > h\}_{k \in \mathbb{N}}$$

Let B_k be the Bohr compactification of M_k for every $k \in \mathbb{N}$, \hat{M}_k the copy of M_k in B_k and let $\hat{\pi}_h^k$ be the homomorphisms induced by the π_h^k .

Then M is isomorphic to $\varprojlim \{\tilde{M}_k, \tilde{\pi}_h^k\}$ since ω_k are natural isomorphism for every $k \in N$.

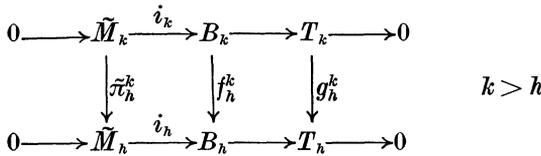
Therefore it will suffice to prove that $\varprojlim \{\tilde{M}_k, \tilde{\pi}_h^k\}$ is algebraically compact. Let's consider the following diagram:



The universal property of B_k , assures the existence of a unique continuous homomorphism f_h^k such that the diagram commutes.

Now, by Theor. 1, \tilde{M}_k is an algebraically compact R -module, so $B_k = \tilde{M}_k \oplus T_k$ for every $k \in N$.

Let us consider the following diagram:



with $f_h^k \circ i_k = i_h \circ \tilde{\pi}_h^k$. Since the rows are exact, there is a unique homomorphism g_h^k such that the diagram commutes.

By the unicity of the g_h^k , the system $\{T_k; g_h^k \ k > h\}$ is an inverse system. Then we have:

$$\varprojlim_k \{\tilde{M}_k; \tilde{\pi}_h^k\} \oplus \varprojlim_k \{T_k; g_h^k\} \cong \varprojlim_k \{B_k; \tilde{\pi}_h^k \oplus g_h^k\}.$$

Now, $\varprojlim B_k$ is a compact module in the topology induced by the product topology of the B_k , then by [W₁] Theor. 2, $\varprojlim M_k$ is algebraically compact. //

2. Complete modules in the R -topology.

Let R be a noetherian commutative ring with unit.

The purpose of this section is to characterize the rings R such that any R -module which is complete and T_2 in the R -topology is algebraically compact.

First of all we consider the case in which the Ω -adic topology on R is the discrete topology.

This hypothesis implies that every R -module is discrete in the Ω -adic topology and so, any R -module is algebraically compact.

In the general case, that is, when the open ideals in the Ω -adic topology on R are always non zero, then the R -topology over any R -module M is finer than the Ω -adic topology.

Now, since R is a noetherian ring, any ideal rR , with $r \neq 0$, contains a finite intersection of powers of prime non zero ideals of R .

Therefore, if R has the following property:

(P) every non zero prime ideal of R is maximal

the Ω -adic topology and the R -topology coincides over any R -module.

Then the results contained in Section 1, allow us to say that (P) is a sufficient condition on R to insure that any complete and T_2 module in the R -topology is algebraically compact.

(The converse has been stated by Warfield, as we have just noted).

Now, we shall prove that (P) is also a necessary condition.

Let us suppose that R has a non zero and non maximal prime ideal \mathfrak{F} .

Let \mathfrak{m} be a maximal ideal of R containing \mathfrak{F} and let T be the localization of R/\mathfrak{F} at $\mathfrak{m}/\mathfrak{F}$; we consider the R -module $A = T[x]$ where x is a transcendental element over T .

Clearly the R -topology on A is discrete, so A is complete in such topology, but we shall prove that A is not algebraically compact.

Infact, if A were algebraically compact, Warfield's results would entail the completeness of A in the Ω -adic topology. But now, it is easy to verify that the Ω -adic topology on A is the same as the \mathfrak{m} -adic topology, so it suffices to find a nonconvergent Cauchy sequence of elements of A .

We denote by \mathfrak{n} the maximal ideal of T .

The powers of \mathfrak{n} give a strictly decreasing chain of ideals of T , since, by Krull Theorem, $\bigcap_i \mathfrak{n}^i = 0$ and by the hypotheses on \mathfrak{F} we cannot have $\mathfrak{n}^i = 0$ for any $i \in \mathbb{N}$.

Let a_i be an element of $\mathfrak{n} \setminus \mathfrak{n}^{i+1}$ for every $i \in \mathbf{N}$, and let us consider the following elements of A :

$$s_k = \sum_{i=0}^k a_i x_i \quad k \in \mathbf{N}.$$

Now it is easy to prove that $\{s_k\}_k$ is a Cauchy sequence of element of A which cannot converge to any element of A .

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