RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

JUTTA HAUSEN
JOHNNY A. JOHNSON

Abelian groups with many automorphisms

Rendiconti del Seminario Matematico della Università di Padova, tome 55 (1976), p. 1-5
<http://www.numdam.org/item?id=RSMUP_1976__55__1_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 55 (1976)

Abelian Groups with Many Automorphisms.

JUTTA HAUSEN - JOHNNY A. JOHNSON (*)

1. – The results.

Given a group G, its automorphism group Aut G acts as a group of permutations on the set of all subgroups of G. Problem 91 of Fuchs [2] proposes to determine the equivalence classes (orbits) of subgroups of G determined by this action of Aut G. In this note we characterize the class of all abelian groups G for which these equivalence classes are precisely the isomorphism classes of subgroups of G. These are the T_1-groups in the sense of Polimeni [4]: G is a T_1-group if, given two subgroups H and K of G which are isomorphic, there exists α in Aut G such that $\alpha(H) = K$. This definition is weaker than the definition of a T_1'-group: G is a T_1'-group if every isomorphism between any two of its subgroups can be extended to an automorphism of G. We shall prove the following theorem which shows among other things that every abelian T_1-group actually is a T_1'-group.

(1.1) THEOREM. For an abelian group G the following statements are equivalent.

(i) Aut G operates transitively on the isomorphism classes of subgroups of G (i.e. G is T_1).

(ii) Either G is a torsion group all of whose p-components are homo-cocyclic of finite rank, or G is a divisible group whose torsion-free rank and p-ranks are all finite.

(*) Indirizzo degli AA.: Dept. of Mathematics, Univ. of Houston, Cullen Boulevard, Houston, Texas 77004, U.S.A.

(iii) G is a characteristic subgroup of a divisible group whose torsion-free rank and p-ranks are all finite.

(iv) Every isomorphism between any two subgroups of G is induced by some automorphism of G (i.e., G is T_1').

Let $\mathcal{A}$ be the class of all abelian groups G such that every automorphism of any subgroup of G can be extended to an automorphism of G. Clearly every abelian T_1'-group (and hence, in view of (1.1), every abelian T_1-group) is contained in $\mathcal{A}$. In [3] Mishina has shown that an abelian torsion group belongs to $\mathcal{A}$ if and only if all of its p-components are homo-cocyclic. This result together with (1.1) implies the following theorem.

(1.2) THEOREM. Let G be an abelian torsion group all of whose p-ranks are finite. Then the following conditions are equivalent.

(i) Aut G operates transitively on the isomorphism classes of subgroups of G.

(ii) All p-components of G are homo-cocyclic.

(iii) G is a characteristic subgroup of a divisible torsion group.

(iv) Every automorphism of any subgroup of G can be extended to an automorphism of G.

(v) Every isomorphism between any two subgroups of G can be extended to an automorphism of G.

2. – The proof.

All groups, except groups of automorphisms, are assumed to be abelian and the notation and terminology will be that of Fuchs [1, 2]. We will make use of the following auxiliary results. Some proofs are left to the reader.

(2.1) LEMMA. Characteristic subgroups of T_1-groups are T_1-groups.

(2.2) LEMMA. Characteristic subgroups of T_1'-groups are T_1'-groups.

(2.3) LEMMA. If G is a T_1-group, then G is not isomorphic to a proper subgroup of itself.

(2.4) COROLLARY. If G is a T_1-group and $G = A \oplus \bigoplus_{i \in I} B_i$, where $0 \neq B_i \simeq B_j$, for all i, j in I, then I is finite.

(2.5) PROPOSITION. Let G be a T_1-group and let $x, y \in G$ such that $o(x) = o(y)$. If $o(x) = \infty$ or G is a p-group, then there exists α in Aut G such that $\alpha(x) = y$.

PROOF. Since $\langle x \rangle \simeq \langle y \rangle$, there exists β in Aut G such that

$$\langle y \rangle = \beta(\langle x \rangle) = \langle \beta(x) \rangle$$

and consequently y generates $\langle \beta(x) \rangle$. If $o(x)$ is infinite, $y = k \cdot \beta(x)$, where $k = \pm 1$; if G is a p-group, then $y = k \cdot \beta(x)$, for some integer k relatively prime to p. In either case, the mapping γ such that $g \to kg$, for all $g \in G$, is an automorphism of G. Hence $\alpha = \gamma \circ \beta \in \operatorname{Aut} G$ and

$$\alpha(x) = (\gamma \circ \beta)(x) = \gamma(\beta(x)) = k \cdot \beta(x) = y$$

as desired.

(2.6) COROLLARY. Let G be a T_1-group and let $x, y \in G$ such that $o(x) = o(y)$. If $o(x) = \infty$ or G is a primary group, then, for all primes p, $h_p(x) = h_p(y)$.

A group is called homo-cocyclic if it is the direct sum of pairwise isomorphic cocyclic groups (cf. [1], p. 16).

(2.7) PROPOSITION. If G is a p-primary T_1-group, then G is homo-cocyclic of finite rank.

PROOF. Since $G[p]$ is a characteristic subgroup of G, (2.1) and (2.4) imply that $G[p]$ is finite and G has finite rank k. By (2.6) any two non-zero elements in $G[p]$ have equal height in G. Hence $G = \bigoplus_{i=1}^{k} Z(p^n)$, for some $1 \leqslant n \leqslant \infty$, as claimed.

(2.8) COROLLARY. If G is a torsion T_1-group, then every p-component of G is homo-cocyclic of finite rank.

(2.9) PROPOSITION. If G is a non-torsion T_1-group, then G is a divisible group whose torsion-free and p-ranks are all finite.

PROOF. In view of (2.4) and the structure theorem for divisible groups (cf. [1], p. 104) it suffices to show that G is divisible, which is equivalent to $h_p(x) = \infty$, for all $x \in G$, and all primes p. Let $x \in G$ and let p be a prime. If $o(x) = \infty$, then, for all integers $n \geqslant 1$,

$$o(x) = o(p^n x) \qquad \text{and} \qquad h_p(x) = h_p(p^n x) \geqslant n$$

by (2.6). Hence, $h_p(x) = \infty$ whenever $x \in G$ has infinite order. Suppose $o(x) < \infty$. Since G is not torsion there exists $y \in G$ such that $o(y) = \infty$. Hence

$$o(x+y) = \infty \qquad \text{and} \qquad h_p(x+y) = h_p(y) = \infty .$$

By ([1], p. 6, exercise 11) $h_p(x)$ cannot be finite and $h_p(x) = \infty$ as claimed.

(2.10) PROPOSITION. If D is a divisible group whose torsion-free rank and p-ranks are all finite, then D is a T_1'-group.

PROOF. Let $\varphi\colon A_1 \to A_2$ be an isomorphism between the two subgroups A_1 and A_2 of D. Let $\bar{D}_i$ be the divisible hull of A_i in D, $i = 1, 2$. Since $\bar{D}_2 \geqslant A_2$ is injective, there exists a homomorphism $\sigma\colon \bar{D}_1 \to \bar{D}_2$ such that $\sigma|A_1 = \varphi$. The fact that $\bar{D}_1$ is an essential extension of A_1 and φ is monic implies σ is monic (cf. [1], p. 106 f). Also, σ is epic since $\sigma(\bar{D}_1)$ is divisible and

$$A_2 = \varphi(A_1 = \sigma(A_1) \leqslant \sigma(\bar{D}_1) \leqslant \bar{D}_2 .$$

Hence $\sigma\colon \bar{D}_1 \to \bar{D}_2$ is an isomorphism extending φ. There exists $D_i \leqslant D$ such that

$$\bar{D}_1 \oplus D_1 = D = \bar{D}_2 \oplus \bar{D}_2 .$$

Since the torsion-free rank and all p-ranks of D are finite, $\bar{D}_1 \simeq \bar{D}_2$ implies $D_1 \simeq D_2$: Let $\tau\colon D_1 \to D_2$ be an isomorphism and define $\alpha\colon D \to D$ by $\alpha|\bar{D}_1 = \sigma$ and $\alpha|D_1 = \tau$. Then $\alpha \in \operatorname{Aut} D$ and $\alpha|A_1 = \sigma|A_1 = \varphi$, completing the proof.

PROOF OF (1.1). The proof is cyclic. (ii) follows from (i) by (2.8) and (2.9). Assume the validity of (ii). Clearly, we may assume that G is torsion. If the p-component G_p of G is not divisible, then $G_p = \bigoplus_{i=1}^{k} Z(p^n)$, for some integer n and $G_p = D_p[p^n]$, where $D_p = \bigoplus_{i=1}^{k} Z(p^\infty)$. Hence, for all p, G_p is a characteristic subgroup of a divisible p-group D_p of finite rank, and $G = \bigoplus_p G_p$ is characteristic in $D = \bigoplus_p D_p$, as stated in (iii). Using (2.2) and (2.10), (iv) follows from (iii). The last implication is trivial.

REFERENCES

[1] L. Fuchs, *Infinite Abelian Groups*, vol. I, Academic Press, New York, 1970.

[2] L. Fuchs, *Infinite Abelian Groups*, vol. II, Academic Press, New York, 1973.

[3] A. P. Mishina, *On automorphisms and endomorphisms of abelian groups*, Vestnik, Moscow Univ. Ser. Matem. i Mekh., no. 4 (1962), 39-43 [Russian].

[4] A. D. Polimeni, *Groups in which* Aut(G) *is transitive on the isomorphism classes of* G, Pacific J. Math., **48** (1973), 473-480.

Manoscritto pervenuto in redazione il 18 dicembre 1974.