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A Selection Theorem.

ARRIGO CELLINA (*)

1. – Introduction.

A well known theorem of Michael states that a lower semi-continuous multi-valued mapping, from a metric space into the non-empty closed and convex subsets of a Banach space, admits a continuous selection. It is also known that, when the multi-valued mapping is instead upper semi-continuous, in general we have only measurable selections.

This paper considers a compact convex valued mapping F of two variables, t and x , that is separately upper semi-continuous in t for every fixed x and lower semi-continuous in x for every fixed t , and proves the existence of a selection $f(t, x)$, separately measurable in t and continuous in x . As a consequence, an existence theorem for solutions of a multi-valued differential equation is presented.

2. – Notations and basic definitions.

In what follows \mathbf{R} are the reals, X a separable metric space and Z a Banach space. We shall denote by $K(Z)$ the set of non-empty compact and convex subsets of Z . $B[A, \varepsilon]$ is an open ball of radius $\varepsilon > 0$ about the set A , \bar{A} is the closure of A . We shall use the symbol $d(\cdot, \cdot)$ both for the metric in X and for the metric inherited from the norm in Z . Also $d(a, B)$ is the distance from the point a to the set B , while

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$\delta^*(A, B) = \sup \{d(a, B) : a \in A\}$ and D is the Hausdorff distance, i.e. $D(A, B) = \sup \{\delta^*(A, B), \delta^*(B, A)\}$. A mapping F from a subset I of the reals into the nonempty compact subsets of Z is called upper semicontinuous (u.s.c.) if $\forall t^0 \in I, \forall \varepsilon > 0, \exists \delta > 0 : |t - t^0| < \delta \Rightarrow F(t) \subset \subset B[F(t_0), \varepsilon]$. A mapping $F : X \rightarrow K(Z)$ is called lower semi-continuous (l.s.c.) if $\forall x^0 \in X, \forall \varepsilon > 0, \exists \delta > 0 : d(x, x^0) < \delta \Rightarrow F(x^0) \subset B[F(x), \varepsilon]$.

3. - Main results.

LEMMA. Let $E \subset \mathbf{R}$ be compact; let X be a separable metric space, Z a Banach space. Let $\Phi : E \times X \rightarrow K(Z)$ be upper semi-continuous in $t \in E$ for every $x \in X$ and lower semi-continuous in x for every $t \in E$. Then for every $\varepsilon > 0$ there exist E_ε , a compact subset of E , with $\mu(E \setminus E_\varepsilon) \leq \varepsilon$ and a single-valued continuous function $f_\varepsilon : E_\varepsilon \times X \rightarrow Z$ such that for $(t, x) \in E_\varepsilon \times X$,

$$d(f_\varepsilon(t, x), \Phi(t, x)) \leq \varepsilon.$$

PROOF. Let $D = \{x_j\}$ be a countable dense subset of X . Set $\Delta = \text{diam}(E)$. For every j set

$$\delta_j(t) = \sup \{\delta : 0 < \delta < \Delta : \exists y \in \Phi(t, x_j) : d(x, x_j) < \delta \Rightarrow d(y, \Phi(t, x)) \leq \varepsilon/2\}$$

Since Φ is l.s.c. in x for every t , the set inside parenthesis is nonempty. The following a) and b) are the two main reasons for the above definition

a) The real valued functions $\delta_j(t)$ are semi-continuous. Fix j and t^0 . We wish to prove that

$$\overline{\lim} \delta_j(t) \leq \delta_j(t^0).$$

Assume this is false; then there exist $\{t_n\}$, $t_n \rightarrow t^0$ and a positive ξ : $\delta_j(t_n) > \delta_j(t^0) + \xi$. By the very definition of δ_j , for every n there exists $y_n \in \Phi(t_n, x_j)$ such that $d(x, x_j) \leq \delta_j(t^0) + \xi/2$ implies $d(y_n, \Phi(\cdot, x)) \leq \varepsilon/2$. Since $\Phi(\cdot, x_j)$ is u.s.c. at t^0 , $d(y_n, \Phi(t^0, x_j)) \rightarrow 0$. Then from the compactness of $\Phi(t^0, x_j)$ it follows easily that there exists a subsequence converging to some $y^0 \in \Phi(t^0, x)$. Now fix any x such that $d(x, x_j) \leq \delta_j(t^0) + \xi/2$. Then

$$d(y^0, \Phi(t^0, x)) \leq d(y^0, y_n) + d(y_n, \Phi(t_n, x)) + \delta^*(\Phi(t_n, x), \Phi(t^0, x)).$$

Since $d(y^0, y_n) \rightarrow 0$, $\delta^*(\Phi(t_n, x), \Phi(t^0, x)) \rightarrow 0$ and $d(y_n, \Phi(t_n, x)) \leq \varepsilon$, it follows that $d(y^0, \Phi(t^0, x)) \leq \varepsilon/2$.

Therefore $\delta_j(t^0) + \xi/2 \leq \delta_j(t^0)$, a contradiction. This proves our claim on $\delta_j(\cdot)$.

The functions $\delta_j(\cdot)$, being semi-continuous, are measurable. Applying Lusin's Theorem we infer the existence of a compact $E_1 \subset E$ with $\mu(E \setminus E_1) \leq \varepsilon/2$ such that on E_1 each $\delta_j(\cdot)$ is continuous.

b) For every $t \in E_1$, $V_{j,t} = \{x: d(x, x_j) < \delta_j(t)/2\}$. Then $\{V_{j,t}\}$ is a covering of X (for each fixed t).

It is enough to show that if $\{x_j\}$ converges to \hat{x} , then $\lim \delta_j(t) > 0$. Consider \hat{x} : since $\Phi(t, \cdot)$ is u.s.c., there exists $\Delta > 0$: $d(x, \hat{x}) < \Delta$ implies $\Phi(t, \hat{x}) \subset B[\Phi(t, x), \varepsilon/4]$. We claim then: x_j sufficiently close to \hat{x} implies $\delta_j(t) \geq \Delta/2$. In fact let $d(x_j, \hat{x}) < \Delta/2$; let $x' \in B[x_j, \Delta/2]$, so that $d(x', \hat{x}) < \Delta$. Take any $y \in \Phi(t, \hat{x})$: there exists $y'_j \in \Phi(t, x_j)$: $d(y, y'_j) < \varepsilon/4$. Hence

$$d(y'_j, \Phi(t, x')) \leq d(y'_j, y) + d(y, \Phi(t, x')) < \varepsilon/4 + \varepsilon/4 = \varepsilon/2.$$

This proves that $\delta_j(t) \geq \Delta/2$ and our point b).

Consider now the mappings $\Psi_j: E \rightarrow 2^z$ defined by

$$\Psi_j(t) = \{y \in \Phi(t, x_j): d(x, x_j) < \delta_j(t) \Rightarrow d(y, \Phi(t, x)) \leq \varepsilon/2\}.$$

By the definition of δ_j , $\Psi_j(t)$ is non-empty. Our next claim is that the restriction of Ψ_j to E_1 is u.s.c. We shall prove first that it has closed graph. Assume this is not true: there exist t^0 and $\{t_n\}$, $t_n \rightarrow t^0$, points y_n and y^0 , with $y_n \in \Psi_j(t_n)$ and $y_n \rightarrow y^0$ such that $y^0 \notin \Psi_j(t^0)$, i.e. there exist $\xi > 0$ and \hat{x} : $d(\hat{x}, x_j) \leq \delta_j(t^0) - \xi$ but

$$d(y^0, \Phi(t^0, \hat{x})) > \varepsilon/2.$$

By the continuity of δ_j , n large implies $\delta_j(t_n) > d(\hat{x}, x_j)$, hence

$$d(y^0, \Phi(t^0, \hat{x})) \leq d(y^0, y_n) + d(y_n, \Phi(t_n, \hat{x})) + \delta^*(\Phi(t_n, \hat{x}), \Phi(t^0, \hat{x})).$$

Since $y_n \in \Psi_j(t_n)$, $d(y^0, \Phi(t^0, \hat{x})) \leq \varepsilon/2$ or

$$y^0 \in \Psi_j(t^0).$$

A contradiction, so Ψ_j has closed graph. We have in addition, that $\Phi(\cdot, x_j)$ is u.s.c. and that its images are compact sets. This implies that $\Phi(E_1, x_j)$ is compact. Finally, Ψ_j , a closed mapping whose range is contained in a compact set, is u.s.c.

Drop an open set of measure at most $\varepsilon/2$ so that on its complement $E_\varepsilon \subset E_1$ (we have $\mu(E \setminus E_\varepsilon) < \varepsilon$) each $\Psi_j(\cdot)$ is continuous. Then for every j , for every $\tau \in E_\varepsilon$, there exist $\varrho(j, \tau) > 0$ and $\eta(j, \tau) : 0 < \varrho(j, \tau) \leq \eta(j, \tau) : |\tau - t| < \varrho(j, \tau)$ implies $D(\Psi_j(t), \Psi_j(\tau)) < \varepsilon/2$ and $|\tau - t| < \eta(j, \tau)$ implies $\delta_j(t) > \frac{1}{2}\delta_j(\tau)$.

Consider the collection $\{O(j, \tau)\}$,

$$O(j, \tau) = \{(t, x) : |t - \tau| < \eta(j, \tau) \text{ and } x \in V_{j,\tau}\}.$$

It is an open covering of the paracompact $E_\varepsilon \times S$. Let $\{V(j, \tau)\}$ be a (precise) locally finite refinement, $\{p_{j,\tau}\}$ a partition of unity subordinate to $V(j, \tau)$; choose $y_{j,\tau} \in \Psi_j(\tau)$ and set

$$f_\varepsilon(t, x) = \sum p_{j,\tau}(t, x) y_{j,\tau}.$$

We claim that the above f_ε has the required properties.

In fact, fix $(t, x) \in E_\varepsilon \times S$. Let j, τ be such that $p_{j,\tau}(t, x) > 0$. Hence $(t, x) \in O(j, \tau)$, i.e.

$$\text{i) } |t - \tau| < \eta(j, \tau) \quad \text{and} \quad \text{ii) } |x - x_j| < \frac{1}{2}\delta_j(\tau).$$

From point i), there exists $\hat{y} \in \Psi_j(t) : d(\hat{y}, y_{j,\tau}) < \varepsilon/2$. Moreover $|t - \tau| < \eta(j, \tau)$ implies $\frac{1}{2}\delta_j(\tau) < \delta_j(t)$. Hence from ii) and the definition of $\Psi_j(t)$, we have

$$d(y_{j,\tau}, \Phi(t, x)) \leq d(y_{j,\tau}, \hat{y}) + d(\hat{y}, \Phi(t, x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

The convexity of $\Phi(t, x)$ implies that the same relation holds for $f_\varepsilon(t, x)$, a convex combination of $y_{j,\tau}$'s. **Q.E.D.**

THEOREM 1. Let $I \subset \mathbf{R}$ be compact, X a separable metric space $F : I \times X \rightarrow K(Z)$ be u.s.c. for every fixed $x \in X$, l.s.c. for every fixed $t \in I$. Then there exists a mapping $f : I \times X \rightarrow Z$ such that

- i) for every $(t, x) \in I \times X$, $f(t, x) \in F(t, x)$,
- ii) for every $x \in X$, $f(\cdot, x) : I \rightarrow Z$ is measurable,
- iii) for every $t \in I$, $f(t, \cdot) : X \rightarrow Z$ is continuous.

PROOF. Let $\varepsilon_n \downarrow 0: \sum \varepsilon_n < \mu(I)$. We claim first: there exist compact $E_n \subset I$ with $\mu(I \setminus E_n) \leq \varepsilon_n$ and continuous $f_n: E_n \times X \rightarrow Z$ such that

$$\begin{aligned} d(f_n(t, x), F(t, x)) &\leq \varepsilon_n, & t \in E_n, \quad n = 1, \dots, \\ d(f_n(t, x), f_{n-1}(t, x)) &\leq \varepsilon_{n-1}, & t \in E_{n-1} \cap E_n, \quad n = 2, \dots. \end{aligned}$$

For $n = 1$ set in the preceding Lemma $\varepsilon = \varepsilon_1$, $E = I$, $\Phi = F$ and call f_1 the f_ε obtained.

Assume we have constructed E_n, f_n up to $n = N - 1$. Consider $I \setminus E_{N-1}$. It is an open set; there exist C_{N-1} , a compact subset of $I \setminus E_{N-1}$, with $\mu((I \setminus E_{N-1}) \setminus C_{N-1}) < \varepsilon_N/3$. In the Lemma set $E = C_{N-1}$, $F = \Phi$, $\varepsilon = \varepsilon_N/3$ to yield:

- a compact subset K_N^1 of C_{N-1} , with $\mu(C_{N-1} \setminus K_N^1) \leq \varepsilon_N/3$ and
- a function $f^1(t, x): K_N^1 \times X \rightarrow Z$ such that

$$d(f^1(t, x), F(t, x)) \leq \varepsilon_N/3 < \varepsilon_N.$$

Consider now the set $E_{N-1} \times X$ and the mapping $\Phi: E_{N-1} \times X \rightarrow K(Z)$ defined by

$$\Phi(t, x) = F(t, x) \cap \overline{B[f_{N-1}(t, x), \varepsilon_{N-1}]}$$

By our induction assumption, $\Phi(t, x)$ is non-empty. Moreover it is compact and convex. In addition it is u.s.c. in $t \in E_{N-1}$ for every fixed $x \in X$ (its graph is the intersection of two closed graphs and the range is contained in a compact set) and l.s.c. in x for every fixed t [1].

Applying the Lemma to Φ , E_{N-1} and ε_N , we infer the existence of a compact $K_N^2 \subset E_{N-1}$, $\mu(E_{N-1} \setminus K_N^2) < \varepsilon_N$ and a $f^2: K_N^2 \times X \rightarrow Z$ such that

$$d(f^2(t, x), \Phi(t, x)) < \varepsilon_N.$$

Hence for f^2 both

$$d(f^2(t, x), f_{N-1}(t, x)) \leq \varepsilon_{N-1}$$

and

$$d(f^2(t, x), F(t, x)) < \varepsilon_N \quad \text{hold}.$$

Set $E_N = K_N^1 \cup K_N^2$ and define $f_N: E_N \times X \rightarrow Z$ by

$$f_N(t, x) = \begin{cases} f^1(t, x), & t \in K_N^1, \\ f^2(t, x), & t \in K_N^2. \end{cases}$$

We have that $\mu(I \setminus E_N) = \mu((E_{N-1} \setminus K_N^2) \cup ((I \setminus E_{N-1}) \setminus K_N^1)) \leq \varepsilon_N/3 + 2\varepsilon_N/3 = \varepsilon_N$, and the claim is proved.

Now set

$$A_N = \bigcup_{n=N}^{\infty} (I \setminus E_n).$$

Then $A_N \subset A_{N-1}$ and $\mu(\bigcap A_N) = \lim \mu(A_N) = 0$. Fix $t \notin \bigcap A_N$. Then $\{f_N(t, \cdot)\}$ is a Cauchy sequence of continuous functions and converges uniformly to a $\varphi(t, x)$, continuous in x . Fix x . Then for every $t \notin \bigcap A_N$, $\varphi(t, x)$ is the pointwise limit of $f_N(t, x)$, hence measurable. For $t \in \bigcap A_N$, let $\hat{\varphi}(t, \cdot)$ be any continuous selection from $F(t, \cdot)$ [1].

The function

$$f(t, x) = \begin{cases} \varphi(t, x), & t \in I \setminus \bigcap A_N, \\ \hat{\varphi}(t, x), & t \in \bigcap A_N, \end{cases}$$

has the required properties. Q.E.D.

From Theorem 1 the following Theorem 2 can easily be proved:

THEOREM 2. Let Z be a finite dimensional space, Ω an open subset of $R \times Z$, $F: \Omega \rightarrow K(Z)$ be u.s.c. in t for every fixed x and l.s.c. in x for every fixed t , t and x in Ω . Moreover assume that the range of F is contained in some compact subset of Z . Let $(t^0, x^0) \in \Omega$. Then the Cauchy problem

$$x' \in F(t, x), \quad x(t^0) = x^0$$

admits at least one solution.

Also, applying a result of Scorza Dragoni [2] to the function f of Theorem 1, the following Corollary can be derived:

COROLLARY. Let $I \subset R$ be compact, X a separable metric space, $F: I \times X \rightarrow K(Z)$ be u.s.c. for every fixed $x \in X$, l.s.c. for every fixed $t \in I$. Then for every $\varepsilon > 0$ there exist K_ε , a compact subset of I and a continuous $f_\varepsilon: K_\varepsilon \times X \rightarrow Z$ that is a selection from F .

REFERENCES

- [1] E. MICHAEL, *Continuous selections*, Ann. Math., **63** (1956), pp. 361-362.
- [2] G. SCORZA DRAGONI, *Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile*, Rend. Sem. Mat. Univ. Padova, **17** (1948), pp. 102-108.

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