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A Variational Principle, in General Relativity, for the Equilibrium of an Elastic Body also Capable of Couple Stresses.

LUCIANO BATTAIA (*)

SUMMARY - We prove a variational theorem for the equilibrium of an elastic body, also capable of couple-stresses but not of heat conduction, in general relativity. It can be considered as the relativization of the classical principle of stationary potential energy.

1. Introduction.

In classical physics equilibrium problems are often proved to be equivalent to variational principles. One of these principles is the principle of stationary potential energy, firstly formulated within the linear theory for infinitesimal deformations and then extended to the non lineary theory for finite deformations but, in both cases, for materials not capable of couple-stresses (non-polar materials)-see for example [12], [14].

In this work we present an extension of this equivalence theorem-to general relativity and we take also polar materials (and finite deformation) into account. The work is based on the Eulerian and Lagrangian theories of continuous media in general relativity as formulated by Bressan (§ 2), and on a certain variational principle, involving the

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variation of world lines. This principle was first formulated by Taub in [13] and improved by Fock cf. [19], § 47; then it was extended by Schöpf and Bressan to the non-polar and polar cases respectively.

As a preliminary, in § 4 we state the dynamical equations in mixed form in general relativity for polar materials, thus extending the corresponding result of Bressan for non-polar materials, cf. [2].

In § 5 we present and prove the equivalence of the aforementioned variational principle to the corresponding conservation equations, which include the dynamical equations of the equilibrium.

In § 6 we compare our principle with the classical principle of stationary potential energy.

2. Preliminaries.

We first recall some fundamental concepts of the Eulerian and Lagrangian theories of continuous media in general relativity presented by Bressan in [1]-[7].

Let S_4 be the space time of general relativity with the metric (1)

$$ds^2 = -g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{1}$$

whose signature is determined by the condition that it can everywhere be reduced locally to the pseudo-Euclidean form:

$$g_{rs} = \delta_{rs}\ ; \qquad g_{\alpha 0} = g_{0\alpha} = -\delta_{\alpha 0}\ , \tag{2}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol.

The co-ordinate system (x) is assumed to be admissible: the hypersurfaces $x_0 =$ const are space like and x^0 increases towards future.

Let $\mathcal{C}$ be a body. We may consider the process $\mathcal{T}$ of the universe in S_4 as consisting only of the motion $\mathcal{M}$ of $\mathcal{C}$, the temperature distribution in the world-tube $W_{\mathcal{C}}$ of $\mathcal{C}$, and the metric tensor field over S_4.

We consider only regular motions of the body $\mathcal{C}$ for which, among other things, $\mathcal{C}$ can be regarded as a collection of material points. We denote the typical one of them by P^*, and the four velocity (intrinsic acceleration) of $\mathcal{C}$ at its material point P^* by $u^\alpha(A^\alpha)$.

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(1) Greek and Latin indices run over 0 1, 2, 3, and 1, 2, 3 respectively.

The frame (x) is called locally natural and proper at the event point x^{ϱ} if there (2) and

$$g_{\alpha\beta,\gamma} = 0\ ; \qquad u^{\alpha} = \delta^{\alpha}{}_{0} \qquad (f,\alpha = \partial f / \partial x^{\alpha}) \tag{3}$$

hold.

We use the notation $T^{\cdots}_{\ldots/\gamma}$ for the covariant derivative:

$$T^{\alpha}{}_{\beta/\gamma} = T^{\alpha}{}_{\beta,\gamma} - \begin{Bmatrix} \sigma \\ \beta\gamma \end{Bmatrix} T^{\alpha}_{\sigma} + \begin{Bmatrix} \alpha \\ \sigma\gamma \end{Bmatrix} T^{\sigma}_{\beta}\,. \tag{4}$$

We also consider the spatial projector

$$\overset{\perp}{g}_{\alpha\beta} = g_{\alpha\beta} + u_{\alpha} u_{\beta} \tag{5}$$

and the following notations

$$T^{\cdots}_{\ldots\overset{\perp}{\alpha}} = \overset{\perp}{g}{}_{\alpha}{}^{\gamma}\, T^{\cdots}_{\ldots\gamma}\,; \qquad \overset{\perp}{T}{}^{\gamma\,\cdot\cdot}_{\alpha\,\cdot\cdot} = T^{\overset{\perp}{\gamma}\cdots}_{\overset{\perp}{\alpha}\cdots}\,, \tag{6}$$

recalling that the index α of the tensor $T^{\cdots}_{\ldots\alpha}$ is said to be spatial if $T^{\cdots}_{\ldots\alpha} u^{\alpha} = 0$ or, equivalently, if $T^{\cdots}_{\ldots\alpha} = T^{\cdots}_{\ldots\overset{\perp}{\alpha}}$.

We also use the spatial derivative and divergence of any tensor:

$$T^{\ldots\beta}_{\ldots\alpha/\overset{\perp}{\gamma}} = \overset{\perp}{g}{}^{\varrho}_{\gamma}\, T^{\ldots\beta}_{\ldots\alpha/\overset{\perp}{\varrho}}\ ; \qquad T^{\ldots\beta}_{\ldots\alpha/\overset{\perp}{\beta}} = \overset{\perp}{g}{}^{\varrho}_{\beta}\, T^{\ldots\beta}_{\ldots\alpha/\overset{\perp}{\varrho}} \tag{7}$$

and the natural decomposition of a tensor with respect to the index α. This decomposition for a vector v_{α} is:

$$v_{\alpha} = v_{\overset{\perp}{\alpha}} - v_{\beta} u^{\beta} u_{\alpha} \tag{8}$$

and $v_{\overset{\perp}{\alpha}}(-v_{\beta} u^{\beta} u_{\alpha})$ is called the spatial (temporal) part of v_{α} (2).

The proper density ϱ of total internal energy is defined by:

$$dm = c^{-2} \varrho\, dC \tag{9}$$

where dm is the proper gravitational mass of the element $d\mathcal{C}$ of $\mathcal{C}$ and dC the actual proper volume of $d\mathcal{C}$.

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(2) For more details on this subject see Cattaneo [8].

Now we fix a process $\mathcal{T}^*$ (reference process) physically possible for the universe (containing $\mathcal{C}$) and we define S_4^*, $W_{\mathcal{C}}^*$, $\mathcal{M}^*$, $g_{\lambda\mu}^*$, dC^* to be corresponding instances of S_4, $W_{\mathcal{C}}$, $\mathcal{M}$, $g_{\alpha\beta}$ and dC. We consider an admissible system of co-ordinates (y) and the intersection S_3^* of $W_{\mathcal{C}}^*$ with the hypersurface $y^0 = 0$. For every material point P^* of $\mathcal{C}$ we shall use to co-ordinate y^L of the intersection of S_3^* with the world line of P^* as the L-th material coordinate. The spatial metric in S_3^* will be called ds^{*2}:

(10) $$ds^{*2} = a_{LM}^* dy^L dy^M \qquad \text{where} \qquad a_{rs}^* = \hat{a}_{rs}(y^L) = g_{rs}^*(0, y', y^2, y^3)\,.$$

The co-ordinates y^L and their increments dy^L then caracterize the material points P^* and the linear infinitesimal material elements at P^*.

Let Σ^* be the intrinsic state of $\mathcal{C}$ in $\mathcal{T}^*$; we call $k^* dC^*$ the proper gravitational mass of $d\mathcal{C}$ in Σ^*. It is related to the actual volume dC by the following definition of the density k:

(11) $$k^* dC^* = k\, dC\ ; \qquad \text{hence } k^* = k\mathfrak{D},\ \ \mathfrak{D} = dC/dC^*.$$

As a consequence of this definition k satisfies the continuity equation

(12) $$(ku^\alpha)_{/\alpha} \equiv \frac{Dk}{Ds} + ku^\alpha{}_{/\alpha} = 0\,.$$

Let us set $c^{-2} wk\, dC = c^{-2} \varrho\, dC - k\, dC$. Then by the equivalence principle of mass and energy $wk\ dC$ can be regarded as the internal energy of $d\mathcal{C}$. Furthermore:

(13) $$\varrho = k(c^2 + w)\,.$$

We also set

(14) $$\varrho^* = \varrho\mathfrak{D} = k^*(c^2 + w)\,.$$

An arbitrary motion of $\mathcal{C}$ in the system of co-ordinates (x) is represented by the equations [3]

(15) $$x^\alpha = \hat{x}^\alpha(t, y^1, y^2, y^3)\,[= \hat{x}^\alpha(t, y^L) = \hat{x}^\alpha(y^\Sigma),\ t \equiv y^0]\,.$$

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(3) Capital and lower case letters represent material and space time indices respectively.

For the regularity conditions on the functions (15) and their indetermination in the choice of the time parameter $t = t(x)$ see Bressan [1]; here we only remember that $t = t(x)$ can be chosen in such a way that

$$u_L^{+} \equiv u^{\varrho} x_L^{\varrho} = 0 \, , \qquad (x_L^{\varrho} = \partial x^{\varrho} / \partial y^{L}) \tag{16}$$

holds at an arbitrarily pre-assigned event $\mathcal{E}$.

Let now $T^{\alpha \dots L \dots}_{\beta \dots M \dots}$ be a double tensor field—cf. Ericksen [9]—associated to the event point x^{ϱ} and the material point y^{L}. We shall think of it as depending from x^{ϱ}, y^{L} and the time parameter t, with x^{ϱ} and y^{L} connected by means of the equations of motions (15). Then the total covariant derivative based on the map (15) is defined by

$$T^{\alpha \dots L \dots}_{\beta \dots M \dots ; P} = T^{\dots}_{\dots / \varrho} x^{\varrho}_{\varrho} + T^{\dots}_{\dots / P} \, . \tag{17}$$

It depends on the particular representation of the motion through the time parameter t. In case (16) holds it is called Lagrangian spatial derivative and denoted by $T^{\dots}_{\dots | P}$.

Then we introduce the first and second deformation gradient

$$\alpha_L^{\varrho} = \overset{\perp}{g}{}^{\varrho}_{\sigma} x_L^{\sigma} \; ; \qquad \alpha^{\varrho}_{LM} = \alpha^{\varrho}_{L|M} \, , \tag{18}$$

the first and second Cauchy-Green tensor

$$C_{ML} = \alpha_{\varrho M} \alpha_L^{\varrho} \; ; \qquad C_{MLB} = \alpha_M^{\varrho} \alpha_{\varrho LB} \, , \tag{19}$$

and lastly the deformation tensor ε_{LM}

$$\overset{*}{a}_{LM} + 2 \varepsilon_{LM} = C_{LM} \, . \tag{20}$$

We have (4)

$$\alpha^{\varrho}_{[AB]} = 0 \; ; \qquad C_{[AB]} = \varepsilon_{[AB]} = 0 \; ; \qquad C_{L[AB]} = 0. \tag{21}$$

The spatial inverse $\mathfrak{D}\gamma_{\varrho}^{L}$ of α_L^{ϱ} is defined by

$$\alpha_L^{\varrho} \gamma_{\sigma}^{L} = \mathfrak{D} \overset{\perp}{g}{}^{\varrho}_{\sigma} \qquad \text{hence } u^{\varrho} \gamma_L^{\varrho} = 0 \, . \tag{22}$$

(4) We use the notations $2T_{(\alpha\beta)} = T_{\alpha\beta} + T_{\beta\alpha}$; $2T_{[\alpha\beta]} = T_{\alpha\beta} - T_{\beta\alpha}$.

If $X^{\varrho\sigma}$ is an arbitrary Eulerian double tensor, its mixed and Lagrangian counterparts are defined by

$$(23)\qquad K^{\varrho M} = X^{\varrho\sigma}\gamma_\sigma^M; \qquad Y^{LM} = \frac{1}{\mathfrak{D}}\gamma_\varrho^L K^{\varrho M}.$$

In the sequel $X^{\varrho\sigma}$ will denote the Eulerian stress tensor so that its mixed and Lagrangian counterparts defined by (23) are called first and second Piola stress tensor, see [15].

We are interested in an elastic body $\mathcal{C}$ capable of couple-stress but not of heat conduction and we assume, also for the sequel the absence of electromagnetic phenomena. We consider an event $\mathcal{E}$ in the world tube $W_\mathcal{C}$ of $\mathcal{C}$ and we suppose that the vector $d\sigma_{\sigma\alpha}$ represents the infinitesimal oriented spatial surface $d\sigma$ at $\mathcal{E}$. We admit that the forces exerted by the material elements contiguous to the negative face of $d\sigma$ on those contiguous to the positive one are caracterizable by means of the resultant dR^α and the intrinsic resultant moment $d\mathcal{M}^\alpha \cdot dR^\alpha$ is expressed by

$$(24)\qquad dR^\alpha = X^{\alpha\beta} d\sigma_\beta \qquad \text{with } X^{\alpha\beta}u_\beta = 0 = u_\alpha X^{\alpha\beta}$$

while $d\mathcal{M}^\alpha$ is expressed by

$$(25)\qquad d\mathcal{M}^\alpha = \overset{\perp}{\varepsilon}_{\alpha\varrho\sigma} m^{\varrho\sigma\beta} d\sigma_\beta \qquad \text{with } m^{(\varrho\sigma)\beta} = 0 = u_\varrho m^{\varrho\sigma\beta} = m^{\varrho\sigma\beta}u_\beta,$$

where $\overset{\perp}{\varepsilon}_{\alpha\varrho\sigma}$ is the spatial Ricci tensor $u^\varrho \varepsilon_{\varrho\alpha\beta\gamma}$ (5).

As well as the stress $X^{\alpha\beta}$, the tensor $m^{\alpha\beta\gamma}$ of couple stress depends on $\mathcal{E}$ but not on $d\sigma_\beta$.

We define

$$(26)\qquad \mathcal{M}^{\alpha\beta} = m^{\alpha\lambda\beta}{}_{/\lambda} + m^{\beta\lambda\alpha}{}_{/\lambda} + \nu^\alpha u^\beta + u^\alpha \nu^\beta \qquad (\nu^\alpha = 2m{:}^{(\alpha\varrho\sigma)} u_{\varrho/\sigma}).$$

Then the total energy tensor is:

$$(27)\qquad \mathcal{U}_{\alpha\beta} = \varrho u_\alpha u_\beta + X_{(\alpha\beta)} + \mathcal{U}_{\alpha\beta}.$$

(5) $\mathcal{E}_{\alpha\beta\gamma\delta}$ is the usual permutation symbol with $\mathcal{E}_{0123} = 1$. The Ricci tensor is defined by $\varepsilon_{\alpha\beta\gamma\delta} = \sqrt{-g}\,\mathcal{E}_{\alpha\beta\gamma\delta}$ $\{g = \det\|g_{\alpha\beta}\| < 0)$.

The temporal part of the conservation equations

$$\mathcal{U}^{\alpha\beta}{}_{/\beta} = 0 \, , \tag{28}$$

can satisfactorily be taken as the relativization of the first principle of thermodynamics and can be put into the form

$$K \frac{Dw}{Ds} + \frac{dl^{(i)}}{Ds} = 0 \tag{29}$$

where $dl^{(i)}$ is the work of internal contact forces

$$\frac{dl^{(i)}}{Ds} = - u_\alpha (X^{(\alpha\beta)} + \mathcal{M}^{\alpha\beta})_{/\beta} \, . \tag{30}$$

The spatial part of (28)

$$\overset{\perp}{g}_{\alpha\gamma} (\varrho u^\gamma u^\beta + X^{(\gamma\beta)} + \mathcal{M}^{\gamma\beta})_{/\beta} = 0 \tag{31}$$

can be taken as the relativization of the first Cauchy equation of continuous media [6].

3. Variation of world lines in the elastic adiabatic case.

Introducing the Lagrangian counterpart.

$$m_*^{BLM} = \mathfrak{D}^{-2} \gamma^B_\beta \gamma^L_\lambda \gamma^M_\mu m^{\beta\lambda\mu} \tag{32}$$

of $m^{\beta\lambda\mu}$, the constitutive equations of the elastic body $\mathcal{C}$ capable of

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[6] For more detail on the definition of $\mathcal{M}^{\alpha\beta}$ and on the acceptability in general relativity of the equations (29), (31) and the expression (30) of the work of internal contact forces see the works of Bressan. When thermodynamic and electromagnetic phenomena are taken into account, Eckart's tensor $Q_{\alpha\beta}$ and the electromagnetic tensor $E^{\alpha\beta}$ should also be added to the expression (27) of $U^{\alpha\beta}$. However no equation involving $Q_{\alpha\beta}$ or $E_{\alpha\beta}$ will be considered in the sequel.

couple stress but not of heat conduction are

$$(33)\qquad \begin{aligned} &w = \hat{w}(\varepsilon_{LM}, C_{BLM}, y^L); \qquad m_*^{BLM} = K^* \frac{\partial w}{\partial C_{[BL]M}} + \lambda \varepsilon^{*BLM} \\ &Y^{(LM)} = -K^* \frac{\partial w}{\partial \varepsilon_{LM}} - 2m_*^{(LAB} \overset{-1}{C}{}^{M)S} C_{SAB}; \qquad \frac{\partial w}{\partial C_{(BL)M}} = 0 \end{aligned}$$

where $\lambda = \lambda(\varepsilon_{LM}, C_{BLM})$ and ε^{*BLM} is the spatial Ricci tensor is S_3^*.

Now we suppose the field $g_{\alpha\beta}$ and the motion $\mathcal{M}$ of $\mathcal{C}$ as given in S_4. We consider a regular motion $\mathcal{M}_\lambda$ depending on the real parameter λ

$$(34)\qquad z^\alpha = {}^*z^\alpha(\lambda, y^\Sigma) \quad \text{with } {}^*z^\alpha(0, y^\Sigma) = \hat{x}^\alpha(y^\Sigma),$$

and the functions $z^\alpha(\lambda, x^\varrho)$ defined by the condition

$$(35)\qquad z^\alpha = z^\alpha[\lambda, \hat{x}^\varrho(y^\Sigma)] = {}^*z^\alpha(\lambda, y^\Sigma);$$

furthermore we set

$$(36)\qquad \xi^\varrho = \xi^\varrho(x) = \left[\frac{\partial z^\varrho(\lambda, x)}{\partial \lambda}\right]_{\lambda=0}.$$

It follows that $\xi^\varrho d\lambda$ is the displacement of the material point y^L (at the instant $t = y^0$) in the correspondence $\mathcal{M}_0 \to \mathcal{M}_{d\lambda}$.

Consider the functional

$$(37)\qquad I = \int_{C_4} \varrho \sqrt{-g}\, dx \qquad (dx = dx_0\, dx^1\, dx^2\, dx^3)$$

and the variation $\xi^\varrho d\lambda$ of the motion $\mathcal{M}$ of $\mathcal{C}$ that is of class $C^{(s)}$ and satisfies

$$(38)\qquad \xi^\varrho = 0 = \xi^\varrho{}_{,\sigma}$$

on the boundary $\mathcal{F}C_4$ of C_4.

Then in [6] it is proved that for these variations we have

$$(39)\qquad \delta \int_{C_4} \varrho \sqrt{-g}\, dx = d\lambda \int_{C_4} (\varrho u^\varrho u^\sigma + X^{(\varrho\sigma)} + \mathcal{M}^{\varrho\sigma})_{/\sigma} \xi_\varrho \sqrt{-g}\, dx.$$

Hence the validity of the variational condition

$$\delta I = 0 \tag{40}$$

is equivalent to the conservation equations (28).

4. Mixed form of the dynamical equations.

In the absence of electromagnetic and heat conduction phenomena the spatial part of $\mathcal{U}^{\alpha\beta}{}_{/\beta} = 0$ is:

$$\overset{\perp}{g}_{\alpha\beta}(\varrho u^{\gamma} u^{\beta} + X^{(\beta\gamma)} + \mathcal{M}^{\gamma\beta})_{/\beta} \equiv (\varrho \overset{\perp}{g}_{\alpha\gamma} + X_{(\gamma\beta)}) A^{\beta} + \overset{\perp}{g}_{\alpha\gamma} X^{(\gamma\beta)}{}_{/\overset{\perp}{\beta}} + \overset{\perp}{g}_{\alpha\gamma} \mathcal{M}^{\gamma\beta}{}_{/\beta} = 0 \, . \tag{41}$$

We call $M^{\alpha} = -\overset{\perp}{g}{}^{\alpha}_{\sigma} \mathcal{M}^{\sigma\beta}{}_{/\beta}$ the resultant of the forces due to couple stresses and, in analogy with Bressan [2], we introduce the resultant

$$M^{*\alpha} = \mathfrak{D} M^{\alpha} \tag{42}$$

of the same forces per unit volume of reference configuration.

By a useful formula proved in [2] we have

$$\mathfrak{D} X^{(\beta\gamma)}{}_{/\overset{\perp}{\beta}} = K^{\gamma M}{}_{|M} \, . \tag{43}$$

Let us multiply (41) by $\mathfrak{D}(\neq 0)$ then by (14), (23), (41), (42) and (43) we have

$$\mathfrak{D} \overset{\perp}{g}_{\alpha\gamma}(\varrho u^{\gamma} u^{\beta} + X^{(\gamma\beta)} + \mathcal{M}^{\gamma\beta})_{/\beta} = (\varrho^{*} \overset{\perp}{g}_{\alpha\beta} + K_{\alpha}{}^{M} \alpha_{\beta M}) A^{\beta} + \overset{\perp}{g}_{\alpha\gamma} K^{\gamma M}{}_{|M} - M^{*}_{\alpha} \, , \tag{44}$$

from which the following mixed form of the dynamical equations follows

$$(\varrho \overset{\perp}{g} g_{\alpha\beta} + K_{\alpha}{}^{M} \alpha_{\beta M}) A^{\beta} + \overset{\perp}{g}_{\alpha\gamma} K^{\gamma M}{}_{|M} - M^{*}_{\alpha} = 0 \; (^{7}). \tag{45}$$

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(7) Recalling that $K_{\alpha}{}^{M} = \alpha_{\alpha L} Y^{LM}$, from (45) we can also deduce the completely Lagrangian form of the same equations.

5. Varational principle for equilibrium problems.

We now suppose that the space-time S_4 is stationary and introduce a stationary system of reference (x). Let C_3 be the intersection of the world tube $W_{\mathcal{C}}$ of $\mathcal{C}$ with the hypersurface $x^0 = 0$.

We want to consider the equilibrium of $\mathcal{C}$ with respect to the stationary frame of reference (x). We identify the arbitrary parameter t in the equations (15) of the motion of $\mathcal{C}$ with x^0 and denote the configuration of $\mathcal{C}$ in C_3 by $x^r = \chi^r(y^L)$. Then the equations of the motion of $\mathcal{C}$ can be put into the form

$$
(46) \qquad \begin{aligned} x_0 &= t \\ x^r &= x^r(t, y^2) = \chi^r(y^2)\,. \end{aligned}
$$

The functions $\chi^r(y^L)$ are such that (46) satisfy the regularity conditions requested for the equations of motion.

The hypothesis of stationarity of (x) implies

$$
(47) \qquad g_{\alpha\beta,0} = 0\,.
$$

Consider the functional

$$
(48) \qquad J^* = J^*(\chi^r) = \int_{S_3^*} \varrho^* \sqrt{a^*}\, dy \qquad (dy = dy^2, dy^2 \cdot dy^2)
$$

where $\varrho^* = \varrho^*(\varepsilon_{LM}, C_{BLM,y})$ and J^* depends of χ^r through ε_{LM} and $C_{[BL]M}$, and an arbitrary variation $\delta\chi^r$ of χ^r, that is of class $C^{(\varepsilon)}$ and vanishes on the boundary $\mathcal{F}S_3^*$ of S_3^* together with $\delta\chi^r{}_{,L}$.

We shall prove the following:

a) the first principle of thermodynamics holds for the body $\mathcal{C}$ (not capable of heat conduction);

b) for the aforementioned variations $\delta\chi^r$ we have

$$
(49) \qquad \delta J^* = \int_{S_3^*} \{(\varrho^* \overset{\perp}{g}_{r\varrho} + K_r^L \alpha_{\varrho L}) A^\varrho + g_{r\varrho} K^{\varrho M}{}_{IM} - M_r^*\} \delta\chi^r \sqrt{a^*}\, dy\,,
$$

Hence the validity of the variational condition

$$
(50) \qquad \delta J^* = 0
$$

is equivalent to the dynamical equations (45) of the equilibrium [8].

Let $\mathcal{C}$ undergo the motion (46). We first prove (*a*). Since heat conduction is absent, the first principle of thermodynamics reads

$$k\frac{Dw}{Ds}+\frac{dl^{(i)}}{Ds}=0\,. \tag{51}$$

By (46) $D\varepsilon_{LM}/Ds=0=DC_{[BL]M}/Ds$. Hence on the one hand from $w=\tilde{w}(\varepsilon_{LM}, C_{[BL]M})$ we deduce $Dw/Ds=0$. With regard to $dl^{(i)}/Ds$ we know from Bressan [6] that [9]

$$\frac{dl^{(i)}}{Ds}=\frac{1}{\mathfrak{D}}\frac{d^*l^{(i)}}{Ds}=\frac{1}{\mathfrak{D}}[Y^{(BR)}+2m_*^{(BLM}\overset{-1}{C}{}^{R)S}C_{SLM}]\cdot \cdot\frac{D\varepsilon_{BR}}{Ds}-\frac{m_*^{BLM}}{\mathfrak{D}}\frac{DC^{[BL]M}}{Ds}\,, \tag{52}$$

so that, on the other hand, we have $dl_{(i)}/Ds=0$.

Now we consider the set C_4 of the points x^ϱ of $W_{\mathcal{C}}$ with $|x^0|\leqslant a+1$ (a real positive). Let C_4^a be the subset of C_4 where $|x^0|\leqslant a$, let C_4^+ be the subset where $a\leqslant x^0\leqslant a+1$, and let C_4^- be the subset were $-(a+1)\leqslant x^0\leqslant -a$.

Then we consider any function, of class $C^{(3)}$, $\varphi(\xi)$, of the real variable ξ, for which

$$\begin{cases}\varphi'(0)=\varphi''(0)=\varphi'''(0)=0\\ \varphi(0)=1;\qquad \varphi(1)=\varphi'(1)=0\\ \int_0^1\varphi(\xi)\,d\xi=0\,.\end{cases} \tag{53}$$

---

(8) It must also be remembered that the vector on the left hand side of (45) is purely spatial, hence it vanishes if and only if its components with $\alpha=1,2,3$ vanish.

(9) If $dl^{(i)}/Ds$ and $d^*l_{(i)}/Ds$ represent the power of internal contact forces for unity of actual and reference configuration respectively, we have

$$d^*l^{(i)}=\mathfrak{D}\,dl^{(i)}\,.$$

Next we introduce the arbitrary variations $\delta\chi^r$, of class $C^{(3)}$, that vanish on $\mathcal{F}S_3^*$ together with $\delta\chi^r{}_{,L}$; we also consider the corresponding δx^r of the x^r as given by (46) and we set $\delta x^0 = 0$. We regard these δx^r as functions $\delta x^r(x^s)$ of the space time co-ordinates x^s by means of the inverses of the functions $x^r = \chi^r(y^L)$. Then $\delta x^r = 0 = \delta x^r{}_{,s}$ on $\mathcal{F}C_3$. We also suppose that, for x^r in C_3, the functions that appear in (39) are given by

$$\text{(54)}\qquad \begin{cases} \xi_0\, d\lambda = 0 \quad \text{in } C_4 \\ \xi^r(x^0, x^r)\, d\lambda = \begin{cases} \delta x^r(x^s) & |x^0| \leqslant a \\ \varphi(x^0 - a)\,\delta x^r(x^s) & a \leqslant x^0 \leqslant a+1 \\ \varphi(-x^0 - a)\,\delta x^r(x^s) & -(a+1) \leqslant x^0 \leqslant -a\,. \end{cases} \end{cases}$$

By (53) these functions are of class $C^{(3)}$ in C_4 and vanish on $\mathcal{F}C_4$ together with $\xi^\varrho{}_{,\sigma}$.

For these variations we have, remembering that $\varrho\sqrt{-g}$ does not depend on x^0,

$$\text{(55)}\qquad \delta\int_{C_4^+} \varrho\sqrt{-g}\,dx = \int_{C_4^+} \frac{\partial\varrho}{\partial x^r}\,\delta x^r \varphi(x^0 - a)\sqrt{-g}\,dx = \\ = \int_{C_3} \frac{\partial\varrho}{\partial x^r}\,\delta x^r\sqrt{-g}\; d_3x \int_a^{a+1} \varphi(x^0 - a)\,dx^0 = 0\,,$$

and likewise

$$\text{(56)}\qquad \delta\int_{C_4^-} \varrho\sqrt{-g}\,dx = 0\,.$$

Then, as $\varrho u_\alpha u_\beta + X_{\alpha\beta} + \mathcal{M}_{\alpha\beta}$ does not depend on x^0,

$$\text{(57)}\qquad d\lambda\int_{C_4^+} U_{\varrho\sigma}{}^{/\sigma}\zeta^\varrho\sqrt{-g} = \int_{C_4^+} (\varrho u_r u_\sigma + X_{(r\sigma)} + \mathcal{M}_{r\sigma})^{/\sigma}\varphi(dx^0 - a)\,\delta x^r\sqrt{-g}\,dx \\ = \int_{C_3} (\varrho u_r u_\sigma + X_{(r\sigma)} + \mathcal{M}_{r\sigma})^{/\sigma}\delta x_r\sqrt{-g}\,d_3x \int_a^{a+1}\varphi(x^0 - a)\,dx^0 = 0\,,$$

and analogously

$$\text{(58)}\qquad d\lambda\int_{C_4^-} (\varrho u_\varrho u_\sigma + X_{(\varrho\sigma)} + \mathcal{M}_{\varrho\sigma})^{/\sigma}\xi^\varrho\sqrt{-g}\,dx = 0\,.$$

We then have

$$(59)\qquad \int_{C_4^+} \varrho\sqrt{-g}\,dx = \int_{-a}^{a} dx^0 \int_{C_3} \varrho\sqrt{-g}\,d_3x = 2a\int_{C_3} \varrho\sqrt{-g}\,d_3x =$$

$$= 2a\int_{S_3^*} \varrho\sqrt{a^*}\,\mathfrak{D}\,dy = 2a\int_{S_3^*} \varrho^*\sqrt{a^*}\,dy\ ,$$

hence, see also (55) and (56),

$$(60)\qquad \delta\int_{C_4} \varrho\sqrt{-g}\,dx = 2a\delta\int_{S_8^*} \varrho^*\sqrt{a^*}\,dy$$

for the variations $\xi^\varrho d\lambda$ and $\delta\chi^r$ considered above. Furthermore, remembering (57), (58), and that the temporal part of $\mathcal{U}^{\alpha\beta}{}_{/\beta}$ vanishes identically (first principle of thermodynamics), we have

$$(61)\qquad d\lambda\int_{C_4} (\varrho u_\varrho u_\sigma + X_{(\varrho\sigma)} + \mathcal{M}_{\varrho\sigma})^{/\sigma}\xi^\varrho\sqrt{-g}\,dx =$$

$$= \int_{C_4^a} (\varrho u_r u_\sigma + X_{(r\sigma)} + \mathcal{M}_{r\sigma})^{/\sigma}\delta x^r\sqrt{-g}\,dx =$$

$$= \int_{C_4^a} \overset{\perp}{g}_{r\varrho}(\varrho u^\varrho u^\sigma + X^{(\varrho\sigma)} + \mathcal{M}^{\varrho\sigma})_{/\sigma}\delta x^r\sqrt{g-}\,dx =$$

$$= \int_{-a}^{a} dx^0 \int \overset{\perp}{g}_{r\varrho}(\varrho u^\varrho u^\sigma X^{(\varrho\sigma)} + \mathcal{M}^{\varrho\sigma})_{/\sigma}\delta x^r\sqrt{-g}\,g\,d_3x =$$

$$= 2a\int_{S_3^*} \overset{\perp}{g}_{r\varrho}(\varrho u^\varrho u^\sigma + X^{(\varrho\sigma)} + \mathcal{M}^{\varrho\sigma})_{/\sigma}\delta\chi^r(y)\mathcal{M}\sqrt{a^*}\,dy =$$

$$= 2a\int_{S_3^*} \{(\varrho^*\overset{\perp}{g}_{r\varrho} + K^L_r\alpha_{\varrho L})A^\varrho + g_{r\varrho}K^{\varrho M}{}_{|M} - M^*_r\}\,\delta\chi^r\sqrt{a^*}\,dy\ .$$

From (60), (61) and (39) we obtain the validity of

$$(62)\qquad \delta\int_{S_3^*} \varrho^*\sqrt{a^*}\,dy = \int_{S_*^3} \{(\varrho^*\overset{\perp}{g}_{r\varrho} + K^L_r\alpha_{\varrho L})A^\varrho + \overset{\perp}{g}_{r\varrho}K^{\varrho M}{}_{|M} - M^*_r\}\,\delta\chi^r\sqrt{a^*}\,dy$$

for the variations $\delta\chi^r$ of χ^r that are of class $C^{(3)}$ and vanish on $\mathcal{F}S_3^*$ together with their first partial derivatives. Q.E.D.

6. Comparison with a classical principle.

We now consider an elastic body $\mathcal{C}$ not capable of couple stresses, in classical physics. If W is the elastic potential energy, the constitutive equations of $\mathcal{C}$ are ([10])

$$Y^{LM} = -\mu^* \frac{\partial W}{\partial \varepsilon_{LM}} \tag{63}$$

where μ^* is the mass density in the reference configuration.

We suppose that the body forces are conservative, that is there exists a potential $U = U(x)$ such that

$$f^r = U_{,}{}^{r} . \tag{64}$$

Then the equations of the equilibrium of $\mathcal{C}$ are

$$\left(\mu^* \frac{\partial W}{\partial \alpha_{rL}}\right)_{;L} + \mu^* U_{,}{}^{r} = 0 . \tag{65}$$

Consider the functional

$$J(x^r) = \int_{\mathcal{C}} \mu^* W \, dC^* - \int_{\mathcal{C}} \mu^* U \, dC^* \tag{66}$$

and any variation δx^r of x^r that vanishes on $\mathcal{FC}$. Then it has been proved that the variational condition

$$\delta J(x^r) = 0 , \tag{67}$$

is equivalent to the equilibrium equations.

If we remember that in general relativity body forces are taken into account by means of the metric, we see that our theorem in § 5 generalizes this classical equivalence theorem involving the statio-

([10]) The quantities Y^{LM}, ε_{LM} etc., occurring in classical physics, are defined in substantially the same way as we did in general relativity. See for example [14], [15].

narity of potential energy. Furthermore in our principle we also take into account possibly vanishing couple stresses and the energy due to the mass of $\mathcal{C}$, as requested by the principle of equivalence of mass and energy.

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