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The number of restricted solutions of some systems of linear congruences

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We shall determine the number of solutions of a system of linear congruences

\[ n_i \equiv x_{i1} + \ldots + x_{it}(\text{mod } r), \quad i = 1, \ldots, t, \]

when the solutions are required to satisfy certain conditions. Two solutions, \( \{x_{ij}\} \) and \( \{x'_{ij}\} \), are counted as the same when and only when

\[ x_{ij} \equiv x'_{ij}(\text{mod } r) \]

for \( i = 1, \ldots, t \) and \( j = 1, \ldots, s \).

For each \( r \), and for \( j = 1, \ldots, s \), let \( T_j(r) \) be a nonempty set of \( t \)-tuples of integers from the set \( \{1, \ldots, r\} \). We shall use the notation \( \langle \ldots \rangle \) for a \( t \)-tuple since we wish to reserve the notation \( (\ldots) \) for greatest common divisor. Let \( M(n_1, \ldots, n_t, r, s) \) be the number of solutions of (1) with \( \langle x_{1j}, \ldots, x_{tj} \rangle \in T_j(r) \) for \( j = 1, \ldots, s \). Under a certain hypothesis this number can be evaluated using only elementary properties of the complex exponential function.

A function \( f(n, r) \) of an integer variable \( n \) and a positive integer variable \( r \) is called an even function (mod \( r \)) if \( f(n, r) = f((n, r), r) \) for all \( n \) and \( r \). A function \( g(n_1, \ldots, n_t, r) \) of \( t \) integer variables \( n_1, \ldots, n_t \) and a positive integer variable \( r \) is called a totally even function (mod \( r \)) if there is an even function (mod \( r \)), say \( f(n, r) \), such that \( g(n_1, \ldots, n_t, r) = f((n_1, \ldots, n_t), r) \) for all \( n_1, \ldots, n_t, \) and \( r \). Even functions and totally even functions (mod \( r \)) were introduced and studied by Cohen, the former in \([3]\) and several other papers and the latter in \([6]\).

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For $j = 1, \ldots, s$ let

$$g_j(n_1, \ldots, n_t, r) = \sum_{\langle x_1, \ldots, x_t \rangle \in T_j(r)} e(n_1 x_1 + \ldots + n_t x_t, r),$$

where $e(n, r) = \exp(2\pi in/r)$.

**Theorem 1.** If $g_j(n_1, \ldots, n_t, r)$ is a totally even function (mod $r$) for $j = 1, \ldots, s$, then

$$M(n_1, \ldots, n_t, r, s) = \frac{1}{r^t} \sum_{d|r} \left\{ \prod_{j=1}^{s} g_j(r/d, r) \right\} c(n_1, \ldots, n_t, d),$$

where $g_j(n, r) = g_j(n_1, \ldots, n_t, r)$ and

$$c(n_1, \ldots, n_t, r) = \sum_{\langle y_1, \ldots, y_t \rangle = 1} e(n_1 y_1 + \ldots + n_t y_t, r).$$

**Proof.** Set $M = M(n_1, \ldots, n_t, r, s)$. Then

$$M = \sum_{1}^{s} \sum_{t} \prod_{j=1}^{s} h_j(x_{1j}, \ldots, x_{tj}),$$

where $\sum$ is the summation over all solutions of the $i$th congruence of (1), and for $j = 1, \ldots, s$,

$$h_j(x_{1j}, \ldots, x_{tj}) = \begin{cases} 1 & \text{if } \langle x_{1j}, \ldots, x_{tj} \rangle \in T_j(r) \\ 0 & \text{otherwise} \end{cases}.$$

Since

$$h_j(x_{1j}, \ldots, x_{tj}) = \frac{1}{r^t} \sum_{\langle y_1, \ldots, y_t \rangle = 1} \sum_{q_{ij}=1}^{r} e((x_{ij} - y_{ij})q_{ij}, r),$$

where $\sum$ is summation over all $\langle y_{1j}, \ldots, y_{tj} \rangle \in T_j(r)$, we have

$$M = \frac{1}{r^{ts}} \sum_{1}^{s} \sum_{t} \prod_{j=1}^{s} \sum_{q_{ij}=1}^{r} e((x_{ij} - y_{ij})q_{ij}, r)$$

$$= \frac{1}{r^{ts}} \sum_{1}^{s} \sum_{t} \prod_{j=1}^{s} \sum_{q_{ij}=1}^{r} e(x_{ij} q_{ij}, r) e(-y_{ij} q_{ij}, r)$$

$$= \frac{1}{r^{ts}} \sum_{1}^{s} \sum_{t} \prod_{j=1}^{s} \left\{ g_j(-q_{1j}, \ldots, -q_{tj}, r) \prod_{i=1}^{t} e(x_{ij} q_{ij}, r) \right\},$$
where \( \sum' \) is summation over all \( ts \)-tuples of integers from the set \( \{1, \ldots, r\} \). Since \( g_i(n_1, \ldots, n_t, r) \) is a totally even function (mod \( r \)), the minus signs can be removed from the arguments of this function. Hence,

\[
M = \frac{1}{r^t} \sum' \left\{ \prod_{j=1}^{s} g_j(q_{ij}, \ldots, q_{tj}, r) \right\} \prod_{i=1}^{t} \sum_{j=1}^{s} e(x_{ij} q_{ij}, r).
\]

By [2, Lemma 3],

\[
\sum_{i} \prod_{j=1}^{s} e(x_{ij} q_{ij}, r) = \begin{cases} r^{t-1} e(n_{i} q_{i}, r) & \text{if } q_{i1} = \ldots = q_{it} = q_i \\ 0 & \text{otherwise} \end{cases}.
\]

Thus,

\[
M = \frac{1}{r^t} \sum'' \left\{ \prod_{j=1}^{s} g_j(q_{ij}, \ldots, q_{tj}, r) \right\} \prod_{i=1}^{t} e(n_{i} q_{i}, r),
\]

where \( \sum'' \) is summation over all \( t \)-tuples \( \langle q_1, \ldots, q_t \rangle \) of integers from the set \( \{1, \ldots, r\} \).

Let \( d \) run over all divisors of \( r \), and for each \( d \) let \( \langle u_1, \ldots, u_t \rangle \) run over all \( t \)-tuples of integers from the set \( \{1, \ldots, r/d\} \) such that \( (u_1, \ldots, u_t, r/d) = 1 \). Then \( \langle u_1 d, \ldots, u_t d \rangle \) runs over all \( t \)-tuples of integers from the set \( \{1, \ldots, r\} \). (See [6, p. 356] and the reference given there, and Proposition 2 below.) Thus,

\[
M = \frac{1}{r^t} \sum_{d|\sigma} \sum_{(u_1, \ldots, u_t, r/d)} \left\{ \prod_{j=1}^{s} g_j(u_{ij} d, \ldots, u_{tj} d, r) \right\} e(n_{i} u_{i1} + \ldots + n_{i} u_{it}, r/d).
\]

Since \( g_j(n_1, \ldots, n_t, r) \) is a totally even function (mod \( r \)), \( g_j(u_{i1} d, \ldots, u_{it} d, r) = g_j(d, r) \). Therefore,

\[
M = \frac{1}{r^t} \sum_{d|\sigma} \left\{ \prod_{j=1}^{s} g_j(d, r) \right\} \sum_{(u_1, \ldots, u_t, r/d)} e(n_{i} u_{i1} + \ldots + n_{i} u_{it}, r/d)
\]

\[
= \frac{1}{r^t} \sum_{d|\sigma} \left\{ \prod_{j=1}^{s} g_j(d, r) \right\} e(n_1, \ldots, n_t, r/d),
\]

which is the same as the formula in the statement of the theorem.

There is a general method for obtaining sets \( T_j(r) \) such that the hypothesis of Theorem 1 is satisfied. For each \( r \), let \( D(r) \) be a nonempty
set of divisors of $r$, and let

$$T(r) = \{ \langle x_1, \ldots, x_t \rangle : 1 \leq x_j \leq r \text{ for } j = 1, \ldots, t \text{ and } (x_1, \ldots, x_t, r) \in D(r) \}.$$ 

We shall show that

$$g(n_1, \ldots, n_t, r) = \sum_{\langle x_1, \ldots, x_t \rangle \in T(r)} e(n_1 x_1 + \ldots + n_t x_t, r)$$

is a totally even function (mod $r$).

**Proposition 1.** [6] $c(n_1, \ldots, n_t, r)$ is a totally even function (mod $r$). In fact,

$$c(n_1, \ldots, n_t, r) = \sum_{d|\langle n_1, \ldots, n_t, r \rangle} d^t \mu(r/d).$$

**Proposition 2.** Let $d$ run over the divisors of $r$ in $D(r)$, and for each $d$ let $\langle u_1, \ldots, u_t \rangle$ run over all $t$-tuples of integers from the set $\{1, \ldots, r/d\}$ such that $(u_1, \ldots, u_t, r/d) = 1$. Then $\langle u_1 d, \ldots, u_t d \rangle$ runs over $T(r)$.

**Proof.** Clearly, every element of $T(r)$ has the stated form, and all such $t$-tuples are in $T(r)$. It remains only to show that the $t$-tuples formed in this way are distinct. Let $d, d' \in D(r)$ and $(u_1, \ldots, u_t, r/d) = 1 = (u'_1, \ldots, u'_t, r/d)$. If $u_i d = u'_i d'$ for $i = 1, \ldots, t$, then $d = (u_1 d, \ldots, u_t d, r) = (u'_1 d', \ldots, u'_t d', r) = d'$ and $u_i = u'_i$ for $i = 1, \ldots, t$.

**Proposition 3.** $g(n_1, \ldots, n_t, r) = \sum_{d \in D(r)} c(n_1, \ldots, n_t, r/d)$. 

**Proof.** We have by Proposition 2,

$$g(n_1, \ldots, n_t, r) = \sum_{d \in D(r)} \sum_{(u_1, \ldots, u_t, r/d) = 1} e(n_1 u_1 + \ldots + n_t u_t, r/d).$$

Following Cohen [6] we shall denote $c(n_1, \ldots, n_t, r)$ by $c^{(t)}(n, r)$ when $n_1 = \ldots = n_t = n$.

**Example 1.** Let $N(n_1, \ldots, n_t, r, s)$ be the number of solutions of (1) with $(x_1, \ldots, x_t, r) = 1$ for $j = 1, \ldots, s$. Then,

$$N(n_1, \ldots, n_t, r, s) = \frac{1}{r^t} \sum_{d|r} c^{(s)}(r/d, r)c(n_1, \ldots, n_t, d).$$
This result is due to Cohen [6, Theorem 8]: in [6] Cohen confined himself to the case \( t = 2 \), but his methods and results extend immediately to the case of an arbitrary number of congruences. The number \( N(n, r, s) \) was evaluated by Ramanathan [8], Cohen [3], and others.

**Example 2.** For \( j = 1, \ldots, s \) let \( D_j(r) \) be the set of all divisors of \( r \) which are \( k \)-free. If \( Q_k(n_1, \ldots, n_t, r, s) \) is the number of solution of (1) with \( (x_{i_1}, \ldots, x_{i_t}, r)_k = 1 \), where \( (x_{i_1}, \ldots, x_{i_t}, r)_k \) is the largest \( k \)-th power common divisor of \( x_{i_1}, \ldots, x_{i_t} \), and \( r \), then

\[
Q_k(n_1, \ldots, n_t, r, s) = \frac{1}{r^t} \sum_{d|r} G_k(r/d, r) c(n_1, \ldots, n_t, d),
\]

where

\[
G_k(n, r) = \sum_{(d,r)_k = 1} c^{(0)}(n, r/d).
\]

We have \( N(n_1, \ldots, n_t, r, s) = Q_1(n_1, \ldots, n_t, r, s) \). The number \( Q_k(n, r^k, s) \) was evaluated by Cohen [4, Theorem 12] and expressed in terms of the extended Ramanujan sum which he introduced in [1].

**Example 3.** Let \( k \) and \( q \) be integers such that \( k \geq 2 \) and \( 0 < q < k \). Let \( S_{k,q} \) be the set of all integers \( n \) such that if \( p^h \) is the highest power of a prime \( p \) dividing \( n \), then \( h = 0, 1, \ldots, \) or \( q - 1 \) (mod \( k \)). For \( j = 1, \ldots, s \) let \( D_j(r) \) be the set of all divisors of \( r \) contained in \( S_{k,q} \), and let \( P_{k,q}(n_1, \ldots, n_t, r, s) \) be the number of solutions of (1) with \( (x_{i_1}, \ldots, x_{i_t}, r) \in S_{k,q} \) for \( j = 1, \ldots, s \). Then

\[
P_{k,q}(n_1, \ldots, n_t, r, s) = \frac{1}{r^t} \sum_{d|r} H_{k,q}(r/d, r) c(n_1, \ldots, n_t, d).
\]

where

\[
H_{k,q}(n, r) = \sum_{d|r, d \in S_{k,q}} c^{(0)}(n, r/d).
\]

When \( t = 1 \), this result is due to Subba Rao and Harris [9, Theorem 7]: Lemma 2 of [9] is a special case of our Theorem 1.

The next example involves unitary divisors of an integer, and the reader is referred to [5] and [7] for many details regarding unitary divisors and associated arithmetical functions.

A divisor \( d \) of \( r \) is called a unitary divisor if \( (d, r/d) = 1 \). We de-
note by \((x, r)_*\) the largest divisor of \(x\) which is a unitary divisor of \(r\), and we set \((x_1, \ldots, x_t, r)_* = ((x_1, \ldots, x_t), r)_*\). For each \(r\) let \(D(r)\) be a set of unitary divisors of \(r\), and

\[ T(r) = \{ \langle x_1, \ldots, x_t \rangle : 1 \leq x_i \leq r \text{ for } i = 1, \ldots, t \text{ and } (x_1, \ldots, x_t, r)_* \in D(r) \}. \]

It turns out that the corresponding function \(g(n_1, \ldots, n_t, r)\) is, in this case also, a totally even function \((\text{mod } r)\).

Set

\[ c^*(n_1, \ldots, n_t, r) = \sum_{(u_1, \ldots, u_t, r)_* = 1} c(n_1y_1 + \ldots + n_ty_t, r), \]

this is the unitary analogue of the function \(c(n_1, \ldots, n_t, r)\). When \(t = 1\) it is the unitary analogue of the Ramanujan sum introduced by Cohen in [5]. Let \(\gamma(r)\) be the core of \(r\), i.e., \(\gamma(1) = 1\), and if \(r > 1\) then \(\gamma(r)\) is the product of the distinct primes which divide \(r\). Let \(d\) run over the divisors of \(r\) such that \(\gamma(d) = \gamma(r)\), and for each \(d\) let \(\langle y_1, \ldots, y_t \rangle\) run over the \(t\)-tuples of integers from the set \(\{1, \ldots, d\}\) such that \((y_1, \ldots, y_t, d) = 1\). Then, \(\langle y_1r/d, \ldots, y_tr/d \rangle\) runs over the \(t\)-tuples \(\langle x_1, \ldots, x_t \rangle\) of integers from the set \(\{1, \ldots, r\}\) such that \((x_1, \ldots, x_t, r)_* = 1\). From this it follows that

\[ c^*(n_1, \ldots, n_t, r) = \sum_{d|r, \gamma(d) = \gamma(r)} c(n_1, \ldots, n_t, d). \]

Therefore, \(c^*(n_1, \ldots, n_t, r)\) is a totally even function \((\text{mod } r)\). If we denote \(c^*(n_1, \ldots, n_t, r)\) by \(c^*(t)(n, r)\) when \(n_1 = \ldots = n_t = n\), then

\[ c^*(t)(n, r) = \sum_{d|r, \gamma(d) = \gamma(r)} c^*(t)(n, r). \]

**Proposition 4.** Let \(d\) run over the divisors of \(r\) in \(D(r)\), and for each \(d\) let \(\langle u_1, \ldots, u_t \rangle\) run over all \(t\)-tuples of integers from the set \(\{1, \ldots, r/d\}\) such that \((u_1, \ldots, u_t, r/d)_* = 1\). Then \(\langle u_1d, \ldots, u_td \rangle\) runs over \(T(r)\).

The proof of this proposition is similar to that of Proposition 2. From it we obtain the following result from which we conclude that \(g(n_1, \ldots, n_t, r)\) is, indeed, a totally even function \((\text{mod } r)\).
PROPOSITION 5. With $D(r)$ and $T(r)$ as in the preceding discussion

\[ g(n_1, \ldots, n_t, r) = \sum_{d \in D(r)} c^*(n_1, \ldots, n_t, r/d) . \]

EXAMPLE 4. If $N^*(n_1, \ldots, n_t, r, s)$ is the number of solutions of (1) with $(x_{1j}, \ldots, x_{tj}, r)_{*} = 1$ for $j = 1, \ldots, s$ then

\[ N^*(n_1, \ldots, n_t, r, s) = \frac{1}{r^t} \sum_{d|r} c^*(r/d, r)c(n_1, \ldots, n_t, d) . \]

When $t = 1$, this number was evaluated by Cohen [7, Theorem 6.1]: his formula is different in form from ours, and each can be obtained from the other by using the relation between $c^*(n, r)$ and $c(n, r)$ [7, Theorem 3.1].

In our examples the restrictions are the same for all values of $j$. Of course, they could be chosen differently for different values of $j$: for example, we could obtain immediately a generalization of [7, Theorem 6.3].

Next we go in another direction and obtain a very general result of the type obtained by Suguamma in [10]. For $i = 1, \ldots, 8$ let $t_i$ be a positive integer and for each $r$ let $T_i(r)$ be a nonempty set of $t_i$-tuples of integers from the set $\{1, \ldots, r\}$. Further, let $g_i(n_1, \ldots, n_{t_i}, r)$ be defined as before. Let $L(n, r, t_1, \ldots, t_s)$ be the number of solutions of

\[ n = \sum_{j=1}^{t_1} x_{1j} + \ldots + \sum_{j=1}^{t_s} x_{sj} \pmod{r} \]

with $\langle x_{11}, \ldots, x_{t_{1i}} \rangle \in T_i(r)$ for $i = 1, \ldots, s$.

THEOREM 2. If $g_i(n_1, \ldots, n_{t_i}, r)$ is a totally even function (mod $r$) for $i = 1, \ldots, s$ then

\[ L(n, r, t_1, \ldots, t_s) = \frac{1}{r} \sum_{d|r} \left( \prod_{i=1}^{s} g_i(r/d, r) \right) c(n, d) , \]

where $g_i(n, r) = g_i(n, \ldots, n, r)$.

PROOF. Let $L = L(n, r, t_1, \ldots, t_s)$. Then

\[ L = \sum_{i=1}^{s} \prod_{j=1}^{t_i} h_i(x_{i1}, \ldots, x_{ij}) , \]
where $\sum'$ is summation over all solutions of (2), and

$$h_i(x_{i1}, \ldots, x_{it_i}) = \frac{1}{r^t} \sum_{i=1}^{t_i} \prod_{j=1}^{t_i} \sum_{a_{ij}=1}^{r} e((x_{ij} - y_{ij})q_{ij}, r),$$

where $\sum$ is summation over all $\langle y_{i1}, \ldots, y_{it_i} \rangle \in T_i(r)$. Let $t = t_1 + \ldots + t_s$. Then

$$L = \frac{1}{r^t} \sum_{i=1}^{s} \prod_{j=1}^{t_i} \sum_{a_{ij}=1}^{r} e((x_{ij} - y_{ij})q_{ij}, r)$$

$$= \frac{1}{r^t} \sum_{i=1}^{s} \prod_{j=1}^{t_i} \sum_{a_{ij}=1}^{r} e(x_{ij}q_{ij}, r) e(-y_{ij}q_{ij}, r),$$

where $\sum^{s}$ is summation over all $t$-tuples of integers from the set $\{1, \ldots, r\}$. Thus,

$$L = \frac{1}{r^t} \sum_{i=1}^{s} \sum' \prod_{j=1}^{t_i} \left\{ g_i(q_{i1}, \ldots, q_{it_i}, r) \prod_{j=1}^{t_i} e(x_{ij}q_{ij}, r) \right\}$$

$$= \frac{1}{r^t} \sum_{i=1}^{s} \prod_{j=1}^{t_i} g_i(q_{i1}, \ldots, q_{it_i}, r) \sum' \prod_{i=1}^{s} \prod_{j=1}^{t_i} e(x_{ij}q_{ij}, r).$$

By [2, Lemma 3] the summation on the right is equal to $r^{t-1}c(nq, r)$ if $q_{ij} = q$ for all $i$ and $j$, and is equal to zero otherwise. Hence,

$$L = \frac{1}{r^t} \sum_{q=1}^{r} \sum_{i=1}^{s} \left\{ g_i(q, r) \right\} c(nq, r).$$

If we proceed as in the final steps of the proof of Theorem 1 we will obtain the formula of Theorem 2.

**Example 5.** If $N'(n, r, t_1, \ldots, t_s)$ is the number of solutions of (2) with $(x_{i1}, \ldots, x_{it_i}, r) = 1$ for $i = 1, \ldots, s$, then

$$N'(n, r, t_1, \ldots, t_s) = \frac{1}{r} \sum_{d|r} \left\{ \prod_{i=1}^{s} c(d)(r/d, r) \right\} c(n, d).$$
EXAMPLE 6. If $Q_k(n, r, t_1, \ldots, t_s)$ is the number of solutions of (2) with $(x_{i1}, \ldots, x_{it_k}, r)_{x_k}=1$ for $i=1, \ldots, s$, then
\[
Q_k(n, r, t_1, \ldots, t_s) = \frac{1}{r} \sum_{d|r} \left\{ \prod_{i=1}^s G_i^{(t_i)}(r/d, r) \right\} c(n, d),
\]
where
\[
G_i^{(t_i)}(n, r) = \sum_{d|r, (d,r)_{x_k}=1} d^{(t_i)}(n, r/d).
\]
Sugunamma evaluated $Q_k(n, r, s, \ldots, t)$ [10, Theorem 5]: his formula is in terms of the extended Ramanujan sum $c_k(n, r)$.

Of course, there is a unitary analogue of Example 5. Also, we can mix the restrictions, and we shall give one example of a result of this kind.

EXAMPLE 7. Let $R(n, r, s, t)$ be the number of solutions of
\[
n \equiv x_1 + \ldots + x_s + y_1 + \ldots + y_t \pmod{r}
\]
with $(x_1, \ldots, x_s, r)_{x_k}=1$ and $(y_1, \ldots, y_t, r)=1$. Then
\[
R(n, r, s, t) = \frac{1}{r} \sum_{d|r} c^{*(s)}(r/d, r)c^{(t)}(r/d, r)c(n, d).
\]

Finally, it is clear that by the same kind of arguments we could give a single result which contains both Theorem 1 and Theorem 2. In the light of these theorems, it is easy to predict what the formula in such a result would be.

BIBLIOGRAPHY


