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The Number of Restricted Solutions of Some Systems of Linear Congruences.

P. J. MC CARTHY (*)

We shall determine the number of solutions of a system of linear congruences

$$n_i \equiv x_{i1} + \ldots + x_{is} (\operatorname{mod} r)\,, \qquad i = 1, \ldots, t, \tag{1}$$

when the solutions are required to satisfy certain conditions. Two solutions, $\{x_{ij}\}$ and $\{x'_{ij}\}$, are counted as the same when and only when $x_{ij} \equiv x'_{ij} (\operatorname{mod} r)$ for $i = 1, \ldots, t$ and $j = 1, \ldots, s$.

For each r, and for $j = 1, \ldots, s$, let $T_j(r)$ be a nonempty set of t-tuples of integers from the set $\{1, \ldots, r\}$. We shall use the notation $\langle \ldots \rangle$ for a t-tuple since we wish to reserve the notation $(\ldots)$ for greatest common divisor. Let $M(n_1, \ldots, n_t, r, s)$ be the number of solutions of (1) with $\langle x_{1j}, \ldots, x_{tj} \rangle \in T_j(r)$ for $j = 1, \ldots, s$. Under a certain hypothesis this number can be evaluated using only elementary properties of the complex exponential function.

A function $f(n, r)$ of an integer variable n and a positive integer variable r is called an even function $(\operatorname{mod} r)$ if $f(n, r) = f\big((n, r), r\big)$ for all n and r. A function $g(n_1, \ldots, n_t, r)$ of t integer variables $n_1, \ldots, n_t$ and a positive integer variable r is called a totally even function $(\operatorname{mod} r)$ if there is an even function $(\operatorname{mod} r)$, say $f(n, r)$, such that $g(n_1, \ldots, n_t, r) = f\big((n_1, \ldots, n_1), r\big)$ for all $n_1, \ldots, n_t$, and r. Even functions and totally even functions $(\operatorname{mod} r)$ were introduced and studied by Cohen, the former in [3] and several other papers and the latter in [6].

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For $j=1, \dots, s$ let

$$g_j(n_1, \dots, n_t, r) = \sum_{\langle x_1, \dots, x_t\rangle \in T_j(r)} e(n_1x_1 + \dots + n_tx_t, r) ,$$

where $e(n, r) = \exp(2\pi in/r)$.

THEOREM 1. If $g_j(n_1, \dots, n_t, r)$ is a totally even function $(\mathrm{mod}\, r)$ for $j=1, \dots, s$, then

$$M(n_1, \dots, n_t, r, s) = \frac{1}{r^t}\sum_{d|r}\left\{\prod_{j=1}^{s} g_j(r/d, r)\right\}c(n_1, \dots, n_t, d) ,$$

where $g_j(n, r) = g_j(n, \dots, n, r)$ and

$$c(n_1, \dots, n_t, r) = \sum_{(y_1, \dots, y_t, r)=1} e(n_1y_1 + \dots + n_ty_t, r) .$$

PROOF. Set $M = M(n_1, \dots, n_t, r, s)$. Then

$$M = \sum_1 \cdots \sum_t \prod_{j=1}^{s} h_j(x_{1j}, \dots, x_{tj}) ,$$

where $\sum_i$ is the summation over all solutions of the ith congruence of (1), and for $j=1, \dots, s$,

$$h_j(x_{1j}, \dots, x_{tj}) = \begin{cases} 1 & \text{if } \langle x_{1j}, \dots, x_{tj}\rangle \in T_j(r) \\ 0 & \text{otherwise} . \end{cases}$$

Since

$$h_j(x_{1j}, \dots, x_{tj}) = \frac{1}{r^t}\sum^{(j)}\prod_{i=1}^{t}\sum_{q_{ij}=1}^{r} e((x_{ij}-y_{ij})q_{ij}, r) ,$$

where $\sum^{(j)}$ is summation over all $\langle y_{1j}, \dots, y_{tj}\rangle \in T_j(r)$, we have

$$\begin{aligned} M &= \frac{1}{r^{ts}}\sum_1 \cdots \sum_t \prod_{j=1}^{s}\sum^{(j)}\prod_{i=1}^{t}\sum_{q_{ij}=1}^{r} e((x_{ij}-y_{ij})q_{ij}, r) \\ &= \frac{1}{r^{ts}}\sum{}'\sum_1 \cdots \sum_t \prod_{j=1}^{s}\sum^{(j)}\prod_{i=1}^{t} e(x_{ij}q_{ij}, r)e(-y_{ij}q_{ij}, r) \\ &= \frac{1}{r^{ts}}\sum{}'\sum_1 \cdots \sum_t \prod_{j=1}^{s}\left\{g_j(-q_{1j}, \dots, -q_{tj}, r)\prod_{i=1}^{t} e(x_{ij}q_{ij}, r)\right\} , \end{aligned}$$

where $\sum'$ is summation over all ts-tuples of integers from the set $\{1, \ldots, r\}$. Since $g_j(n_1, \ldots, n_t, r)$ is a totally even function (mod r), the minus signs can be removed from the arguments of this function. Hence,

$$M = \frac{1}{r^{ts}} \sum' \left\{\prod_{j=1} g_j(q_{1j}, \ldots, q_{tj}, r)\right\} \prod_{i=1}^{t} \sum_{i} \prod_{j=1}^{s} e(x_{ij} q_{ij}, r) .$$

By [2, Lemma 3],

$$\sum_{i} \prod_{j=1}^{s} e(x_{ij} q_{ij}, r) = \begin{cases} r^{s-1} e(n_i q_i, r) & \text{if } q_{i1} = \ldots = q_{it} = q_i \\ 0 & \text{otherwise} . \end{cases}$$

Thus,

$$M = \frac{1}{r^t} \sum'' \left\{\prod_{j=1}^{s} g_j(q_1, \ldots, q_t, r)\right\} \prod_{i=1}^{t} e(n_i q_i, r) ,$$

where $\sum''$ is summation over all t-tuples $\langle q_1, \ldots, q_t \rangle$ of integers from the set $\{1, \ldots, r\}$.

Let d run over all divisors of r, and for each d let $\langle u_1, \ldots, u_t \rangle$ run over all t-tuples of integers from the set $\{1, \ldots, r/d\}$ such that $(u_1, \ldots, u_t, r/d) = 1$. Then $\langle u_1 d, \ldots, u_t d \rangle$ runs over all t-tuples of integers from the set $\{1, \ldots, r\}$. (See [6, p. 356] and the reference given there, and Proposition 2 below.) Thus,

$$M = \frac{1}{r^t} \sum_{d|r} \sum_{(u_1, \ldots, u_t, r/d) = 1} \left\{\prod_{j=1}^{s} g_j(u_1 d, \ldots, u_t d, r)\right\} e(n_1 u_1 + \ldots + n_t u_t, r/d) .$$

Since $g_j(n_1, \ldots, n_t, r)$ is a totally even function (mod r), $g_j(u_1 d, \ldots, u_t d, r) = g_j(d, r)$. Therefore,

$$M = \frac{1}{r^t} \sum_{d|r} \left\{\prod_{j=1}^{s} g_j(d, r)\right\} \sum_{(u_1, \ldots, u_t, r/d) = 1} e(n_1 u_1 + \ldots + n_t u_t, r/d)$$

$$= \frac{1}{r^t} \sum_{d|r} \left\{\prod_{j=1}^{s} g_j(d, r)\right\} c(n_1, \ldots, n_t, r/d) ,$$

which is the same as the formula in the statement of the theorem.

There is a general method for obtaining sets $T_j(r)$ such that the hypothesis of Theorem 1 is satisfied. For each r, let $D(r)$ be a nonempty set of divisors of r, and let

$$T(r)=\{\langle x_1, \ldots, x_t\rangle : 1\leqslant x_j \leqslant r \text{ for } j=1, \ldots, t \text{ and } (x_1, \ldots, x_t, r)\in D(r)\} .$$

We shall show that

$$g(n_1, \ldots, n_t, r)=\sum_{\langle x_1,\ldots,x_t\rangle\in T(r)} e(n_1x_1+\ldots+n_tx_t, r)$$

is a totally even function (mod r).

PROPOSITION 1. [6] $c(n_1, \ldots, n_t, r)$ is a totally even function (mod r). In fact,

$$c(n_1, \ldots, n_t, r)=\sum_{d|(n_1,\ldots,n_t,r)} d^t\mu(r/d) .$$

PROPOSITION 2. Let d run over the divisors of r in $D(r)$, and for each d let $\langle u_1, \ldots, u_t\rangle$ run over all t-tuples of integers from the set $\{1, \ldots, r/d\}$ such that $(u_1, \ldots, u_t, r/d)=1$. Then $\langle u_1d, \ldots, u_td\rangle$ runs over $T(r)$.

PROOF. Clearly, every element of $T(r)$ has the stated form, and all such t-tuples are in $T(r)$. It remains only to show that the t-tuples formed in this way are distinct. Let $d, d'\in D(r)$ and $(u_1, \ldots, u_t, r/d)= 1=(u_1', \ldots, u_t', r/d)$. If $u_id=u_i'd'$ for $i=1, \ldots, t$, then $d=(u_1d, \ldots, u_td, r)=(u_1'd', \ldots, u_t'd', r)=d'$ and $u_i=u_i'$ for $i=1, \ldots, t$.

PROPOSITION 3. $g(n_1, \ldots, n_t, r)=\sum_{d\in D(r)} c(n_1, \ldots, n_t, r/d)$.

PROOF. We have by Proposition 2,

$$g(n_1, \ldots, n_t, r)=\sum_{d\in D(r)}\ \sum_{(u_1,\ldots,u_t,r/d)=1} e(n_1u_1+\ldots+n_tu_t, r/d) .$$

Following Cohen [6] we shall denote $c(n_1, \ldots, n_t, r)$ by $c^{(t)}(n, r)$ when $n_1=\ldots=n_t=n$.

EXAMPLE 1. Let $N(n_1, \ldots, n_t, r, s)$ be the number of solutions of (1) with $(x_{1j}, \ldots, x_{tj}, r)=1$ for $j=1, \ldots, s$. Then,

$$N(n_1, \ldots, n_t, r, s)=\frac{1}{r^t}\sum_{d|r} c^{(t)}(r/d, r)^s c(n_1, \ldots, n_t, d) .$$

This result is due to Cohen [6, Theorem 8]: in [6] Cohen confined himself to the case $t = 2$, but his methods and results extend immediately to the case of and arbitrary number of congruences. The number $N(n, r, s)$ was evaluated by Ramanathan [8], Cohen [3], and others.

EXAMPLE 2. For $j = 1, \ldots, s$ let $D_j(r)$ be the set of all divisors of r which are k-free. If $Q_k(n_1, \ldots, n_t, r, s)$ is the number of solution of (1) with $(x_{1j}, \ldots, x_{tj}, r)_k = 1$, where $(x_{1j}, \ldots, x_{tj}, r)_k$ is the largest k-th power common divisor of $x_{1j}, \ldots, x_{tj}$, and r, then

$$Q_k(n_1, \ldots, n_t, r, s) = \frac{1}{r^t} \sum_{d|r} G_k(r/d, r)^s c(n_1, \ldots, n_t, d) ,$$

where

$$G_k(n, r) = \sum_{\substack{d|r \\ (d,r)_k = 1}} c^{(t)}(n, r/d) .$$

We have $N(n_1, \ldots, n_t, r, s) = Q_1(n_1, \ldots, n_t, r, s)$. The number $Q_k(n, r^k, s)$ was evaluated by Cohen [4, Theorem 12] and expressed in terms of the extended Ramanujan sum which he introduced in [1].

EXAMPLE 3. Let k and q be integers such that $k \geqslant 2$ and $0 < q < k$. Let $S_{k,q}$ be the set of all integers n such that if p^h is the highest power of a prime p dividing n, then $h \equiv 0, 1, \ldots,$ or $q-1 \pmod{k}$. For $j = 1, \ldots, s$ let $D_j(r)$ be the set of all divisors of r contained in $S_{k,q}$, and let $P_{k,q}(n_1, \ldots, n_t, r, s)$ be the number of solutions of (1) with $(x_{1j}, \ldots, x_{tj}, r) \in S_{k,q}$ for $j = 1, \ldots, s$. Then

$$P_{k,q}(n_1, \ldots, n_t, r, s) = \frac{1}{r^t} \sum_{d|r} H_{k,q}(r/d, r)^s c(n_1, \ldots, n_t, d) .$$

where

$$H_{k,q}(n, r) = \sum_{\substack{d|r \\ d \in S_{k,q}}} c^{(t)}(n, r/d) .$$

When $t = 1$, this result is due to Subba Rao and Harris [9, Theorem 7]: Lemma 2 of [9] is a special case of our Theorem 1.

The next example involves unitary divisors of an integer, and the reader is referred to [5] and [7] for many details regarding unitary divisors and associated arithmetical functions.

A divisor d of r is called a unitary divisor if $(d, r/d) = 1$. We denote by $(x, r)_*$ the largest divisor of x which is a unitary divisor of r, and we set $(x_1, \ldots, x_t, r)_* = \big((x_1, \ldots, x_t), r\big)_*$. For each r let $D(r)$ be a set of unitary divisors of r, and

$$T(r) = \{\langle x_1, \ldots, x_t\rangle : 1 \leqslant x_i \leqslant r \text{ for } i = 1, \ldots, t \text{ and } (x_1, \ldots, x_t, r)_* \in D(r)\}.$$

It turns out that the corresponding function $g(n_1, \ldots, n_t, r)$ is, in this case also, a totally even function (mod r).

Set

$$c^*(n_1, \ldots, n_t, r) = \sum_{(y_1, \ldots, y_t, r)_* = 1} e(n_1 y_1 + \ldots + n_t y_t, r) ,$$

this is the unitary analogue of the function $c(n_1, \ldots, n_t, r)$. When $t = 1$ it is the unitary analogue of the Ramanujan sum introduced by Cohen in [5]. Let $\gamma(r)$ be the core of r, i.e., $\gamma(1) = 1$, and if $r > 1$ then $\gamma(r)$ is the product of the distinct primes which divide r. Let d run over the divisors of r such that $\gamma(d) = \gamma(r)$, and for each d let $\langle y_1, \ldots, y_t\rangle$ run over the t-tuples of integers from the set $\{1, \ldots, d\}$ such that $(y_1, \ldots, y_t, d) = 1$. Then, $\langle y_1 r/d, \ldots, y_t r/d\rangle$ runs over the t-tuples $\langle x_1, \ldots, x_t\rangle$ of integers from the set $\{1, \ldots, r\}$ such that $(x_1, \ldots, x_t, r)_* = 1$. From this it follows that

$$c^*(n_1, \ldots, n_t, r) = \sum_{\substack{d \mid r \\ \gamma(d) = \gamma(r)}} c(n_1, \ldots, n_t, d) .$$

Therefore, $c^*(n_1, \ldots, n_t, r)$ is a totally even function (mod r). If we denote $c^*(n_1, \ldots, n_t, r)$ by $c^{*(t)}(n, r)$ when $n_1 = \ldots = n_t = n$, then

$$c^{*(t)}(n, r) = \sum_{\substack{d \mid r \\ \gamma(d) = \gamma(r)}} c^{(t)}(n, r) .$$

PROPOSITION 4. Let d run over the divisors of r in $D(r)$, and for each d let $\langle u_1, \ldots, u_t\rangle$ run over all t-tuples of integers from the set $\{1, \ldots, r/d\}$ such that $(u_1, \ldots, u_t, r/d)_* = 1$. Then $\langle u_1 d, \ldots, u_t d\rangle$ runs over $T(r)$.

The proof of this proposition is similar to that of Proposition 2. From it we obtain the following result from which we conclude that $g(n_1, \ldots, n_t, r)$ is, indeed, a totally even function (mod r).

PROPOSITION 5. With $D(r)$ and $T(r)$ as in the preceeding discussion

$$g(n_1, \ldots, n_t, r) = \sum_{d \in D(r)} c^*(n_1, \ldots, n_t, r/d) .$$

EXAMPLE 4. If $N^*(n_1, \ldots, n_t, r, s)$ is the number of solutions of (1) with $(x_{1j}, \ldots, x_{tj}, r)_* = 1$ for $j = 1, \ldots, s$ then

$$N^*(n_1, \ldots, n_t, r, s) = \frac{1}{r^t} \sum_{d|r} c^{*(t)}(r/d, r)^s c(n_1, \ldots, n_t, d) .$$

When $t = 1$, this number was evaluated by Cohen [7, Theorem 6.1]: his formula is different in form from ours, and each can be obtained from the other by using the relation between $c^*(n, r)$ and $c(n, r)$ [7, Theorem 3.1].

In our examples the restrictions are the same for all values of j. Of course, they could be chosen differently for different values of j: for example, we could obtain immediately a generalization of [7, Theorem 6.3].

Next we go in another direction and obtain a very general result of the type obtained by Sugunamma in [10]. For $i = 1, \ldots, s$, let t_i be a positive integer and for each r let $T_i(r)$ be a nonempty set of t_i-tuples of integers from the set $\{1, \ldots, r\}$. Further, let $g_i(n_1, \ldots, n_{t_i}, r)$ be defined as before. Let $L(n, r, t_1, \ldots, t_s)$ be the number of solutions of

$$n \equiv \sum_{j=1}^{t_1} x_{1j} + \ldots + \sum_{j=1}^{t_s} x_{s_j} (\operatorname{mod} r) \tag{2}$$

with $\langle x_{i1}, \ldots, x_{it_i} \rangle \in T_i(r)$ for $i = 1, \ldots, s$.

THEOREM 2. If $g_i(n_1, \ldots, n_{t_i}, r)$ is a totally even function $(\operatorname{mod} r)$ for $i = 1, \ldots, s$ then

$$L(n, r, t_1, \ldots, t_s) = \frac{1}{r} \sum_{d|r} \left\{ \prod_{i=1}^{s} g_i(r/d, r) \right\} c(n, d) ,$$

where $g_i(n, r) = g_i(n, \ldots, n, r)$.

PROOF. Let $L = L(n, r, t_1, \ldots, t_s)$. Then

$$L = \sum{}' \prod_{i=1}^{s} h_i(x_{i1}, \ldots, x_{it_i}) ,$$

where $\sum'$ is summation over all solutions of (2), and

$$h_i(x_{i1}, \ldots, x_{it_i}) = \frac{1}{r^{t_i}} \sum^{(i)} \prod_{j=1}^{t_i} \sum_{q_{ij}=1}^{r} e\big((x_{ij} - y_{ij})q_{ij}, r\big)\big) ,$$

where $\sum^{(i)}$ is summation over all $\langle y_{i1}, \ldots, y_{it_i}\rangle \in T_i(r)$. Let $t = t_1 + \ldots + t_s$. Then

$$L = \frac{1}{r^t} \sum' \prod_{i=1}^{s} \sum^{(i)} \prod_{j=1}^{t_i} \sum_{q_{ij}=1}^{r} e\big((x_{ij} - y_{ij})q_{ij}, r\big)$$

$$= \frac{1}{r^t} \sum'' \sum' \prod_{i=1}^{s} \sum^{(i)} \prod_{j=1}^{t_i} e(x_{ij}q_{ij}, r) e(-y_{ij}q_{ij}, r) ,$$

where $\sum''$ is summation over all t-tuples of integers from the set $\{1, \ldots, r\}$ Thus,

$$L = \frac{1}{r^t} \sum'' \sum' \prod_{i=1}^{s} \left\{ g_i(q_{i1}, \ldots, q_{it_i}, r) \prod_{j=1}^{t_i} e(x_{ij}x_{ij}q_{ij}, r) \right\}$$

$$= \frac{1}{r^t} \sum'' \left\{ \prod_{i=1}^{s} g_i(q_{i1}, \ldots, q_{it_i}, r) \right\} \sum' \prod_{i=1}^{s} \prod_{j=1}^{t_i} e(x_{ij}q_{ij}, r) .$$

By [2, Lemma 3] the summation on the right is equal to $r^{t-1}e(nq, r)$ if $q_{ij} = q$ for all i and j, and is equal to zero otherwise. Hence,

$$L = \frac{1}{r} \sum_{q=1}^{r} \left\{ \prod_{i=1}^{s} g_i(q, r) \right\} e(nq, r) .$$

If we proceed as in the final steps of the proof of Theorem 1 we will obtain the formula of Theorem 2.

Example 5. If $N'(n, r, t_1, \ldots, t_s)$ is the number of solutions of (2) with $(x_{i1}, \ldots, x_{it_i}, r) = 1$ for $i = 1, \ldots, s$, then

$$N'(n, r, t_1, \ldots, t_s) = \frac{1}{r} \sum_{d|r} \left\{ \prod_{i=1}^{s} c^{(t_i)}(r/d, r) \right\} c(n, d) .$$

EXAMPLE 6. If $Q'_k(n, r, t_1, \ldots, t_s)$ is the number of solutions of (2) with $(x_{i1}, \ldots, x_{it_i}, r)_k = 1$ for $i = 1, \ldots, s$, then

$$Q'_k(n, r, t_1, \ldots, t_s) = \frac{1}{r} \sum_{d|r} \left\{ \prod_{i=1}^{s} G_i^{(t_i)}(r/d, r) \right\} c(n, d) ,$$

where

$$G_i^{(t_i)}(n, r) = \sum_{\substack{d|r \\ (d,r)_k = 1}} c^{(t_i)}(n, r/d) .$$

Sugunamma evaluated $Q'_k(n, r^k, t, \ldots, t)$ [10, Theorem 5]: his formula is in terms of the extended Ramanujan sum $c_k(n, r)$.

Of course, there is a unitary analogue of Example 5. Also, we can mix the restrictions, and we shall give one example of a result of this kind.

EXAMPLE 7. Let $R(n, r, s, t)$ be the number of solutions of

$$n \equiv x_1 + \ldots + x_s + y_1 + \ldots + y_t \pmod r$$

with $(x_1, \ldots, x_s, r)_* = 1$ and $(y_1, \ldots, y_t, r) = 1$. Then

$$R(n, r, s, t) = \frac{1}{r} \sum_{d|r} c^{*(s)}(r/d, r) c^{(t)}(r/d, r) c(n, d) .$$

Finally, it is clear that by the same kind of arguments we could give a single result which contains both Theorem 1 and Theorem 2. In the light of these theorems, it is easy to predict what the formula in such a result would be.

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