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Abelian Groups whose Endomorphism Ring is Linearly Compact.

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If G is any abelian group, the finite topology of the endomorphism ring $E(G)$ has the family of all $U_X = \{\varphi \in E(G) | \varphi(X) = 0\}$ with X a finite subset of G as a basis of neighbourhoods of 0. It is well known [F 1] that, with respect to this topology, $E(G)$ is a complete Hausdorff topological ring. It has been suggested [F 2] to characterize the groups G whose endomorphism rings have topological properties stronger than completeness, such as compactness, linear compactness, etc. E.g., it has been proved that $E(G)$ is compact (in the finite topology) if and only if G is a torsion group whose primary components are finite direct sums of cyclic and quasi-cyclic groups [F 1].

In the first part of this paper we determine the groups G such that $E(G)$ is linearly compact (in the finite topology); i.e. such that every family of closed linear varieties having the finite intersection property has nonempty intersection. In fact, we prove that $E(G)$ is linearly compact if and only if $G = H \oplus D$, with D a divisible group which has finitely many non-zero p-components if G is not a torsion group, H has no elements of infinite height and $H = \bigoplus_{p \in P} H_p$, where, for every prime p, H_p is either a torsion-complete p-group or a direct sum $C_p \oplus B_p$ of a torsion-free J_p-module C_p complete in the p-adic topology and a bounded p-group B_p.

According to [B, Ex. 19, p. 110], if A is a topological ring and E is a Hausdorff linear topological A-module, E is strictly linearly compact

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if it is linearly compact and every continuous A-homomorphism from E is an open map. With arguments very similar to those used in the first part of the paper, we show that $E(G)$ is strictly linearly compact (in the finite topology) if and only if $G = (\bigoplus_{\Lambda} Q) \oplus (\bigoplus_{p \in P} (B_p \oplus D_p))$ where B_p is a bounded p-group, D_p is a divisible p-group, and only finitely many D_p's are non-zero if $\Lambda \neq \emptyset$.

In [L 1] Liebert defined the p-finite topology for the endomorphism ring $E(M)$ of a J_p-module M without elements of infinite height, and determined all torsion [L 1] and torsion-free [L 2] J_p-modules M such that $E(M)$ is complete in this topology. It turns out that they are precisely the torsion and torsion-free reduced J_p-modules such that $E(M)$ is linearly compact in the finite topology. The situation is different in the mixed case, where we prove that $E(M)$ is complete in the p-finite topology if and only if either M is complete in its p-adic topology or M is a p-pure fully invariant subgroup of the p-adic completion $\widehat{t(M)}$ of its torsion subgroup $t(M)$.

1. Throughout the paper, « group » means « abelian group ». If G is a group, $t(G)$ is the torsion subgroup of G, $t_p(G)$ is the p-component of $t(G)$, $G_\infty = \bigcap_{n \in N} nG$, $p^\infty G = \bigcap_{n \in N} p^n G$, $G[n] = \{g \in G | ng = 0\}$; $E = E(G)$ is the endomorphism ring of G. If $g \in G$ the orbit Eg of g is the E-submodule $\{\varphi(g) | \varphi \in E\}$; the annihilator U_g is the left ideal $\{\varphi \in E | \varphi(g) = 0\}$ (obviously $Eg \cong_E E/U_g$); the annihilator U_X of the subset X of G is the left ideal $\{\varphi \in E | \varphi(X) = 0\}$; $o(g)$ is the order of g and if G is a p-group $o(g) = p^{e(g)}$, $h(g)$ is the p-height of g; N is the set of natural numbers, Z the ring of integers, Q the (additive) group of rational numbers, J_p the ring of p-adic integers, J the natural completion of Z, P the set of prime numbers.

It is well known that $E = E(G)$, being complete in the finite topology, is linearly compact if and only if E/U_X is linearly compact as a discrete E-module for every finite subset X of G; and E is strictly linearly compact in the finite topology if and only if E/U_X is an artinian E-module [B, Ex. 16 δ, p. 109 and Ex. 19 γ, p. 111].

LEMMA 1.1. *$E = E(G)$ is linearly compact in the finite topology if and only if for every $g \in G$ Eg is linearly compact as a discrete E-module. E is strictly linearly compact in the finite topology if and only if for every $g \in G$ Eg is an artinian E-module.*

PROOF. If E is (strictly) linearly compact, then Eg, being isomorphic to the quotient module E/U_g, is linearly compact in the discrete topology (artinian). Assume now that for every $g \in G$ Eg is linearly compact as a discrete E-module (an artinian E-module); if $X = \{g_1, \dots, g_n\}$ is any finite subset of G, U_X is the kernel of the diagonal map $\varphi: E \to (E/U_{g_1}) \times \dots \times (E/U_{g_n})$ of the natural homomorphisms $\varphi_i: E \to E/U_{g_i}$. So E/U_X is E-isomorphic to a submodule of a finite product of linearly compact discrete (artinian) E-modules, and is itself linearly compact in the discrete topology (artinian).

LEMMA 1.2. Let H be a fully invariant subgroup of G, and assume $E(G)$ is linearly compact in the finite topology. If H is contained in an orbit, then it is complete (not necessarily Hausdorff) in every topology which has a basis of neighbourhoods of 0 consisting of E-modules.

PROOF. Since $H \subseteq Eg$ is a linearly compact discrete E-module, the lemma follows from [B, Ex. 16 γ, p. 109].

Lemma 1.2 will be used to infer completeness of H in the topology induced on H by the natural (or p-adic) topology of 0 (it will be Hausdorff if and only if $H \cap G_\infty = 0$, or $H \cap p^\infty G = 0$) and in its own natural (or p-adic) tonology (it will be Hausdorff if and only if $H_\infty = 0$, or $p^\infty H = 0$).

2. In this section we begin the discussion of the groups such that $E(G)$ is linearly compact in the finite topology; our first goal is to get rid of the elements of infinite height.

LEMMA 2.1. If G is any group, $G[p^k]$ is contained in an orbit for every prime p and natural number k.

PROOF. If G has a cyclic direct summand $\langle a \rangle$ such that $o(a) = p^k$ then $G[p^k] \subseteq Ea$. Otherwise $t_p(G) = B \oplus D$ where $p^{k-1}B = 0$ and D is a divisible p-group, and $G[p^k] \subseteq E(b + d)$ where b is an element of maximum order in B and d is either 0 (in case $D = 0$) or an element of order p^k in a quasi-cyclic direct summand of D.

LEMMA 2.2. If G is not a torsion group, then the divisible part D of G is contained in Eg for every $g \in G$, $g \notin t(G)$.

PROOF. For every $d \in D$ and rational number $r \neq 0$ there is a homomorphism $\alpha_r: Q \to D$ such that $\alpha_r(r) = d$. For $g \in G$, $g \notin t(G)$ there is a monomorphism $\varphi: (\langle g \rangle + t(G)/t(G)) \to Q$ which extends to a homomorphism ψ: $G/t(G) \to Q$; if π: $G \to G/t(G)$ is the canonical map, then $r = \psi(\pi(g)) \neq 0$, and $\beta = \alpha_r \psi \pi$ is an element of $E(G)$ such that $d = \beta(g) \in Eg$.

LEMMA 2.3. If $E(G)$ is linearly compact in the finite topology, then G_∞ is divisible.

PROOF. Suppose $a \in G_\infty$; we must show that for every prime p and natural number k there is $b \in G_\infty$ such that $p^k b = a$. Since $a \in G_\infty$, $a = p^k g$ for some $g \in G$; moreover, for every $n \in N$, there is $g_n \in G$ with $a = p^k(n!)g_n$: but $p^k(g - n!g_n) = 0$, i.e. $\{g - (n!)g_n\}$ is a Cauchy sequence in $G[p^k]$ with respect to the topology induced in $G[p^k]$ by the natural topology of G. By Lemma 1.2 and Lemma 2.1 $\{g - (n!)g_n\}$ has a limit $h \in G[p^k]$; if we put $b = g - h$, then $b \in G_\infty$ and $p^k b = p^k g = a$.

We can now prove the reduction to the Hausdorff case.

THEOREM 2.4. Let $G = H \oplus D$ where D is the divisible part of G. The following statements are equivalent:

i) $E(G)$ is linearly compact in the finite topology;

ii) $E(H)$ is linearly compact in the finite topology, H has no elements of infinite height, and, if G is not a torsion group, then D has only finitely many non-zero p-components.

PROOF. i) $\Rightarrow$ ii): $E(H)$ can be identified as the subring of $E(G)$ consisting of those $\varphi \in E(G)$ such that $\varphi(H) \subseteq H$, $\varphi(D) = 0$. The finite topology of $E(G)$ induces on $E(H)$ its own finite topology: thus, if V_X is the annihilator in $E(H)$ of the finite subset X of H and if U_Y is the annihilator in $E(G)$ of the finite subset $Y = \{y_1 = h_1 + d_1, \ldots, y_n = h_n + d_n\}$ of $G (h_i \in H, d_i \in D)$, then $V_X = E(H) \cap U_X$ and $E(H) \cap U_Y = V_{\{h_1, \ldots, h_n\}}$. Furthermore, $E(H)$ is closed in $E(G)$: assume $\varphi \in \bigcap_Y (E(H) + U_Y) = \overline{E(H)}$, where Y runs in the family of finite subsets of G; then for $d \in D$, $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 \in E(H)$ and $\varphi_2 \in U_d$, whence $\varphi(d) = \varphi_1(d) + \varphi_2(d) = 0$; and for $h \in H$, $\varphi = \varphi_3 + \varphi_4$ with $\varphi_3 \in E(H)$ and $\varphi_4 \in U_h$, whence $\varphi(h) = \varphi_3(h) \in H$, i.e. $\varphi \in E(H)$. This proves that $E(H)$, as a closed submodule of a linearly compact Hausdorff module, is itself linearly compact; H being reduced, Lemma 2.3 implies $H_\infty = 0$. Suppose now that G is not a torsion group; by Lemma 2.3 D is a linearly compact discrete E-module, so it cannot contain an infinite direct sum of E-submodules [B, Ex. 20, p. 111]; in particular, D has only finitely many non-zero p-components.

ii) $\Rightarrow$ i): We have to prove that for every $g \in G$, $E(G)g$ is linearly compact as a discrete $E(G)$-module; if $g = h + d$ with $h \in H$, $d \in D$,

$E(G)\,g \subseteq E(H)\,h \oplus D'$, where $D' = D$ if g is torsion-free, and $D' = \bigoplus_{p|o(g)} t_p(D)$ if g is torsion; D' is in either case an artinian $E(G)$-module, hence a linearly compact discrete one; $E(H)h$ is likewise linearly compact as a discrete $E(H)$-module. Let $\{x_i + E_i\}_{i\in I}$ be a family of linear $E(G)$-varieties contained in $E(G)$ with the finite intersection property; if $\pi_H: G \to H$ and $\pi_D: G \to D$ are the projections, then $E_i = \pi_H(E_i) \oplus \pi_D(E_i)$, $\pi_H(E_i) = E_i \cap H$ is an $E(H)$-submodule of $E(H)h$, $\pi_D(E_i) = E_i \cap D$ is an $E(G)$-submodule of D', $\{\pi_H(x_i + E_i)\}_{i\in I}$ and $\{\pi_D(x_i + E_i)\}_{i\in I}$ have the finite intersection property, so there are $h' \in \bigcap_{i\in I} \pi_H(x_i + E_i)$, $d' \in \bigcap_{i\in I} \pi_D(x_i + E_i)$, and $h' + d' \in \bigcap_{i\in I} (x_i + E_i)$.

3. In view of Theorem 2.4, we shall now proceed to classify all groups G with $G_\infty = 0$ such that $E(G)$ is linearly compact.

Lemma 3.1. *Assume $E = E(G)$ is linearly compact in the finite topology, and $G_\infty = 0$. Then G is a J-module.*

Proof. Suppose $\alpha \in J$ is the limit of $\{n_k\}_{k\in N}$, a Cauchy sequence of integers with respect to the natural topology of Z. Then $\{n_k g\}_{k\in N}$ is a Cauchy sequence in Eg with respect to the relative topology of the natural topology of G; from Lemmas 1.1 and 1.2 it follows that Eg is complete (and Hausdorff, since $G_\infty = 0$) in this topology, so we can define $\alpha g = \lim n_k g$. It is easily checked that this product is well defined, and that the module axioms are indeed fulfilled.

Lemma 3.2. *Under the hypotheses of Lemma 3.1, if $\varepsilon_p = (0, \dots, 0, 1, 0, \dots) \in J$ (1 is in the p-th place), and if $g \in G$, then $\varepsilon_p g = 0$ for almost all primes p.*

Proof. Otherwise Eg would contain the infinite direct sum of E-submodules $\bigoplus_{p\in P} \varepsilon_p Eg$.

Theorem 3.3. *Let G be a group without elements of infinite height. $E(G)$ is linearly compact in the finite topology if and only if $G = \bigoplus_{p\in P} G_p$ where, for each prime p, G_p is a J_p-module without elements of infinite height such that $E(G_p)$ is linearly compact in the finite topology.*

Proof. Assume that $E(G)$ is linearly compact in the finite topology. For $\varepsilon_p = (0, \dots, 0, 1, 0, \dots) \in J$ (with 1 in the p-th place) put $G_p = \varepsilon_p G$; then G_p is a J_p-module, $p^\infty G_p = 0$, and $G \supseteq \bigoplus_{p\in P} G_p$. From

Lemma 3.2, for every $g \in G$ we can write $g = 1g = \sum_{p \in P} \varepsilon_p g$, which shows that $G = \bigoplus_{p \in P} G_p$. Furthermore, since $\mathrm{Hom}_Z(G_p, G_q) = 0$ if $p \neq q$, $E(G) \cong \prod_{p \in P} E(G_p)$, the finite topology of $E(G)$ coincides with the product topology of the finite topologies of the $E(G_p)$'s, and, considering $E(G_p)$ as an $E(G)$-module, the $E(G)$-submodules are precisely the $E(G_p)$-submodules. It follows that $E(G_p)$ is linearly compact in the finite topology for every prime p. Conversely, if $G = \bigoplus_{p \in P} G_p$ with $p^\infty G_p = 0$ and $E(G_p)$ linearly compact in the finite topology, then clearly $G_\infty = 0$ and $E(G)$, being algebraically and topologically isomorphic to $\prod_{p \in P} E(G_p)$, is linearly compact in the finite topology.

4. We are thus led to determine which ones of the J_p-modules M without elements of infinite height are such that $E(M)$ $(= E_{J_p}(M))$ is linearly compact in the finite topology. We shall deal separately with the torsion, torsion-free, and mixed case.

THEOREM 4.1. Let M be a p-group without elements of infinite height. The following statements are equivalent:

1) $E = E(M)$ is linearly compact in the finite topology.

2) M is torsion-complete.

PROOF. 1) $\Rightarrow$ 2): For every $k \in N$, $M[p^k]$ is complete in the relative topology of the p-adic topology of M by Lemmas 1.2 and 2.1; M is therefore torsion-complete [F 1, 70.7, p. 28].

2) $\Rightarrow$ 1): If $a \in M$, with $o(a) = p^k$, then

$$Ea = \{x \in M \mid h(p^i x) \geqslant h(p^i a),\ i = 0, \ldots, k\} = \Big(\bigcap_{i=0}^{k-1} (p^i)^{-1}(p^{h(p^i a)} M)\Big) \cap M[p^k]$$

[F 1, 65.5, p. 4]; according to Lemma 1.1, we shall show that every orbit Ea is a linearly compact discrete E-module, by proving that for every $n \in N$ $M[p^n]$ is linearly compact. If M is bounded, then it is easily seen that there are only finitely many orbits, so M as well as every $M[p^n]$ is even an artinian E-module. If M is not bounded, for every $k \in N$ select $a_k \in M[p^n] \cap p^k N$ with $o(a_k) = p^n$. Obviously

$$Ea_k \subseteq M[p^n] \cap p^k M; \text{ moreover } Ea_k = \Big(\bigcap_{i=0}^{n-1} (p^i)^{-1}(p^{h(p^i a_k)} M)\Big) \cap M[p^n]$$

is open in the topology τ_n induced in $M[p^n]$ by the p-adic topology of M, so $\{Ea_k\}_{k\in N}$ is a basis for τ_n. $M[p^n]/Ea_k$ has only finitely many E-submodules, and $(M[p^n], \tau_n) = \varprojlim M[p^n]/Ea_k$ because $(M[p^n], \tau_n)$ is complete. It follows that $(M[p^n], \tau_n)$ is a linearly compact E-module.

Now if H is an E-submodule of $M[p^n]$ with $p^r H = 0$, $r \leqslant h$, and if $a \in H$ has maximum order $H \supseteq Ea = \left(\bigcap_{i=0}^{r-1} (p^i)^{-1}(p^{h(p^i a)} M)\right) \cap M[p^r]$ is open, hence closed in $M[p^r]$ with respect to the topology τ_r, so that it is closed in τ_n as well. Hence, if $\{x_\lambda + H_\lambda\}_{\lambda\in\Lambda}$ is a family of cosets of $M[p^n]$ modulo E-submodules H_λ having the finite intersection property, every coset $x_\lambda + H_\lambda$ is τ_n-closed and $\bigcap_{\lambda\in\Lambda} (x_\lambda + H_\lambda) \neq \emptyset$ because $(M[p^n], \tau_n)$ is linearly compact.

THEOREM 4.2. Let M be a reduced torsion-free J_p-module. The following statements are equivalent:

1) $E = E(M)$ is linearly compact in the finite topology.

2) M is complete in the p-adic topology.

PROOF. If $a \in p^k M$, but $a \notin p^{k-1} M$, then $a = p^k g$ with $M = J_p g \oplus \oplus M'$ [K, p. 32]; it follows that $Ea = p^k M$ and that for every orbit linear compactness is equivalent to completeness in the p-adic topology. Since M itself is an orbit, this remark proves the Theorem.

THEOREM 4.3. Let M be a mixed J_p-module without elements of infinite height. The following statements are equivalent:

1) $E = E(M)$ is linearly compact in the finite topology.

2) $M = B \oplus C$, where B is a bounded p-group and C is a reduced torsion-free J_p-module complete in the p-adic topology.

PROOF. 1) $\Rightarrow$ 2): We shall show first that $M/t(M)$ cannot be divisible. Thus, by Lemmas 1.2. and 2.1., for every $n \in N$, $M[p^n]$ is complete in the relative topology of the p-adic topoloty of M, so $t(M) = T$ is the torsion subgroup of its p-adic completion $\hat{T}$, and if M/T is divisible, then $T \subseteq M \subseteq \hat{T}$. If $B = \bigoplus_{n\in N} B_n$ is a basic subgroup of T (where B_n is either 0 or a direct sum of cyclic groups of order p^n), then B is not bounded and $\hat{T}$ can be viewed as the subgroup of $\prod_{n\in N} B_n$ consisting of all sequences $(x_n)_{n\in N}$ such that $\{n - e(x_n)\}$ tends to infinity; in this representation T is identified with the subgroup of $\hat{T}$ consisting of all sequences $(x_n)_{n\in N}$ where $\{e(x_n)\}$ is bounded. For every $i \in N$ the projection $\prod_{n\in N} B_n \to B_i$ induces an endomorphism of M; it follows that if $g \in M$, $g \notin T$, then $t(Eg)$ is unbounded. On the other hand, $t(Eg)$ is a linearly compact discrete E-module, so it is complete in its p-adic topology, hence it is bounded: this contradiction proves that $M/t(M)$ is not divisible. But then there is $g \in M$, $g \notin T$, such that $M = J_p g \oplus M$; for this choice of g, $Eg = M$ and Lemmas 1.2 and 2.1 imply that M and $t(M)$ are complete in the p-adic topology, so $t(M) = B$ is bounded and $M = B \oplus C$ where C is likewise complete in the p-adic topology. 2) $\Rightarrow$ 1): If $M = B \oplus C$ with B and C satisfying the above conditions, then for $g \in B$, Eg is artinian, while if $g = b + c$ with $b \in D$, $0 \neq c \in C$, $h(c) = k$, $Eg \subseteq B \oplus p^k C$ is linearly compact in the discrete topology, since so are B and $B + Eg/B \cong p_k C$.

Remark. From 4.1. and 4.3. it follows that if T is a torsion-complete unbounded p-group and $\hat{T}$ is its p-adic completion, then $E(\hat{T})$, which is isomorphic to $E(T)$, is not linearly compact in the finite topology, while $E(T)$ is.

5. In this section M will always denote a J_p-module without elements of infinite height; the p-finite topology of $E(M)$ ([L 1] and [L 2]) has the family of all left ideals $U_X^n = \{f \in E(M) | f(M) \subseteq p^n M\}$, where $n \in N$ and X is a finite subset of G, as a basis of neighbourhoods of 0; it is a Hausdorff topology since $p^\infty M = 0$.

Lemma 5.1. If $E(M)$ is linearly compact in the finite topology, then it is complete in the p-finite topology.

Proof. The p-finite topology is weaker than the finite topology; so [B, Ex. 16 γ, p. 109] applies.

Lemma 5.2. If M is either torsion or torsion-free, then $E(M)$ is linearly compact in the finite topology if and only if it is complete in the p-finite topology.

Proof. Compare 4.1 and 4.2 above with [L 1] and [L 2].

Lemma 5.3. If M is complete in the p-adic topology, then $E(M)$ is complete in the p-finite topology.

Proof. E with the p-finite topology is embedded as a closed subspace in M^M with the product topology of the p-adic topologies of the factors.

THEOREM 5.4. Let M be a mixed J_p-module. The following statements are equivalent:

1) $E(M)$ is complete in the p-finite topology.
2) $T = t(M)$ is torsion-complete and either M is complete in the p-adic topology, or M is a p-pure fully invariant subgroup of the p-adic completion $\hat{T}$ of T.

PROOF. 1) $\Rightarrow$ 2): That T is torsion-complete can be seen exactly as in [L 1]. Since $p^n E \subseteq U_X^n$ for every natural number n and finite subset X of G, the p-adic topology of E is stronger than the p-finite topology; it follows that E is complete in the p-adic topology. For every $g \in M$, $\pi_g\colon E \to Eg$ defined by $\pi_g(\varphi) = \varphi(g)$ for every $\varphi \in E$ is a continuous map with respect to the p-adic topologies of E and Eg, $\operatorname{Ker} \pi_g = U_g$ is closed, since $p^n(Eg) = (p^n E) g = \pi_g(p^n E)$ π_g is open, so that the p-adic topology of Eg is the quotient topology of π_g. It follows that, for every $g \in M$, Eg is complete in the p-adic topology. In particular, if M/T is not divisible, there is $g \in M$, g torsion-free, such that $M = J_p g \oplus M'$, and $M = Eg$ is complete in the p-adic topology. If M/T is divisible, then $T \subseteq M \subseteq \hat{T}$, M is p-pure in $\hat{T}$ and the restriction map $\varrho\colon E(M) \to E(T)$ is injective; we shall prove that it is also surjective. Take a basic subgroup $B = \bigoplus_{n \in N} B_n$ of T, where for each $n \in N$ B_n is either 0 or a direct sum of cyclic groups of order p^n; M admits the decompositions $M = B_1 \oplus \ldots \oplus B_n \oplus K_n$ with $K_n = B_{n+1} \oplus K_{n+1}$. Let $\varphi \in E(T)$; for each $n \in N$ define $\varphi_n\colon M \to M$ by: $\varphi_{n|B_1 \oplus \ldots \oplus B_n} = \varphi_{|B_1 \oplus \ldots \oplus B_n}$, $\varphi_{n|K_n} = 0$. For every $x \in M$ and $i \in N$ there is $j \in M$ such that for $n \geqslant j$, if $x = b_1 + \ldots + b_n + k_n$ with $b_i \in B_i$, $k_n \in K_n$, then $b_n \in p^i B$; this implies that for $n \geqslant j$ $(\varphi_n - \varphi_{n-1})(x) = \varphi(b_n) \in p^i M$, i.e. $\{\varphi_n\}$ is Cauchy in the p-finite topology of $E(M)$; if $\psi = \lim \varphi_n$, then $\psi_{|T} = \varphi$. If now α is an arbitrary endomorphism of $\hat{T}$, there is $\beta \in E(M)$ such that $\beta_{|T} = \alpha_{|T}$; $\alpha_{|M} - \beta\colon M \to \hat{T}$ is 0 on the dense subset T of M, so $\alpha_{|M} \in E(M)$, and M is fully invariant in $\hat{T}$.

2) $\Rightarrow$ 1): Assume first that $t(M) = T$ is torsion-complete and M is p-pure and fully invariant in $\hat{T}$. In this case we make the identifications: $E = E(\hat{T}) = E(M) = E(T)$, and remark that the p-finite topology of E does not depond on which group E operates: thus for $g \in T$ and for every $n \in N$, U_g^n is open in all three topologies; and if $g \notin T$ for every $n \in N$ there is $t_n \in T$ such that $g - t_n \in p^n \hat{T}$ ($g - t_n \in p^n M$ if $g \in M$): from $\varphi \in U_{t_n}^n$ it follows that

$$\varphi(g) = \varphi(t_n) + \varphi(g - t_n) \in p^n \hat{T}\big(\varphi(g) \in p^n M \text{ if } g \in M\big)\,,$$

i.e. $U_g^n \supseteq U_{t_n}^n$ and U_g^n is open in the p-finite topology of E regarded as the ring of endomorphisms of T. To end the proof, take $G = M$ if M is complete in the p-adic topology, $G = \hat{T}$ if M is a p-pure fully invariant subgroup of $\hat{T}$; $E = E(M) = E(G)$, M and G induce on E the same p-finite topology, so E is complete in the p-finite topology by Lemma 5.1.

REMARK. p-pure fully invariant subgroups of the p-adic completion $\hat{T}$ of a Hausdorff p-group T have been determined by Mader in [M]: there it is proved that the Ulm sequences of the elements of $\hat{T}$ are a meet-semilattice H (meets are taken pointwise) with respect to the obvious ordering; if N is a fully invariant subgroup of $\hat{T}$, $\{H(x) | x \in N\}$ is a filter of H, and in this way one gets an isomorphism of the lattice of fully invariant subgroups of $\hat{T}$ onto the lattice of all filters of H; a filter Φ of H corresponds to a p-pure fully invariant subgroup if and only if for every $n \in N$ and $h = (h_0, h_1, \ldots, h_n, \ldots) \in \Phi$, $n + h = (h_n, h_{n+1}, \ldots) \in \Phi$; a filter of H is principal exactly when it corresponds to an orbit of $\hat{T}$.

6. In this last section we characterize the groups G whose endomorphism ring is strictly linearly compact in the finite topology.

THEOREM 6.1. A necessary and sufficient condition for $E(G)$ to be strictly linearly compact in the finite topology is that

$$G = \Big(\bigoplus_{\Lambda} Q\Big) \oplus \Big(\bigoplus_{p \in P} (B_p \oplus D_p)\Big)$$

where, for every prime p, B_p is a bounded p-group, D_p is a divisible p-group, and almost all D_p's are 0 if $\Lambda \neq \emptyset$.

PROOF. Assume $E = E(G)$ is strictly linearly compact. If p is a prime, $G[p]$ is an orbit, so the descending chain $\{G[n] \cap p^n t_p(G)\}$ of E-submodules becomes stationary after a finite number of steps by Lemma 1.1; this implies that $t_p(G) = B_p \oplus D_p$ where B_p is bounded and D_p is divisible, and so $t_p(G)$ is a direct summand of G. If $\psi_p: G \to t_p(G)$ is a projection, then for every $g \in G$, $\psi_p(g) \in t_p(Eg)$; the artinian E-module Eg cannot contain an infinite direct sum of E-submodules, so $t_p(Eg) = 0$ for almost all $p \in P$, hence $\psi_p(g) = 0$ for almost all $p \in P$. We can define $\psi: G \to t(G)$ by letting $\psi(g) = \sum_{p \in P} \psi_p(g)$; ψ is a projection, proving that $t(G)$ is a direct summand of G. $G/t(G)$ is divisible: thus if $g \in G$ the descending chain $\{n!Eg\}_{n \in N}$ of E-modules is stationary, so there is $r \in N$ such that, for every $m \in N$, $r!Eg = (rm)!Eg$; in particular $r!g = (rm)!\varphi(g)$ with $\varphi \in E$, hence $r!(g - mg') = 0$ for a suitable $g' \in G$. To end the proof that the condition is necessary, we only need to show that if $G \neq t(G)$ the D_p's are almost always 0; but this follows from Theorem 2.4. Conversely, let G be as in the Theorem. If $\Lambda = \emptyset$, G is a torsion group; we have only to prove that $E(t_p(G))$ is strictly linearly compact in the finite topology for every $p \in P$; but this is obvious since $t_p(G)$ is an artinian $E(t_p(G))$-module. If $\Lambda \neq \emptyset$, write any $g \in G$ as $g = g_1 + g_2$ with $g_1 \in \bigoplus_{\Lambda} Q$, $g_2 \in \bigoplus_{p \in J} t_p(G)$ where J is a finite subset of P which we may assume to contain all the primes p such that $D_p \neq 0$; it follows that $Eg \subseteq \left(\bigoplus_{\Lambda} Q\right) \oplus \oplus \left(\bigoplus_{p \in J} t_p(G)\right)$, an artinian E-module, and the proof is complete.

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