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Multi-valued contraction mappings in complete metric spaces

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Multi-Valued Contraction Mappings in Complete Metric Spaces.

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Let \((X, d)\) be a metric space. For any nonempty subsets \(A, B\) of \(X\), we define

\[
D(A, B) = \inf \{d(a, b) | a \in A, b \in B\},
\]

\[
\delta(A, B) = \sup \{d(a, b) | a \in A, b \in B\},
\]

\[
H(A, B) = \max \{\sup \{D(a, B) | a \in A\}, \sup \{D(A, b) | b \in B\}\}.
\]

Let \(CB(X)\) be the set of all nonempty closed and bounded subsets of \(X\). The space \(CB(X)\) is a metric space with respect to the above defined distance \(H\) (see K. Kuratowski [1] p. 214). Then we have the following theorem which is a generalization of S. Reich result [2] (or see I. Rus [3]).

**Theorem 1.** Let \((X, d)\) be a complete metric space, and let \(f : X \rightarrow CB(X)\) be a multi-valued mapping with the following condition: for every \(x, y \in X\),

\[
H(f(x), f(y)) \leq \alpha \left( D(x, f(x)) + D(y, f(y)) \right) + \\
+ \beta \left( D(x, f(y)) + D(y, f(x)) \right) + \gamma D(x, y),
\]

where \(\alpha, \beta, \gamma\) are non-negative and \(2\alpha + 2\beta + \gamma < 1\). Then \(f\) has a fixed point, i.e. there is a point \(x\) such that \(x \in f(x)\).

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PROOF. Let \( x_0 \) be a point in \( X \), and \( x_1 \in f(x_0) \). If

\[
H(f(x_0), f(x_1)) = 0,
\]

then we have \( f(x_0) = f(x_1) \), since \( H \) is a metric on \( CB(X) \). Therefore we have \( x_1 \in f(x_1) \). This contains the proof of the case \( \alpha = \beta = \gamma = 0 \).

Next we suppose \( 0 < 2\alpha + 2\beta + \gamma \) and

\[
H(f(x_0), f(x_1)) > 0.
\]

Put \( p = (2\alpha + 2\beta + \gamma)^{\frac{1}{4}} \), then \( 0 < p < 1 \).

Let \( h = H(f(x_0), f(x_1))/p \), then we have

\[
h > H(f(x_0), f(x_1)).
\]

By the definition of \( H \), we have

\[
h > H(f(x_0), f(x_1)) > D(x_1, f(x_1)).
\]

Therefore there is a point \( x_2 \) of \( f(x_1) \) such that

\[
h > d(x_1, x_2).
\]

Hence

\[
d(x_1, x_2) < p^{-1}H(f(x_0), f(x_1))
\]

\[
< p^{-1}\left\{\alpha[D(x_0, f(x_0)) + D(x_1, f(x_1))] + \beta[D(x_0, f(x_1)) + D(x_1, f(x_0))] + \gamma D(x_0, x_1)\right\}
\]

\[
< p^{-1}\left\{\alpha[d(x_0, x_1) + d(x_1, x_2)] + \beta[d(x_0, x_2) + d(x_1, x_1)] + \gamma d(x_0, x_1)\right\}
\]

\[
< p^{-1}\left\{\alpha[d(x_0, x_1) + d(x_1, x_2)] + \beta[d(x_0, x_1) + d(x_1, x_2)] + \gamma d(x_0, x_1)\right\}.
\]

Hence we have

\[
(p - (\alpha + \beta))d(x_1, x_2) < (\alpha + \beta + \gamma)d(x_0, x_1).
\]
Therefore

\[ d(x_1, x_2) < q d(x_0, x_1), \]

where \( q = (\alpha + \beta + \gamma)(p - (\alpha + \beta)) \) and \( 0 < q < 1 \).

For \( x_1, x_2 \), we have two cases:

1) \( H(f(x_1), f(x_2)) = 0 \),

2) \( H(f(x_1), f(x_2)) > 0 \).

If we have the first case, then \( x_2 \in f(x_2) \), which completes the proof.

If \( H(f(x_1), f(x_2)) > 0 \), by a similar method, there is a point \( x_3 \) of \( f(x_2) \) such that

\[ d(x_2, x_3) < q d(x_1, x_2). \]

In general, if \( H(f(x_i), f(x_{i+1})) = 0 \) for some \( i \), then \( x_i \in f(x_i) \). If, for all \( i \) (\( i = 0, 1, \ldots \)), \( H(f(x_i), f(x_{i+1})) > 0 \), there is a point \( x_{i+2} \in f(x_{i+1}) \) satisfying

\[ d(x_{i+1}, x_{i+2}) < q d(x_i, x_{i+1}). \]

Hence, for \( n > m \),

\[ d(x_n, x_m) < \frac{q^m}{1 - q} d(x_0, x_1). \]

This shows that \( \{x_n\} \) is a Cauchy sequence. The completeness of \( X \) implies the existence of the limit of \( \{x_n\} \). Let \( x' \) be the limit of \( \{x_n\} \); then

\[ D(x', f(x')) < d(x', x_{n+1}) + d(x_{n+1}, f(x')) \]

\[ < d(x', x_{n+1}) + H(f(x_n), f(x')). \]

Hence, by the assumption, we have

\[
(1) \quad D(x', f(x')) < d(x', x_{n+1}) \\
+ \alpha [D(x_n, f(x_n)) + D(x', f(x'))] \\
+ \beta [D(x_n, f(x')) + D(x', f(x_n))] + \gamma D(x_n, x') \\
< d(x', x_{n+1}) + \alpha [d(x_n, x_{n+1}) + D(x', f(x'))] \\
+ \beta [D(x_n, f(x')) + d(x', x_{n+1})] + \gamma d(x_n, x').
\]
Let \( n \rightarrow \infty \), then (1) implies the following relation.

\[
D(x', f(x')) \leq \alpha D(x', f(x')) + \beta D(x', f(x')).
\]

From \( 1 - \alpha - \beta > 0 \), we have

\[
D(x', f(x')) = 0,
\]

which means \( x' \in f(x') \). This completes the proof.

Let \( BN(X) \) be the set of all nonempty bounded subset of \( X \). Then we have a fixed point theorem.

**Theorem 2.** Let \( (X, d) \) be a complete metric space. If \( f: X \rightarrow BN(X) \) is a function which satisfies

\[
\delta(f(x), f(y)) \leq \alpha[H(x, f(x)) + H(y, f(y))] + \beta[H(x, f(y)) + H(y, f(x))] + \gamma d(x, y),
\]

for every \( x, y \) in \( X \), where \( \alpha, \beta, \gamma \) are non-negative and \( 2\alpha + 4\beta + \gamma < 1 \), then \( f \) has a unique fixed point, i.e. for some \( x' \), \( f(x') = \{x'\} \).

Theorem 2 is a generalization of S. Reich result [2]. The proof is due to an idea by S. Reich [2].

**Proof.** If \( \alpha = \beta = \gamma = 0 \), then the result is trivial. We suppose \( 0 < 2\alpha + 4\beta + \gamma \). Now put \( p = (2\alpha + 4\beta + 1)^{1/2} \). Then we have \( p < 1 \). Hence there is a single-valued function \( g: X \rightarrow X \) such that \( g(x) \) is a point \( y \) in \( f(x) \) which satisfies

\[
d(x, y) = d(x, g(x)) \geq pH(x, f(x)).
\]

For such a function \( g \),

\[
d(g(x), g(y)) \leq \delta(f(x), f(y))
\]

\[
\leq \alpha[H(x, f(x)) + H(y, f(y))]
\]

\[
+ \beta[H(x, f(y)) + H(y, f(x))]
\]

\[
+ \gamma d(x, y)
\]

\[
+ \alpha p^{-1}[d(x, g(x)) + d(y, g(y))]
\]

\[
+ \beta p^{-1}[2d(x, y) + d(x, g(x)) + d(y, g(y))] + \gamma d(x, y)
\]

\[
\leq (\alpha + \beta) p^{-1}[d(x, g(x)) + d(y, g(y))] + (2\beta p^{-1} + \gamma) d(x, y).
\]
Hence we have
\[
(2) \quad d(g(x), g(y)) \leq (\alpha + \beta) p^{-1}[d(x, g(x)) + d(y, g(y))] \\
+ (2\beta p^{-1} + \gamma) d(x, y).
\]

The assumption $2\alpha + 4\beta + \gamma < 1$ implies $2(\alpha + \beta)p^{-1} + 2\beta p^{-1} + + \gamma < 1$. By a well known theorem, $g$ has a fixed point $x'$, i.e. $g(x') = x'$. For the point $x'$,
\[
0 = (x', g(x')) \geq pH(x', f(x')).
\]
Hence $x' \in f(x')$.

If $z \in f(z)$, and $H(z, f(z)) > 0$, then
\[
\delta(f(y), f(y)) \leq 2(\alpha + \beta) H(y, f(y)) < H(y, f(y)),
\]
which is impossible. Hence we have $f(z) = \{z\}$.

To show $z = x'$, consider
\[
\delta(f(z), f(x')) \leq \beta [H(z, f(x')) + H(x', f(z))] + \gamma d(z, x') \leq (2\beta + \gamma) d(z, x').
\]
Hence we have $z = x'$, which shows that $f$ has a unique fixed point. The proof is complete.

REFERENCES


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