Multi-valued contraction mappings in complete metric spaces

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Multi-Valued Contraction Mappings in Complete Metric Spaces.

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Let \((X, d)\) be a metric space. For any nonempty subsets \(A, B\) of \(X\), we define
\[
D(A, B) = \inf \{d(a, b) | a \in A, b \in B\},
\]
\[
\delta(A, B) = \sup \{d(a, b) | a \in A, b \in B\},
\]
\[
H(A, B) = \max \{\sup \{D(a, B) | a \in A\}, \sup \{D(A, b) | b \in B\}\}.
\]

Let \(CB(X)\) be the set of all nonempty closed and bounded subsets of \(X\). The space \(CB(X)\) is a metric space with respect to the above defined distance \(H\) (see K. Kuratowski [1] p. 214). Then we have the following theorem which is a generalization of S. Reich result [2] (or see I. Rus [3]).

**THEOREM 1.** Let \((X, d)\) be a complete metric space, and let \(f : X \to CB(X)\) be a multi-valued mapping with the following condition: for every \(x, y \in X\),
\[
H(f(x), f(y)) < \alpha(D(x, f(x)) + D(y, f(y))) + \beta(D(x, f(y)) + D(y, f(x))) + \gamma D(x, y),
\]
where \(\alpha, \beta, \gamma\) are non-negative and \(2\alpha + 2\beta + \gamma < 1\). Then \(f\) has a fixed point, i.e. there is a point \(x\) such that \(x \in f(x)\).

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PROOF. Let $x_0$ be a point in $X$, and $x_1 \in f(x_0)$. If
\[ H(f(x_0), f(x_1)) = 0 , \]
then we have $f(x_0) = f(x_1)$, since $H$ is a metric on $CB(X)$. Therefore
we have $x_1 \in f(x_1)$. This contains the proof of the case $\alpha = \beta = \gamma = 0$.
Next we suppose $0 < 2\alpha + 2\beta + \gamma$ and
\[ H(f(x_0), f(x_1)) > 0 . \]
Put $p = (2\alpha + 2\beta + \gamma)^{\frac{1}{2}}$, then $0 < p < 1$.
Let $h = H(f(x_0), f(x_1))/p$, then we have
\[ h > H(f(x_0), f(x_1)) . \]
By the definition of $H$, we have
\[ h > H(f(x_0), f(x_1)) > D(x_1, f(x_1)) . \]
Therefore there is a point $x_2$ of $f(x_1)$ such that
\[ h > d(x_1, x_2) . \]
Hence
\[ d(x_1, x_2) < p^{-1}H(f(x_0), f(x_1)) \]
\[ < p^{-1}\left\{ \alpha[D(x_0, f(x_0)) + D(x_1, f(x_1))] \right. \]
\[ + \beta[D(x_0, f(x_1)) + D(x_1, f(x_0))] + \gamma D(x_0, x_1) \left. \right\} \]
\[ < p^{-1}\left\{ \alpha[d(x_0, x_1) + d(x_1, x_2)] \right. \]
\[ + \beta[d(x_0, x_2) + d(x_1, x_1)] + \gamma d(x_0, x_1) \left. \right\} \]
\[ < p^{-1}\left\{ \alpha[d(x_0, x_1) + d(x_1, x_2)] \right. \]
\[ + \beta[d(x_0, x_2) + d(x_1, x_1)] + \gamma d(x_0, x_1) \left. \right\} . \]
Hence we have
\[ (p - (\alpha + \beta))d(x_1, x_2) < (\alpha + \beta + \gamma)d(x_0, x_1) . \]
Therefore
\[ d(x_1, x_2) < qd(x_0, x_1), \]
where \( q = (\alpha + \beta + \gamma)(p - (\alpha + \beta)) \) and \( 0 < q < 1 \).

For \( x_1, x_2 \), we have two cases:

1) \( H(f(x_1), f(x_2)) = 0 \),
2) \( H(f(x_1), f(x_0)) > 0 \).

If we have the first case, then \( x_2 \in f(x_0) \), which completes the proof.
If \( H(f(x_1), f(x_2)) > 0 \), by a similar method, there is a point \( x_3 \) of \( f(x_2) \) such that
\[ d(x_3, x_2) < qd(x_1, x_2). \]

In general, if \( H(f(x_i), f(x_{i+1})) = 0 \) for some \( i \), then \( x_i \in f(x_i) \). If, for all \( i (i = 0, 1, \ldots) \), \( H(f(x_i), f(x_{i+1})) > 0 \), there is a point \( x_{i+2} \in f(x_{i+1}) \) satisfying
\[ d(x_{i+1}, x_{i+2}) < qd(x_i, x_{i+1}). \]

Hence, for \( n > m \),
\[ d(x_n, x_m) \leq \frac{q^m}{1 - q} d(x_0, x_1). \]

This shows that \( \{x_n\} \) is a Cauchy sequence. The completeness of \( X \) implies the existence of the limit of \( \{x_n\} \). Let \( x' \) be the limit of \( \{x_n\} \), then
\[ D(x', f(x')) \leq d(x', x_{n+1}) + d(x_{n+1}, f(x')) \]
\[ < d(x', x_{n+1}) + H(f(x_n), f(x')). \]

Hence, by the assumption, we have

\[ D(x', f(x')) < d(x', x_{n+1}) \]
\[ + \alpha[D(x_n, f(x_n)) + D(x', f(x'))] \]
\[ + \beta[D(x_n, f(x')) + D(x', f(x_n))] + \gamma D(x_n, x') \]
\[ < d(x', x_{n+1}) + \alpha[d(x_n, x_{n+1}) + D(x', f(x'))] \]
\[ + \beta[D(x_n, f(x')) + d(x', x_{n+1})] + \gamma d(x_n, x'). \]
Let $n \to \infty$, then (1) implies the following relation.

$$D(x', f(x')) \leq \alpha D(x', f(x')) + \beta D(x', f(x')) .$$

From $1 - \alpha - \beta > 0$, we have

$$D(x', f(x')) = 0 ,$$

which means $x' \in f(x')$. This completes the proof.

Let $BN(X)$ be the set of all nonempty bounded subset of $X$. Then we have a fixed point theorem.

**THEOREM 2.** Let $(X, d)$ be a complete metric space. If $f : X \to BN(X)$ is a functions which satisfies

$$\delta(f(x), f(y)) \leq \alpha[H(x, f(x)) + H(y, f(y))] + \beta[H(x, f(y)) + H(y, f(x))] + \gamma d(x, y) ,$$

for every $x, y$ in $X$, where $\alpha, \beta, \gamma$ are non-negative and $2\alpha + 4\beta + \gamma < 1$, then $f$ has a unique fixed point, i.e. for some $x'$, $f(x') = \{x'\}$.

Theorem 2 is a generalization of S. Reich result [2]. The proof is due to an idea by S. Reich [2].

**PROOF.** If $\alpha = \beta = \gamma = 0$, then the result is trivial. We suppose $0 < 2\alpha + 4\beta + \gamma$. Now put $p = (2\alpha + 4\beta + 1)^{1/4}$. Then we have $p < 1$. Hence there is a single-valued function $g : X \to X$ such that $g(x)$ is a point $y$ in $f(x)$ which satisfies

$$d(x, y) = d(x, g(x)) \geq pH(x, f(x)) .$$

For such a function $g$,

$$d(g(x), g(y)) \leq \delta(f(x), f(y))$$

$$\leq \alpha[H(x, f(x)) + H(y, f(y))] + \beta[H(x, f(y)) + H(y, f(x))] + \gamma d(x, y)$$

$$\leq \alpha p^{-1}[d(x, g(x)) + d(y, g(y))] + \beta p^{-1}[2d(x, y) + d(x, g(x)) + d(y, g(y))] + \gamma d(x, y)$$

$$\leq (\alpha + \beta) p^{-1}[d(x, g(x)) + d(y, g(y))] + (2\beta p^{-1} + \gamma) d(x, y) .$$
Hence we have

\[ d(g(x), g(y)) \leq (\alpha + \beta)p^{-1}[d(x, g(x)) + d(y, g(y))] \]
\[ + (2\beta p^{-1} + \gamma)d(x, y). \]

The assumption \(2\alpha + 4\beta + \gamma < 1\) implies \(2(\alpha + \beta)p^{-1} + 2\beta p^{-1} + \gamma < 1\). By a well known theorem, \(g\) has a fixed point \(x'\), i.e. \(g(x') = x'\).

For the point \(x'\),

\[ 0 = (x', g(x')) \geq pH(x', f(x')). \]

Hence \(x' \in f(x')\).

If \(z \in f(z)\), and \(H(z, f(z)) > 0\), then

\[ \delta(f(y), f(y)) \leq 2(\alpha + \beta)H(y, f(y)) < H(y, f(y)) , \]

which is impossible. Hence we have \(f(z) = \{z\}\).

To show \(z = x'\), consider

\[ \delta(f(z), f(x')) \leq \beta[H(z, f(x')) + H(x', f(z))] + \gamma d(z, x') \leq (2\beta + \gamma)d(z, x'). \]

Hence we have \(z = x'\), which shows that \(f\) has a unique fixed point. The proof is complete.

REFERENCES


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