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A Common Fixed Point Theorem.

KIYOSHI ISEKI (*)

In this note, we shall prove a fixed point theorem which is a generalization of M. G. Maia's theorem [1].

THEOREM. *Let X be a metric space with two metrics d and δ . If X satisfies the following conditions:*

- 1) $d(x, y) \leq \delta(x, y)$ for every x, y in X ,
- 2) X is complete with respect to d ,
- 3) two mappings $f, g: X \rightarrow X$ are continuous with respect to the metric d , and

$$\begin{aligned} \delta(f(x), g(y)) \leq & \alpha \delta(x, y) + \beta [\delta(x, f(x)) + \delta(y, g(y))] \\ & + \gamma [\delta(x, g(y)) + \delta(y, f(x))] \end{aligned}$$

for every x, y in X , where α, β, γ are non-negative and $\alpha + 2\beta + 2\gamma < 1$, then f, g have a unique common fixed point.

PROOF. Let x_0 be a point in X , put

$$x_1 = f(x_0), \quad x_2 = g(x_1), \quad \dots, \quad x_{2n} = g(x_{2n-1}), \quad x_{2n+1} = f(x_{2n}), \quad \dots$$

Then

$$\delta(x_n, x_{n+1}) \leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} \right)^n \delta(x_0, x_1).$$

(*) Indirizzo dell'A.: Kobe University, Rokko Nada Kobe, Giappone.

(see I. Rus [2], p. 20). Therefore, by $\bar{d} \leq \delta$,

$$d(x_n, x_{n+1}) \leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} \right)^n \delta(x_0, x_1).$$

This shows that the sequence $\{x_n\}$ is a Cauchy sequence with respect to $\bar{d}$. Since X is complete with respect to $\bar{d}$, $\{x_n\}$ has a limit point y_0 in X , i.e. $x_n \xrightarrow{\bar{d}} y_0$. Hence, by the continuity of f with respect to the metric $\bar{d}$,

$$y_0 = \lim_{n \rightarrow \infty} x_{2n+1} = \lim_{n \rightarrow \infty} f(x_{2n}) = f\left(\lim_{n \rightarrow \infty} x_{2n}\right) = f(y_0).$$

Similarly we have $y_0 = g(y_0)$. Therefore y_0 is a common fixed point of f and g .

Let z_0 be a common fixed point of f, g , then

$$\begin{aligned} \delta(y_0, z_0) &= \delta(f(y_0), g(z_0)) \\ &\leq \alpha \delta(y_0, z_0) + 2\gamma \delta(y_0, z_0), \end{aligned}$$

which implies $y_0 = z_0$. Hence f, g have a unique common fixed point in X , which completes the proof.

REMARK. In Theorem, let $f(x) = g(x)$, $\beta = \gamma = 0$, then we have a theorem by M. G. Maia [1].

REFERENCES

- [1] M. G. MAIA, *Un'osservazione sulle contrazioni metriche*, Rend. Sem. Mat. Padova, **40** (1968), pp. 139-143.
- [2] I. A. RUS, *Teoria punctului fix*, II. Cluj (1973).
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