RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

ANNA ZAPPA

The center of the convolution algebra $C_u(G)$

Rendiconti del Seminario Matematico della Università di Padova, tome 52 (1974), p. 71-83

<http://www.numdam.org/item?id=RSMUP_1974__52__71_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 52 (1974)

The Center of the Convolution Algebra $C_u(G)^*$.

ANNA ZAPPA

1. Introduction.

Let G be a locally compact Abelian (LCA) group. Let $C_u(G)$ be the space of bounded complex-valued uniformly continuous functions. The dual space $C_u(G)^*$ can be given in a natural way the structure of a convolution algebra ([3], section 19).

In this paper we characterize for some groups G the center of the convolution algebra $C_u(G)^*$. More specifically we prove that the center is the algebra $M(G)$ of bounded regular measures on G, when G is either R or a discrete subgroup of R. We also prove the same result for the discrete group $G = \sum_{i=1}^{\infty} \oplus \boldsymbol{Z}(2)_i$, which is the weak direct product of countably many copies of $\boldsymbol{Z}(2)$, the cyclic group of order two.

It is likely that our result is true in general for every LCA group, but our present techniques do not seem to be susceptible of generalizations.

In section 2 we introduce the notation and the terminology.

In section 3 we prove the main theorems.

In section 4 we state a conjecture for arbitrary discrete groups and we prove a partial result for countable torsion groups, namely that a positive element of the center of $C_u(G)^*$ belongs to $M(G)$.

2. Preliminaries.

We recall here some basic definitions and facts concerning $C_u(G)$ and its dual $C_u(G)^*$ ([3], section 19).

(*) Indirizzo dell'A.: Istituto Matematico - Università di Genova.

First of all we want to define the operation of convolution between two elements of $C_u(G)^*$. This is done as in [3] as follows.

For a function $f \in C_u(G)$, let ${}_xf(y) = f(x+y)$, for all $x, y \in G$. Then the map $x \mapsto {}_xf$ is a uniformly continuous map from G to $C_u(G)$.

DEFINITION 1. *Let $\psi \in C_u(G)^*$ and $f \in C_u(G)$. Define*

$$T_\psi f(x) = \langle {}_xf, \psi\rangle .$$

We notice that, since the map $x \mapsto {}_xf$ is uniformly continuous, $T_\psi f \in C_u(G)$.

DEFINITION 2. *Let $\varphi, \psi \in C_u(G)^*$; we define $\varphi * \psi$ to be the element of $C_u(G)^*$, defined by the equation:*

$$\langle f, \varphi * \psi\rangle = \langle T_\psi f, \varphi\rangle , \qquad \text{for all } f \in C_u(G).$$

It is easy to verify that $C_u(G)^*$ becomes a Banach algebra with respect to the convolution so defined and that $M(G)$ is a subalgebra of $C_u(G)^*$, where $\langle f, \mu\rangle = \int_G f(y)\, d\mu(y)$, for G all $f \in C_u(G)$.

We can also verify immediately, as in [2], pg. 177, that the map $\psi \mapsto T_\psi$ maps isometrically $C_u(G)^*$ onto the algebra $M_u(G)$ of bounded linear operators on $C_u(G)$, which commute with translations. The isometry is an algebra isomorphism carrying convolution into the ordinary product of operators.

We also recall that $C_u(G)$ is a commutative C^*-algebra and therefore ([5], pg. 78) it is isomorphic to the algebra $C(\mathcal{M})$ of all continuous functions on its compact maximal ideals space $\mathcal{M}$. As a consequence, $C_u(G)^*$ may be identified with the space of all bounded regular Borel measures on $\mathcal{M}$. In particular, for $\mu \in M(G)$,

$$T_\mu(f)(x) = \int_G f(x+y)\, d\mu(y) = \check{\mu} * f(x) , \qquad \text{for all } f \in C_u(G),\ x \in G$$

where $\check{\mu}(E) = \mu(-E)$, for alla μ-measurable sets E.

It is clear that $G \subseteq \mathcal{M}$, and therefore $M(G)$ may be regarded as the subspace of $M(\mathcal{M})$ consisting of the measures carried by G.

In the sequel we will use freely the identifications

$$C_u(G)^* \simeq M_u(G) \simeq M(\mathcal{M}) .$$

3. The main result.

Let Z be the center of the algebra $C_u(G)^*$. Our first result is valid for all LCA groups G.

LEMMA 1. *Every bounded regular Borel measure $\mu \in M(G)$ defines an element of $C_u(G)^*$, which belongs to the center Z.*

PROOF. Let $\mu \in M(G)$, $\varphi \in C_u(G)^*$ and $f \in C_u(G)$. We shall prove that $\langle f, \varphi * \mu - \mu * \varphi \rangle$ can be made arbitrarily small in absolute value. Let K be a compact subset of G, such that $|\mu|(G \setminus K) < \varepsilon$.

Let U be a symmetric neighborhood of $0 \in G$, such that, for $x - y \in U$, $\|{}_xf - {}_yf\| < \varepsilon$. Let $x_1, \dots, x_n$ be finitely many points of K such that

$$K \subseteq (U + x_1) \cup \dots \cup (U + x_n)$$

and

$$K \cap (U + x_j) = \square\,, \qquad \forall j = 1, \dots, n.$$

Let $E_1 = U + x_1$, $E_k = (U + x_k) \setminus (E_1 \cup \dots \cup E_{k-1})$, $k = 2, \dots, n$. Then, for all $x \in G$,

$$\Big| \int_G f(x+y)\, d\mu(y) - \sum_{j=1}^{n} \mu(E_j) f(x + x_j) \Big| \leqslant$$

$$\leqslant \sum_{j=1}^{n} \Big| \int_{E_j} [f(x+y) - f(x+x_j)]\, d\mu(y) \Big| + \int_{G \setminus K} |f(x+y)|\, d|\mu|(y) \leqslant$$

$$\leqslant \varepsilon \sum_{j=1}^{n} |\mu|(E_j) + \varepsilon \|f\|_u \leqslant \varepsilon(\|\mu\| + \|f\|_u)\,,$$

because $|f(x+y) - f(x+x_j)| < \varepsilon$, if $y \in E_j \subset U + x_j$.

Let $g(x) = \sum_{j=1}^{n} \mu(E_j) f(x + x_j)$. We have proved then that

$$\|\check{\mu} * f - g\|_u < \varepsilon(\|\mu\| + \|f\|_u)\,.$$

Therefore $|\langle\check{\mu} * f, \varphi\rangle - \langle g, \varphi\rangle| < \varepsilon(\|\mu\| + \|f\|_u)\|\varphi\|$. On the other hand

$$|\langle T_\varphi f, \mu\rangle - \langle g, \varphi\rangle| = \Big|\int_G T_\varphi f(+y)\, d\mu(y) - \sum_{j=1}^{n} \mu(E_j)\, T_\varphi f(+x_j)\Big| \leqslant$$

$$\leqslant \sum_{j=1}^{n} \Big|\int_{E_j} [T_\varphi f(+y) - T_\varphi f(+x_j)]\, d\mu(y)\Big| + \Big|\int_{G\setminus K} T_\varphi f(+y)\, d\mu(y)\Big| \leqslant$$

$$\leqslant \varepsilon\|\varphi\| \sum_{j=1}^{n} |\mu(E_j)| + \varepsilon\|\varphi\|\ \|f\|_u \leqslant \varepsilon\|\varphi\|(\|f\|_u + \|u\|)\,,$$

because, for $y \in E_j$, $|T_\varphi f(+y) - T_\varphi f(+x_j)| \leqslant \|T_\varphi({}_{+y}f - {}_{+x_j}f)\|_u \leqslant \varepsilon\|\varphi\|$.

In conclusion

$$|\langle f, \varphi * \mu\rangle - \langle f, \mu * \varphi\rangle| \leqslant |\langle\check{\mu} * f, \varphi\rangle - \langle g, \varphi\rangle| + |\langle g, \varphi\rangle - \langle T_\varphi f, \mu\rangle| \leqslant$$

$$\leqslant \varepsilon(\|f\|_u + \|\mu\|)\|\varphi\|\,.$$

Since ε is arbitrary, we conclude that

$$\langle f, \varphi * \mu\rangle = \langle f, \mu * \varphi\rangle\,.$$

REMARK 1. It is not difficult to show that, for any LCA group, $Z \neq C_u(G)^*$. Indeed we know ([1], pg. 68) that there exists a bounded uniformly continuous function $f \in C_u(G)$, which is not weakly almost periodic. By definition this means that there exists a net $\{x_\alpha\}$ of elements of G, with the property that the net ${}_{x_\alpha}f$ of elements of $C_u(G)$ convergent subnet. But since the point measures δ_{x_α} are in $C_u(G)^*$ admits no weakly and $\|\delta_{x_\alpha}\| = 1$, some subnet (which we shall still denote by δ_{x_α}) converges in the weak* topology to an element $\psi \in C_u(G)^*$. This means that, for every $\varphi \in C_u(G)^*$,

$$\langle {}_{x_\alpha}f, \varphi\rangle = T_\varphi f(x_\alpha) \xrightarrow[\alpha]{} \langle T_\varphi f, \psi\rangle\,.$$

Now, if $\varphi * \psi = \psi * \varphi$, then $\langle T_\varphi f, \varphi\rangle = \langle T_\psi f, \psi\rangle$, and therefore

$$\langle {}_{x_\alpha}f, \varphi\rangle \xrightarrow[\alpha]{} \langle T_\psi f, \varphi\rangle\,,$$

for every $\varphi \in C_u(G)^*$. But this implies that ${}_{x_\alpha}f$ converges weakly, which is a contradiction.

For general groups this is as much as one can say at the moment (see however Section 4). The next Theorem, which is our main result requires rather stringent hypothesis on G.

REMARK 2. A LCA group G is ordered group ([6], chapter 8) if it is possible choose a semigroup P in G, which is closed and has two additional properties: $P \cap (-P) = \{0\}$, $P \cup (-P) = G$. An order is said to be *archimedean* if it has the following property: to every pair of elements x, y of G, $x > 0$, $y > 0$, there corresponds a positive integer n such that $nx > y$.

Let G be a LCA group with an archimedean order; then there is an orderpreserving isomorphism of G onto R, if G is not discrete, onto a discrete subgroup of R, if G is discrete.

THEOREM 1. *Let G be a ordered archimedean non discrete or countable discrete* LCA *groups. Then the center Z of the algebra $C_u(G)^*$ coincides with $M(G)$.*

PROOF. In view of our remarks in the introduction of this paper and in view of Lemma 1, we only need to show that, if $\varphi \in C_u(G)^*$ belongs to the center Z and φ corresponds to a measure $\mu \in M(\mathcal{M})$, then μ is concentrated on G. Since measures concentrated on G are in fact elements of the center, it will suffice to prove that any measure concentrated on a set disjoint from G is not in the center.

Let us denote A and B the sets $\{x \in G, x > 0\}$ and $\{x \in G, x \leqslant 0\}$ respectively, and $\bar{A}$ and $\bar{B}$ the weak $*$ closures of A and B respectively; we have $\bar{A} \cap \bar{B} = \square$, $\bar{A} \cup \bar{B} = \mathcal{M}$.

Consequently, a measure μ on $\mathcal{M}$ concentrated on $G' = \mathcal{M} \setminus G$ can be written as the sum of a measure μ_A concentrated on $\bar{A} \cap G'$ and of a measure μ_B concentrated on $\bar{B} \cap G'$: $\mu = \mu_A + \mu_B$.

CASE 1. Assume μ concentrated on $\bar{A} \cap G'$.

Let $f_0 \in C_u(G)$ a function such that $\langle f_0, \mu \rangle \neq 0$.

Let $\{x_N\}$ a sequence of elements of A, with the following properties:

i) $x_N > x_{N-1}$, for all $N = 1, 2, 3, \dots$

ii) for all $x \in A$, there exists N_x, such that $x < x_N$ for all $N > N_x$.

Let us denote $\{Y_N\}$ the sequence of positive elements thus defined

$$\begin{cases} Y_1 = x_1 \\ Y_N = Y_{N-1} + x_{N-1} + x_N, \quad \text{for } N = 2, 3, \dots . \end{cases}$$

Let us consider the function $h \in C_u(G)$, defined as follows:
if G is a non discrete group

$$\begin{cases} h(x) = 0 & \text{for } x \geqslant 0 \\ h(x) = f_0(x + y_N)\,, & \text{for } x \in [-y_N - x_N,\ -y_N + x_N - x_1] \\ h(x) \text{ linear for } x \in (-y_N + X_N - X_1,\ Y_{N-1} - X_{N-1})\,; \end{cases}$$

if G is a discrete group

$$\begin{cases} h(x) = 0 & \text{for } x \geqslant 0 \\ h(x) = f_0(x + y_N)\,, & \text{for } x \in [-y_N - x_N,\ -y_N + x_N) \end{cases}$$

The function h above defined is uniformly continuous, bounded on G and also $\|h\|_u = \|f_0\|_u$. We prove that:

$$T_\mu h(x) = 0\,, \quad \text{for all } x \in G\,.$$

Let $\varphi \in \bar{A} \cap G'$; there exists a net $\{x_\alpha\} \subset A$, such that $\delta_{x_\alpha} \xrightarrow[\alpha]{w^*} \varphi$; in particular, for all $x \in G$, we have

$$\langle {}_x h, \delta_{x_\alpha} \rangle = h(x + x_\alpha) \xrightarrow[\alpha]{} \langle {}_x h, \varphi \rangle\,.$$

Since φ does not belongs to G, for each $x \in G$ there exists β_x, such that $x_\alpha > -x$, for all $\alpha \geqslant \beta_x$; therefore $h(x + x_\alpha) = 0$ for each $x \in G$ and for $\alpha > \beta_x$, whence

$$\langle {}_x h, \varphi \rangle = 0\,.$$

Then

$$T_\mu h(x) = \langle {}_x h, \mu \rangle = \int_{\mathcal{M}} \langle {}_x h, \varphi \rangle\, d\mu(\varphi) = 0$$

because μ is concentrated on $\bar{A} \cap G'$.

Since $\mathcal{M}$ is compact in the weak $*$ topology, some subnet of the sequence $\{\delta_{-y_N}\}$ (which we shall still denote by $\{\delta_{-y_N}\}$) converges in the weak $*$ topology to an element $\Phi \in \mathcal{M}$. We prove that

$$T_\Phi h(x) = f_0(x)\,, \quad \text{for all } x \in G.$$

In fact, for all $x \in G$

$$\langle {}_x h, \delta_{-y_N} \rangle \underset{N}{\rightarrow} T_\Phi h(x) ,$$

and also

$$\langle {}_x h, \delta_{-y_N} \rangle = {}_{-y_N}h(x) \underset{N}{\rightarrow} f_0(x) ,$$

because ${}_{-y_N}h(x) = f_0(x)$ for $x \in [-x_N, x_N - x_1]$.

We conclude that

$$\langle h, \Phi * \mu \rangle = \langle T_\mu h, \Phi \rangle = \langle 0, \Phi \rangle = 0 ,$$

$$\langle h, \mu * \Phi \rangle = \langle T_\Phi h, \mu \rangle = \langle f_0, \mu \rangle \neq 0 .$$

CASE 2. Assume μ concentrated on $\bar{B} \cap G'$; the proof is the same as that of part 1, as long as we replace A by B and $\{-y_N\}$ by $\{y_N\}$.

CASE 3. Assume $\mu = \mu_A + \mu_B$, where μ_A, μ_B are measures concentrated on $\bar{A} \cap G'$ and $\bar{B} \cap G'$ respectively, and also $\mu_A \neq 0$, $\mu_B \neq 0$.

Then for $\varepsilon > 0$, there exists $f_0 \in C_u(G)$, $\|f_0\|_u \leqslant 1$, such that

$$\langle f_0, \mu_A \rangle \geqslant \|\mu_A\| - \varepsilon/2 , \qquad \langle f_0, \mu_B \rangle \geqslant \|\mu_B\| - \varepsilon/2 .$$

Let h, Φ be the function of $C_u(G)$ and the functional on $C_u(G)$ respectively, defined in part. 1. Then

$$\langle h, \mu * \Phi \rangle \geqslant \|\mu_A\| + \|\mu_B\| - \varepsilon$$

$$\langle h, \Phi * \mu \rangle = \langle h, \Phi * \mu_B \rangle \leqslant \|\Phi\| \, \|\mu_B\| \, \|h\|_u = \|\mu_B\| \, \|f_0\|_u \leqslant \|\mu_B\| .$$

If we choose $\varepsilon = \|\mu_A\|/2$, it follows that

$$\langle h, \mu * \Phi \rangle > \langle h, \Phi * \mu \rangle .$$

The arguments used for the proof of the Theorem 1, can be extended also to a particular non ordered discrete group.

Let us consider the discrete Abelian group $G = \sum_{i=1} \oplus \boldsymbol{Z}(2)_i$, which is the weak direct product of countably many copies of $\boldsymbol{Z}(2)$, the cyclic group of order two; G is a countable torsion group, therefore not ordered ([4]).

THEOREM 2. *Let* $G = \sum_{i=1}^{\infty} \oplus \boldsymbol{Z}(2)_i$. *The center* Z *of the algebra* $C_u(G)^*$ *coincides with* $M(G)$.

PROOF. Also in this case, it will suffice to prove that any measure concentrated on a set disjoint from G is not in the center.

For all $n \geqslant 1$, we define $H_n = \langle \xi_1, \ldots, \xi_n \rangle =$ group generated by the elements $\xi_1, \ldots, \xi_n$, where

$$\xi_i = \{0, 0, \ldots, \underset{(i)}{1}, 0, \ldots,\}$$

For all $n \geqslant 1$, this is a finite subgroup of G, and also

$$H_1 \subset H_2 \subset \ldots \subset H_{n+1} \subset \ldots \subset G$$

Furthermore $G = \bigcup_{n=1}^{\infty} H_n$, because the elements ξ_i are a system of generators for the group G. Since G is a countable product of cyclic groups of order two, we have also

$$H_n = H_{n-1} \cup (\xi_n + H_{n-1}), \quad \text{for all } n \geqslant 2 .$$

Let us define

$$H_p = (H_2 \setminus H_1) \cup (H_4 \setminus H_3) \cup \ldots \cup (H_{2s} \setminus H_{2s-1}) \cup \ldots$$

$$H_d = H_1 \cup (H_3 \setminus H_2) \cup \ldots \cup (H_{2s+1} \setminus H_{2s}) \cup \ldots ;$$

then

$$H_d \cap H_p = \square , \quad H_p \cup H_d = G .$$

Since

$$\bar{H}_d \cap \bar{H}_p = \square , \quad \bar{H} \cup \bar{H}_p = \mathcal{M} ,$$

where $\bar{H}_d$, $\bar{H}_p$ are the weak* closures of H_d, H_p respectively, a measure on $\mathcal{M}$ concentrated on G' can be written as the sum of a measure μ_d concentrated on $\bar{H}_d \cap G'$ and of a measure μ_p concentrated on $\bar{H}_d \cap G'$.

We prove the theorem for a measure μ concentrated on $\bar{H}_p \cap G'$; the other cases follow from this one, as in Theorem 1.

Let $f_0 \in C_u(G)$ a function such that $\langle f_0, \mu \rangle \neq 0$.

Let us consider the function $h \in C_u(G)$, defined as follows:

$$\begin{cases} h(x) = 0 & \text{for all } x \in H_p \cup H_1 \\ h(x) = f_0(x + \xi_{2n+1}), & \text{for all } x \in H_{2n+1} \diagdown H_{2n}. \end{cases}$$

We prove that

$$T_\mu h(x) = 0\,, \qquad \text{for all } x \in G.$$

Let $\varphi \in \bar{H}_p \cap G'$; there exists a net $\{x_\alpha\} \subset H_p$, such that $\delta_{x_\alpha} \xrightarrow[\alpha]{w^*} \varphi$; in particular for all $x \in G$, we have

$$\langle {}_x h, \delta_{x_\alpha} \rangle = h(x + x_\alpha) \xrightarrow[\alpha]{} \langle {}_x h, \varphi \rangle\,.$$

Since φ does not belongs to G, for all s, there exists β_s such that

$$x_\alpha \in H_{2s} \diagdown H_{2s-1}\,, \qquad \text{for } \alpha \geqslant \beta_s:$$

Furthermore, if $x \in H_k$, for all $s \geqslant 1 + k/2$, we have

$$x + (H_{2s} \diagdown H_{2s-1}) \subset H_{2s} \diagdown H_{2s-1} \subset H_p\,.$$

It follows that, for all $x \in G$, there exists β_x such that

$$x + x_\alpha \in H_p \qquad \text{for } \alpha \geqslant \beta_x.$$

Therefore,

$$h(x + x_\alpha) = 0 \qquad \text{for } x \in G,\ \alpha \geqslant \beta_x.$$

We conclude that $\langle {}_x h, \varphi \rangle = 0$. Then

$$T_\mu h(x) = \langle {}_x h, \mu \rangle = \int_{\mathcal{M}} \langle {}_x h, \varphi \rangle \, d\mu(\varphi) = 0\,,$$

because μ is concentrated on $\bar{H}_p \cap G'$.

Since $\mathcal{M}$ is compact in the weak $*$ topology, some subnet of the sequence $\{\xi_{2n+1}\}$ (which we shall still denote by $\{\xi_{2n+1}\}$) converge in the weak $*$ topology to an element $\Phi \in \mathcal{M}$. We prove that

$$T_\Phi h(x) = f_0(x)\,, \qquad \text{for all } x \in G.$$

In fact, for all $x \in G$,

$$\langle {}_x h, \xi_{2n+1} \rangle \underset{n}{\to} T_\Phi h(x) ,$$

and also

$$\langle {}_x h, \xi_{2n+1} \rangle = {}_{\xi_{2n+1}} h(x) \underset{n}{\to} f_0(x) ,$$

because

$${}_{\xi_{2n+1}} h(x) = f_0(x) \qquad \text{for } x \in H_{2n} .$$

We conclude that

$$\langle h, \Phi * \mu \rangle = \langle T_\mu h, \Phi \rangle = \langle 0, \Phi \rangle = 0 ,$$
$$\langle h, \mu * \Phi \rangle = \langle T_\Phi h, \mu \rangle = \langle f_0, \mu \rangle \neq 0 .$$

4. Further remarks and open problems.

In view of Lemma 1 and the Remark 1 of section 2, it is natural to conjecture that $M(G)$ is always the center of the convolution algebra $C_u(G)^*$. In fact a more general result might be true for groups which are not necessarily commutative. In the case of discrete (not necessarily commutative) groups, we conjecture that the left convolution operators by elements of $l^1(G)$ are exactly the operators commuting with all the bounded operators on $l^\infty(G)$ which commute with left translation.

In support of the above conjecture, at least for the case of an Abelian group, we have a partial result for countable Abelian torsion groups: — *a positive element of $M(\mathcal{M})$, not concentrated on G, does not belongs to the center of $M(\mathcal{M})$.*

Before proving this fact, let us mention some preliminary concepts which will be useful for this purpose.

Let G a countable Abelian torsion group ([4]). There exists a strictly increasing sequence $\{H_n\}$ of finite subgroups of G such that $G = \bigcup_{n=1}^{\infty} H_n$. We can rearrange the elements of G in the following way

$$H_1 = \{a_1 = e, \ldots, a_{k_1}\}$$
$$H_2 = \{a_1 = e, \ldots, a_{k_1}, a_{k_1+1}, \ldots, a_{k_2}\}$$
$$\ldots$$
$$H_n = \{a_1 = e, \ldots, a_{k_1}, a_{k_1+1}, \ldots, a_{k_{n-1}+1}, \ldots, a_{k_n}\} ,$$

where k_n is the order of H_n; we can suppose, possibly considering a subsequence, that $k_n \geqslant 3k_{n-1}$.

Let us define

$$H_p = (H_2 \setminus H_1) \cup (H_4 \setminus H_3) \cup \ldots \cup (H_{2s} \setminus H_{2s-1}) \cup \ldots$$

$$H_d = H_1 \cup (H_3 \setminus H_2) \cup \ldots \cup (H_{2s+1} \setminus H_{2s}) \cup \ldots$$

Let $B^r(G) = \{f\colon G \to R,\ f \text{ bounded}\}$. It is known ([3], section 17) that $B^r(G)$ admits more than an invariant mean. Namely for each $\alpha\colon 1/3 \leqslant \alpha \leqslant 2/3$ there exists an invariant mean $M_\alpha^{(d)}$ for $B^r(G)$, such that

$$\begin{cases} M_\alpha^{(d)}(f) = 0, & \text{for all } f \in \mathcal{U} \\ M_\alpha^{(d)}(\chi_d) = \alpha & \\ M_\alpha^{(d)}(f) \leqslant p(f), & \text{for all } f \in B^r(G), \end{cases}$$

and also an invariant mean $M^{(p)}$ for ${}^r(G)$, such that

$$\begin{cases} M_\alpha^{(p)}(f) = 0, & \text{for all } f \in \mathcal{U} \\ M_\alpha^{(p)}(\chi_p) = \alpha & \\ M_\alpha^{(p)}(f) \leqslant p(f), & \text{for all } f \in B^r(G), \end{cases}$$

where χ_d, χ_p are the characteristic functions of H_d and H_p respectively, $p\colon B^r(G) \to \boldsymbol{R}$ is the real functional defined as:

$$p(f) = \overline{\lim_{n\to\infty}} \left[\frac{1}{k_n} \sum_{x \in H_n} f(x)\right],$$

for all $f \in B^r(G)$ and $\mathcal{U}$ is the closed linear subspace of $B^r(G)$:

$$\mathcal{U} = \left\{ f \in B^r(C)\colon \lim_{n\to\infty} \left[\frac{1}{k_n} \sum_{x \in H_n} f(x)\right] = 0 \right\}.$$

Let us consider the following linear functionals on $C_u(G)$

$$L_\alpha^{(d)}, \quad L_\alpha^{(p)} \quad (\tfrac{1}{3} \leqslant \alpha \leqslant \tfrac{2}{3}),$$

where

$$\left.\begin{aligned}\langle f, L_\alpha^{(d)}\rangle &= M_\alpha^{(d)}(\operatorname{Re} f) + iM_\alpha^{(d)}(\operatorname{Im} f)\\ \langle f, L_\alpha^{(p)}\rangle &= M_\alpha^{(p)}(\operatorname{Re} f) + iM_\alpha^{(p)}(\operatorname{Im} f)\end{aligned}\right\}\text{ for all } f \in C_u(G).$$

Let us prove the above proposition. Let μ be a measure on $\mathcal{M}$ *positive* concentrated on a set disjoint from G (it will suffice to consider this case, because measures concentrated on G are in fact elements of the center); if $\bar{H}_p$, $\bar{H}_d$ are the weak $*$ closure of H_p, H_d respectively we have $\mu = \mu_p + \mu_d$, where μ_p and μ_d are positive measures concentrated on $\bar{H}_p \cap G'$ and $\bar{H}_d \cap G'$ respectively.

CASE 1. Suppose μ concentrated on $\bar{H}_p \cap G'(\mu_d = 0)$.

Since $\chi_d(x) = 0$ for all $x \in H_p$, by the same argument used in theorem 2, we prove that

$$T_\mu \chi_d(x) = 0 \qquad \text{for all } x \in G.$$

Moreover, by definition of $L_\alpha^{(d)}$, $\frac{1}{3} \leqslant \alpha \leqslant \frac{2}{3}$, it follows that

$$\Lambda_\alpha^{(d)} \chi_d(x) = \langle {}_x\chi_d, L_\alpha^{(d)}\rangle = \alpha\,, \qquad \text{for all } x \in G.$$

We conclude that, for each α, $\frac{1}{3} \leqslant \alpha \leqslant \frac{2}{3}$,

$$\langle \chi_d, L_\alpha^{(d)} * \mu\rangle = \langle 0, L_\alpha^{(d)}\rangle = 0\,,$$

$$\langle \chi_d, \mu * L^{(d)}\rangle = \langle \alpha, \mu\rangle = \alpha\mu(\mathcal{M}) \neq 0\,.$$

CASE 2. If μ is concentrated on $\bar{H}_\alpha \cap G'$, similarly we prove that

$$\langle \chi_p, L_\alpha^{(p)} * \mu\rangle = \langle T_\mu \chi_p, L_\alpha^{(p)}\rangle = 0$$

$$\langle \chi_p, \mu * L_\alpha^{(p)}\rangle = \langle \alpha, \mu\rangle = \alpha\mu(\mathcal{M}) \neq 0\,.$$

CASE 3. Suppose $\mu = \mu_p + \mu_d$, where $\mu_p \neq 0$, $\mu_d \neq 0$.

Since $\chi_d(x) = 1$ for all $x \in H_d$, by the same argument used in case 1, we prove that, for all $\varphi \in \bar{H}_d \cap G'$,

$$\langle {}_x(\chi_d), \varphi\rangle = 1 \qquad \text{for all } x \in G,$$

whence

$$T_{\mu_d}\chi_d(x) = \mu_d(\mathcal{M}) \qquad \text{for all } x \in G.$$

Therefore

$$\langle \chi_d, L_\alpha^{(d)} * \mu \rangle = \langle \chi_d, L_\alpha^{(d)} * \mu_d \rangle = \mu_d(\mathcal{M}) \langle 1, L_\alpha^{(d)} \rangle = \mu_d(\mathcal{M}) ,$$

because $\langle 1, L_\alpha^{(d)} \rangle = \langle 1, M_\alpha^{(d)} \rangle = 1$.

$$\langle \chi_d, \mu * L_\alpha^{(d)} \rangle = \alpha\big(\mu_p(\mathcal{M}) + \mu_d(\mathcal{M})\big) .$$

Then it is possible to choose α in such a way that

$$\langle \chi_d, \mu * L_\alpha^{(d)} \rangle \neq \langle \chi_d, L^{(d)} * \mu \rangle .$$

REMARK 3. It is obvious that, if $|\mu| \in Z$, then $\mu \in Z$, but the inverse is not known. If the conjecture $\mu \in Z \Rightarrow |\mu| \in Z$ is true, from the above proposition it follows that also for a countable Abelian torsion group, the center Z of the algebra $C_u(G)^*$ coincides with $M(G)$.

BIBLIOGRAPHY

[1] R. B. BURCKEL, *Weakly almost periodic functions on semigroups*, Gordon and Breach Publishers.

[1] P. C. CURTIS jr. - A. FIGÀ-TALAMANCA, *Factorization theorems for Banach algebras*, in *Function Algebras*, Scott, Foresman & Co., 1966.

[3] E. HEWITT - K. A. ROSS, *Abstract Harmonic Analysis, I*, Springer-Verlag, 963.

[4] I. KAPLANSKI, *Infinite Abelian groups*, University of Michigan Press, Ann Arbor.

[5] L. H. LOOMIS, *An introduction to abstract harmonic analysis*, Van Nostrand Co., 1953.

[6] W. RUDIN, *Fourier analysis on groups*, Interscience, 1962.

Manoscritto pervenuto in redazione il 12 dicembre 1973.