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Saalschützian Transforms.

L. CARLITZ (*)

1. Put

$$(a)_n = a(a+1), \ldots, (a+n-1)\ , \qquad (a)_0 = 1\ .$$

Saalschütz's theorem [1, p. 9], [3, p, 48] reads

$$\sum_{k=0}^{n} \frac{(-n)_k(a)_k(b)_k}{k!(c)_k(d)_k} = \frac{(c-a)_n(c-b)}{(c)_n(c-a-b)_n}\,, \tag{1.1}$$

where

$$c+d = a+b-n+1\ . \tag{1.2}$$

If we replace b by $c-b$, (1.1) becomes

$$\sum_{k=0}^{n} \frac{(-n)_k(a)_k(c-b)_k}{k!(c)_k(a-b-n+1)_k} = \frac{(c-a)_n(b)_n}{(c)_n(b-a)_n}\,.$$

Since

$$(a-b-n+1)_k = (-1)^k(b-a+n-k)_k\ ,$$

$$\frac{(b-a)_n}{(a-b+n-1)_k} = (-1)^k(b-a)_{n-k}\ ,$$

we get

$$\frac{1}{(b)_n}\sum_{k=0}^{n}\binom{n}{k}(a)_k(b-a)_{n-k}\frac{(c-b)_k}{(c)_k} = \frac{(c-a)_n}{(c)_n}\,. \tag{1.3}$$

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We may think of (1.3) as a linear transformation from the set of rational functions

$$\frac{(c-b)_n}{(c)_n} \qquad (n=0,1,2,\ldots)$$

to the set

$$\frac{(c-a)_n}{(c)_n} \qquad (n=0,1,2,\ldots).$$

We accordingly define the linear transformation

$$(1.4) \qquad T_{a,b}\colon x_n = \frac{1}{(b)_n}\sum_{k=0}^{n}\binom{n}{k}(a)_k(b-a)_{n-k}y_k \qquad (n=0,1,2,\ldots).$$

We shall assume that neither a nor b is equal to zero or a negative integer.

It follows at once from (1.4) that

$$(1.5) \qquad T_{a,a} = I,$$

the identity transformation. We shall show that

$$(1.6) \qquad T_{a,b}T_{b,a} = I$$

and that

$$(1.7) \qquad T_{a,b}T_{c,d} = T_{c,d}T_{a,b}.$$

Moreover a relation of the form

$$(1.8) \qquad T_{a_1,b_1}T_{a_2,b_2}\ldots T_{a_n,b_n} = I$$

holds only in a « trivial » way.

In the second part of the paper we show that analogous results are implied by the q-analog of Saalschütz's theorem:

$$(1.9) \qquad \sum_{k=0}^{n}\frac{(q^{-n})_k(a)_k(b)_k}{(q)_k(c)_k(d)_k}q^k = \frac{(c/a)_n(c/b)_n}{(c)_n(c/ab)_n},$$

where now

$$(a)_n = (1-a)(1-qa)\ldots(1-q^{n-1}a), \qquad (a)_0 = 1$$

and

(1.10) $$cd = q^{1-n}ab\,.$$

2. Returning to the definition (1.4), we ask when is

(2.1) $$T_{a,b} = T_{c,d}?$$

Clearly this requires

$$\frac{1}{(b)_n}(a)_k(b-a)_{n-k} = \frac{1}{(d)_n}(c)_k(d-c)_{n-k} \qquad (n = 0, 1, 2, \ldots;\ k = 0, 1, \ldots, n)\,.$$

For $k = n$ this becomes

$$\frac{(a)_n}{(b)_n} = \frac{(c)_n}{(d)_n} \qquad (n = 0, 1, 2, \ldots)\,.$$

It follows that

$$\frac{a}{b} = \frac{c}{d}, \qquad \frac{a+1}{b+1} = \frac{c+1}{d+1}, \qquad \frac{a+2}{b+2} = \frac{c+2}{d+2}, \ldots$$

and therefore

$$(a+x)(d+x) \equiv (b+x)(x+x)\,.$$

For $x = -a$, this implies $b = a$ or $c = a$. If $b = a$ we get $T_{c,d} = I$ so that $c = d$. If $c = a$ it follows that $b = d$. Hence (2.1) holds if and only if

(i) $$a = b\,, \quad c = d$$

or

(ii) $$a = c, \quad b = d\,.$$

We show now that

(2.2) $$T_{a,b}T_{b,c} = T_{a,c}\,.$$

PROOF. Put

(2.3) $$y_k = \frac{1}{(c)_k}\sum_{j=0}^{k}\binom{k}{j}(b)_j(c-b)_{k-j}z_j \qquad (k = 0, 1, 2, \ldots)\,.$$

Then by (1.4) and (2.3)

$$x_n = \frac{1}{(b)_n}\sum_{j=0}^{n}\binom{n}{k}(a)_k(b-a)_{n-k}\cdot\frac{1}{(c)_k}\sum_{j=0}^{k}\binom{k}{j}(b)_j(c-b)_{k-j}z_j = \tag{2.4}$$

$$= \frac{1}{(b)_n}\sum_{j=0}^{n}\binom{n}{j}(b)_j z_j \sum_{k=j}^{n}\binom{n-j}{k-j}(a)_k(b-a)_{n-k}\frac{(c-b)_{k-j}}{(c)_k}.$$

The inner sum is equal to

$$\sum_{k=0}^{n-j}\binom{n-j}{k} a_{j+k}(b-a)_{n-j-k}\frac{(c-b)_k}{(c)_{j+k}} =$$

$$= \frac{(a)_j}{(c)_j}\sum_{k=0}^{n-j}\binom{n-j}{k}\frac{(a+j)_k(c-b)_k}{(c+j)_k}(b-a)_{n-j-k} =$$

$$= \frac{(a)_j(b-a)_{n-j}}{(c)_j}\sum_{k=0}^{n}\frac{(-n+j)_k(a+j)_k(c-b)_k}{k!(c+j)_k(a-b-n+j+1)_k} =$$

$$= \frac{(a)_j(b-a)_{n-j}}{(c)_j}\frac{(c-a)_{n-j}(b+j)_{n-j}}{(c+j)_{n-j}(b-a)_{n-j}} = \frac{(a)j(c-a)_{n-j}(b+j)_{n-j}}{(c)_n}.$$

Hence (2.4) becomes

$$x_n = \frac{1}{b_n}\sum_{j=0}^{n}\binom{n}{j}\frac{(a)_j(b)_j(c-a)_{n-j}(b+j)_{n-j}}{(c)_n}z_j = \frac{1}{(c)_n}\sum_{j=0}^{n}\binom{n}{j}(a)_j(c-a)_{n-j}z_j .$$

This evidently proves (2.2).

As an immediate corollary of (2.2) we have

$$T_{a,b}T_{b,a} = I . \tag{2.5}$$

We show next that

$$T_{b,c}T_{a,b} = T_{a,c} . \tag{2.6}$$

Proof. Put

$$x_n = \frac{1}{(c)_n}\sum_{k=0}^{n}\binom{n}{k}(b)_k(c-b)_{n-k}y_k ,$$

$$y_k = \frac{1}{(b)_k}\sum_{j=0}^{k}\binom{k}{j}(a)_j(b-a)_{k-j}z_j .$$

Then

$$x_n = \frac{1}{(c)_n} \sum_{k=0}^{n} \binom{n}{k} (b)_k (c-b)_{n-k} \cdot \frac{1}{(b)_k} \sum_{j=0}^{k} \binom{k}{j} (a)_j (b-a)_{k-j} z_j =$$

$$= \frac{1}{(c)_n} \sum_{j=0}^{n} \binom{n}{j} (a)_j z_j \sum_{k=j}^{n} \binom{n-j}{k-j} (c-b)_{n-k} (b-a)_{k-j} .$$

The inner sum is equal to

$$\sum_{k=0}^{n-j} \binom{n-j}{k} (c-b)_{n-j-k} (b-a)_k = (c-b)_{n-j} \sum_{k=0}^{n-j} \frac{(-n+j)_k (b-a)_k}{k!(b-c-n+j+1)_k} =$$

$$= (c-b)_{n-j} \frac{(a-c-n+j+1)_{n-j}}{(b-c-n+j+1)_{n-j}} = (c-b)_{n-j} \frac{(c-a)_{n-j}}{(c-b)_{n-j}} = (c-a)_{n-j}$$

by Vandermonde's theorem. Thus

$$x_n = \frac{1}{(c)_n} \sum_{j=0}^{n} \binom{n}{j} (a)_j (c-a)_{n-j} z_j ,$$

which proves (2.6).

It is now easy to prove that

$$T_{a,b} T_{c,d} = T_{c,d} T_{a,b} . \tag{2.7}$$

Indeed, by (2.2) and (2.7),

$$\begin{aligned} T_{a,b} T_{c,d} &= T_{c,b} T_{a,c} T_{c,d} \\ &= T_{c,b} T_{a,d} \\ &= T_{c,b} T_{b,d} T_{a,b} \\ &= T_{c,d} T_{a,b} . \end{aligned}$$

We shall however give a second proof of (2.7) that makes use of an explicit formula for the transformation $T_{a,b} T_{c,d}$. Put

$$x_n = \frac{1}{(b)_n} \sum_{k=0}^{n} \binom{n}{k} (a)_k (b-a)_{n-k} y_k ,$$

$$y_k = \frac{1}{(d)_k} \sum_{j=0}^{k} \binom{k}{j} (c)_j (d-c)_{k-j} z_j .$$

Then

$$(2.8)\qquad x_n = \frac{1}{(b)_n}\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_k(b-a)_{n-k}}{(d)_k}\sum_{j=0}^{k}\binom{k}{j}(c)_j(d-c)_{k-j}z_j =$$

$$= \frac{1}{(b)_n}\sum_{j=0}^{n}\binom{n}{j}(c)_j z_j \sum_{k=j}^{n}\binom{n-j}{k-j}(a)_k(b-a)_{n-k}\frac{(d-c)_{k-j}}{(d)_k} =$$

$$= \frac{1}{(b)_n}\sum_{j=0}^{n}\binom{n}{j}(c)_j z_j \sum_{k=0}^{n-j}\binom{n-j}{k}(a)_{j+k}(b-a)_{n-j-k}\frac{(d-c)_k}{(d)_{j+k}} =$$

$$= \frac{1}{(b)_n}\sum_{j=0}^{n}\binom{n}{j}\frac{(a)_j(c)_j}{(d)_j}z_j \sum_{k=0}^{n-j}\binom{n-j}{k}(a+j)_k(b-a)_{n-j-k}\frac{(d-c)_k}{(d+j)_k}\,.$$

By Vandermode's theorem

$$\frac{(d-c)_k}{(d+j)_k} = \sum_{s=0}^{k}\frac{(-k)_s(c+j)_s}{s!(d+j)_s} = \sum_{s=0}^{k}(-1)^s\binom{k}{c}\frac{(c+j)_s}{(d+j)_s}\,.$$

Thus

$$\sum_{k=0}^{n-j}\binom{n-j}{k}(a+j)_k(b-a)_{n-j-k}\frac{(d-c)_k}{(d+j)_k} =$$

$$= \sum_{k=0}^{n-j}\binom{n-j}{k}(a+j)_k(b-a)_{n-j-k}\sum_{s=0}^{k}(-1)^s\binom{k}{c}\frac{(c+j)_s}{(d+j)_s} =$$

$$= \sum_{s=0}^{n-j}(-1)^s\binom{n-j}{s}\frac{(c+j)_s}{(d+j)_s}\sum_{k=s}^{n-j}\binom{n-j-s}{k-s}(a+j)_k(b-a)_{n-j-k} =$$

$$= \sum_{s=0}^{n-j}(-1)^s\binom{n-j}{s}\frac{(c+j)_s}{(d+j)_s}\sum_{k=0}^{n-j-s}\binom{n-j-s}{k}(a+j)_{s+k}(b-a)_{n-j-s-k} =$$

$$= \sum_{s=0}^{n-j}(-1)^s\binom{n-j}{s}\frac{(a+j)_s(c+j)_s}{(d+j)_s}(b-a)_{n-j-s}\cdot$$

$$\cdot\sum_{k=0}^{n-j-s}\frac{(-n+j+s)_k(a+j+s)_k}{k!(a-b-n+j+s+1)_k} = \sum_{s=0}^{n-j}(-1)^s\binom{n-j}{s}\cdot$$

$$\cdot\frac{(a-j)_s(c+j)_s}{(d+j)_n}(b-a)_{n-j-s}\frac{(-b-n+1)_{n-j-s}}{(a-b-n+j+s+1)_{n-j-s}} =$$

$$= \sum_{s=0}^{n-j}(-1)^s\binom{n-j}{s}\frac{(a+j)_s(c+j)_s}{(d+j)_s}(b-a)_{n-j-s}\frac{(b+j+s)_n}{(b-a)_{n-s}} =$$

$$= (b)_n\sum_{s=0}^{n-j}(-1)^s\binom{n-j}{s}\frac{(a+j)_s(c+j)_s}{(b)_{j+s}}\cdot$$

Therefore (2.8) becomes

$$(2.9)\qquad x_n=\sum_{j=0}^{n}\binom{n}{j}\frac{(a)_j(c)_j}{(b)_j(d)_j}z_j\sum_{s=0}^{n-j}(-1)^s\binom{n-j}{s}\frac{(a+j)_s(c+j)_s}{(b+j)_s(d+j)_s}$$

or, if we prefer,

$$(2.10)\qquad x_n=\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_k(c)_k}{(b)_k(d)_k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}z_j\,.$$

Clearly (2.9) represents both $T_{a,b}T_{c,d}$ and $T_{c,d}T_{a,b}$. For that matter it also represents both $T_{a,d}T_{b,c}$ and $T_{b,c}T_{a,d}$. In particular, (2.10) implies (2.7).

3. It is natural to ask whether

$$(3.1)\qquad T_{a,b}T_{c,d}=T_{e,f}\,,$$

for properly chosen e, f. By (2.8) this is equivalent to

$$(3.2)\qquad \frac{(a)_j(c)_j}{(b)_n(d)_j}\sum_{k=0}^{n-j}\binom{n-j}{k}(a+j)_k(b-a)_{n-j-k}\frac{(d-c)_k}{(d+j)_k}=\frac{1}{(f)_n}(e)_j(f)_{n-j}\,.$$

For $j=n$, (3.2) reduces to

$$(3.3)\qquad \frac{(a)_n(c)_n}{(b)_n(d)_j}=\frac{(e)_n}{(f)_n}\,.$$

This implies

$$\frac{ac}{bd}=\frac{e}{f},\qquad \frac{(a+1)(c+1)}{(b+1)(d+1)}=\frac{e+1}{f+1},\qquad \frac{(a+2)(c+2)}{(b+2)(d+2)}=\frac{e+2}{f+2},\ \cdots$$

and therefore

$$(3.4)\qquad (a+x)(c+x)(f+x)\equiv(b+x)(d+x)(e+x)\,.$$

For $x=-0$, (3.4) becomes

$$(b-a)(d-a)(e-a)=0\,.$$

(i) If $b=a$, (3.1) reduces to $T_{c,d}=T_{e,f}$, so that

$$c=d\,,\quad e=f\qquad\text{or}\qquad c=e\,,\quad d=f\,;$$

(ii) if $d = a$, (3.1) becomes $T_{c,b} = T_{e,f}$, so that

$$c = b\,,\quad e = f \quad \text{or} \quad c = e\,,\quad b = f\,;$$

(iii) if $e = a$, (3.1) reduces to $T_{c,d} = T_{b,f}$, so that

$$c = d\,,\quad b = f \quad \text{or} \quad c = b\,,\quad d = f\,.$$

Thus in every case, (3.1) holds only in a «trivial» way, that is, as a consequence of (2.2) or (2.6).

Consider next the equation

$$T_{a,b}T_{c,d} = T_{e,f}T_{g,h}\,. \tag{3.5}$$

By (2.10), (3.5) is equivalent to

$$\frac{(a)_k(c)_k}{(b)_k(d)_k} = \frac{(e)_k(g)_k}{(f)_k(h)_k}\,. \tag{3.6}$$

Exactly as above, (3.6) implies

$$(a + x)(c + x)(f + x)(h + x) \qquad (b + x)(d + x)(c + x)(g + x)\,. \tag{3.7}$$

Taking $x = -a$, it follows that $a = b, d, e$ or g. If $a = b$, (3.5) reduces to

$$T_{c,d} = T_{e,f}T_{g,h}\,;$$

if $a = d$, it reduces to

$$T_{d,b} = T_{e,f}T_{g,h}\,;$$

if $a = e$, we get

$$T_{c,d} = T_{b,f}T_{g,h}\,;$$

if $a = g$, we get

$$T_{c,d} = T_{e,f}T_{b,h}\,.$$

Thus in every case, (3.5) reduces to (3.1).

The most general relation can be reduced to the form

$$T_{a_1,b_1}T_{a_2,b_2} \ldots T_{a_n,b_n} = I\,. \tag{3.8}$$

As above, (3.8) implies

$$(a_1 + x) \ldots (a_n + x) \equiv (b_1 + x) \ldots (b_n + x)\,. \tag{3.9}$$

It follows from (3.10) that $a_n = b_i$ for some i. If $i = n$, (3.8) reduces to

$$T_{a_1,b_1} \cdots T_{a_{n-1},b_{n-1}} = I .$$

If $i \neq n$, we may assume $i = n-1$, in which case (3.8) becomes

$$T_{a_1,b_1} \cdots T_{a_{n-2},b_{n-2}} T_{a_{n-1}b_n} = I .$$

Again the number of factors on the left has been reduced.

We conclude that (3.8) holds only in a « trivial » way.

4. We may write (1.3) in the following form.

$$(4.1) \qquad (x-a)_n = \frac{1}{(b)_n} \sum_{k=0}^{n} \binom{n}{k} (a)_k (b-a)_{n-k} (x-b)_k (x+k)_{n-k} .$$

This suggests the following interpolation problem. Let $f(x)$ be a polynomial of degree $\leqslant n$. We consider the representation of $f(x)$ in the form

$$(4.2) \qquad f(x) = \sum_{k=0}^{n} A_{n,k} (x-a)_k (x+k)_{n-k} ,$$

where a is an arbitrary constant and $A_{n,k} = A_{n,k}(a)$ is independent of x.

To determine the coefficients $A_{n,k}$ we first take $x = a$. This gives

$$(4.3) \qquad (a)_n A_{n,0} = f(a) .$$

Next, for $x = a-1$, (4.2) reduces to

$$f(a-1) = (a-1)_n A_{n,0} - (a)_{n-1} A_{n,1} ,$$

so that

$$(4.4) \qquad (a)_n A_{n,1} = (a-1) f(a) - (a+n-1) f(a-1) .$$

For $x = a-2$, (4.2) becomes

$$f(a-2) = (a-2)_n A_{n,0} - 2(a-1)_n A_{n,1} + 2(a)_{n-2} A_{n,2} .$$

This yields

$$(4.5) \qquad 2(a)_{n,2} = a(a-1) f(a) - 2(a-1)(a+n-1) f(a-1) + \\ + (a+n-1)(a+n-2) f(a-2) .$$

At the next step we get

$$(4.6)\quad 6(a)_n A_{n,3} = a(a-1)(a+1)f(a) - 3a(a-1)a(a+n-1)f(a-1) + \\ + 3(a-1)(a+n-1)(a+n-2)f(a-2) - \\ - (a+n-1)(a+n-2)(a+n-3)f(a-3)\,.$$

This suggests the general result

$$(4.7)\qquad k!(a)_n A_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (a-1)_{k-j}(a+n-j)_j f(a-j)\,.$$

To prove (4.7) we note first that

$$(4.8)\qquad 1 = \frac{1}{(a)_k} \sum_{j=0}^{k} (-1)^k \binom{k}{j} (x-a)_j (x+j)_{k-j}\,.$$

Indeed

$$\frac{1}{(x)_k} \sum_{j=0}^{k} (-1) \binom{k}{j} (x-a)_j (x+j)_{k-j} = \\ = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(x-a)_j}{(x)_j} = \sum_{j=0}^{k} \frac{(-k)_j (x-a)_j}{j!(a)_j} = \frac{(a)_k}{(x)_k}\,,$$

by Vandermonde's theorem. This evidently proves (4.8).

Now replace x by $x+m$ in (4.8) and then multiply both sides by $(x-a)_m$. We get

$$(x-a)_m = \frac{1}{(a)_k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x-a)_{m+j}(x+m+j)_{k-j} = \\ = \frac{1}{(a)_k} \sum_{j=m}^{k+m} (-1)^{j-m} \binom{k}{j-m} (x-a)_j (x+j)_{k+m-j}\,.$$

Replacing k by $n-m$, this becomes

$$(4.9)\qquad (x-a)_m = \frac{1}{(a)_{n-m}} \sum_{j=m}^{m} (-1)^{j-m} \binom{n-m}{j-m} (x-a)_j (x+j)_{n-j}\,.$$

In the next place if $f(x)$ is a polynomial of degree $\leqslant n$, we may put

$$(4.10)\qquad f(x) = \sum_{m=0}^{n} C_m (x-a)_m$$

where

$$C_m = \frac{1}{m!}\,\delta^m f(a) \tag{4.11}$$

and

$$\delta f(x) \;= f(x) - f(x-1)\,,$$
$$\delta^k f(x) = \delta\cdot\delta^{k-1} f(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x-j)\,.$$

By (4.9) and (4.10)

$$f(x) = \sum_{m=0}^{n} \frac{1}{(a)_{n-m}}\, C_m \sum_{j=m}^{n} (-1)^{j-m} \binom{n-m}{j-n} (x-a)_j (x+j)_{n-j} = \tag{4.12}$$
$$= \sum_{j=0}^{n} (x-a)_j (x+j)_{n-j} \sum_{m=0}^{j} (-1)^{j-m} \binom{n-m}{j-m} \frac{C_m}{(a)_{n-m}}\,.$$

By (4.11),

$$\sum_{m=0}^{j} (-1)^{j-m} \binom{n-m}{j-m} \frac{C_m}{(a)_{n-m}} =$$
$$= \sum_{m=0}^{j} (-1)^{j-m} \binom{n-m}{j-m} \frac{1}{m!(a)_{n-m}} \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(a-k) =$$
$$= (-1)^j \sum_{k=0}^{j} f(a-k) \sum_{m=k}^{j} (-1)^{m-k} \binom{n-m}{j-m}\binom{m}{k} \frac{1}{m!(a)_{n-m}}\,.$$

Since

$$\sum_{m=k}^{j} (-1)^{m-k} \binom{n-m}{j-m}\binom{m}{k} \frac{1}{m!(a)_{n-m}} =$$
$$= \sum_{m=0}^{j-k} (-1)^m \binom{n-k-m}{j-k-m}\binom{k+m}{k} \frac{1}{(k+m)!(a)_{n-k-m}} =$$
$$= \sum_{m=0}^{j-k} (-1)^m \frac{(n-k-m)!}{(n-j)!(j-k-m)k!m!(a)_{n-k-m}} =$$
$$= \frac{(n-k)!}{k!(n-j)!(j-k)!(a)_{n-k}} \sum_{m=0}^{j-k} \frac{(-j+k)_m(-a-n+k+1)_m}{m!(-n+k)_m} =$$
$$= \frac{(n-k)!}{k!(n-j)!(j-k)!(a)_{n-k}} \frac{(a-1)_{j-k}}{(-n+k)_{j-k}} = (-1)^{j-k} \frac{(a-1)_{j-k}}{k!(j-k)!(a)_{n-k}}\,,$$

we have

$$\sum_{m=0}^{j} (-1)^{j-m} \binom{n-m}{j-m} \frac{C_m}{(a)_{n-m}} = \sum_{k=0}^{j} (-1)^k \frac{(a-1)_{j-k}}{k!(j-k)!(a)_{n-k}} f(a-k) .$$

Hence (4.12) becomes

$$f(x) = \sum_{j=0}^{n} (x-a)_j (x+j)_{n-j} \cdot$$

$$\cdot \frac{1}{j!(a)_n} \sum_{k=0}^{j} (-1)^k \binom{j}{k} (a-1)_{j-k} (a+n-k)_k f(a-k) ,$$

in agreement with (4.7).

It remains to show that the representation (4.2) is unique. We shall assume that the parameter a is not equal to zero or a negative integer. If the representation (4.2) is not unique, then for some n, there exist numbers

$$B_k = B_k(a) \qquad (0 \leqslant k \leqslant n) ,$$

not all zero, such that

$$\sum_{k=0}^{n} B_k (x-a)_k (x+k)_{n-k} = 0 . \tag{4.13}$$

For $x = a$, (4.13) reduces to

$$(a)_n B_0 = 0 ,$$

so that $B_0 = 0$. Let

$$B_0 = \ldots = B_{m-1} = 0 , \qquad B_m \neq 0$$

and take $x = a - m$. Then (4.13) reduces to

$$\sum_{k=m}^{n} B_k (-m)_k (a-m+k)_{n-k} = 0 ,$$

so that

$$(-1)^m m!(a)_{n-m} B_m = 0 , \qquad B_m = 0 .$$

We have therefore proved the following

THEOREM A. *Let $f(x)$ be a polynomial of degree $\leqslant n$. Assume that a is not equal to zero or a negative integer. Then $f(x)$ is uniquely repre-*

sentable in the form

$$f(x) = \sum_{k=0}^{n} A_{n,k}(x-a)_k(x+k)_{n-k}, \tag{4.14}$$

where the coefficients $A_{n \cdot k}$ are determined by (4.7).

If a is equal to zero or a negative integer, the representation (4.14) is in general not possible. For example, for $a=0$, (4.14) reduces to

$$f(x) = \sum_{k=0}^{n} A_{n,k}(x)_n,$$

so that $f(x)$ is a constant multiple of $(x)_n$. If $x=-m$, $m \geqslant 0$, (4.14) becomes

$$f(x) = \sum_{k=0}^{n} A_{n,k}(x+m)_k(x+k)_{n-k},$$

so that

$$(x)_m f(x) = \sum_{k=0}^{n} A_{n,k}(x)_{m+k}(x+k)_{n-k} = (x)_n \sum_{k=0}^{n} A_{n,k}(x+k)_m .$$

Hence, for $m \geqslant 0$, $f(x)$ is divisible by $(x+m)_{n-m}$.

5. We turn now to the q-analog of Saalschütz's theorem [1, p. 68], [3, p. 97]

$$\sum_{k=0}^{n} \frac{(q^{-n})_k(a)_k(b)_k}{(q)_k(c)_k(d)_k} q^k = \frac{(c/a)_n(c/b)_n}{(c)(c/ab)_n} \tag{5.1}$$

where now

$$(a)_n = (1-a)(1-qa)\ldots(1-q^{n-1}a), \quad (a)_0 = 1 \tag{5.2}$$

and

$$cd = q^{1-n}ab . \tag{5.3}$$

The following corollaries of (5.1) will be used. First, taking $b=d=0$, we get

$$\sum_{k=0}^{n} \frac{(q^{-n})_k(a)_k}{(q)_k(c)_k} q^k = \frac{(c/a)_n}{(c)_n} a^n . \tag{5.4}$$

This may be called the q-analog of Vandermonde's theorem.

Next, if we let $n \to \infty$, (5.1) reduces to

$$\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(q)_k(c)_k}\left(\frac{c}{ab}\right)^k = \prod_{k=0}^{\infty} \frac{(1-q^k c/a)(1-q^k c/b)}{(1-q^k c)(1-q^k c/ab)} . \tag{5.5}$$

This is the q-analog of Gauss's theorem. Note that for $b = q^{-n}$, (5.5) becomes

$$\sum_{k=0}^{n} \frac{(q^{-n})_k(a)_k}{(q)_k(c)_k}\left(\frac{c}{ab}\right)^k = \frac{(c/a)_n}{(c)_n} . \tag{5.6}$$

Replacing k by $n-k$, (5.6) becomes

$$\sum_{k=0}^{n} \frac{(q^{-n})_k(q^{1-n}/c)_k}{(q)_k(q^{1-n}/a)_k} = (-1)^n q^{-\frac{1}{2}n(n-1)} \frac{(c-a)_n}{(c)_n} .$$

If we now replace q^{1-n}/c and q^{1-n}/a by a and c, respectively, we get (5.4).

In (5.1) replace b by c/b. Then (5.1) becomes

$$\sum_{k=0}^{n} \frac{(q^{-n})_k(a)_k(c/b)_k}{(q)_k(c)_k(q^{1-n}a/b)_k} q^k = \frac{(c/a)_n(b)_n}{(c)_n(b/a)_n} .$$

Since

$$(q^{-n})_k = (-1)^k q^{-nk+\frac{1}{2}k(k-1)}(q)_n/(q)_{n-k} ,$$

$$(q^{1-n}a/b)_k = (-1)_k q^{-nk+\frac{1}{2}k(k+1)}(q^{n-k}b/a)_k(a/b)^k ,$$

$$\frac{(b/a)_n}{(q^{1-n}a/b)_k} = (-1)^k q^{nk-\frac{1}{2}k(k+1)}(b/a)_{n-k}(b/a)^k ,$$

we get

$$\frac{1}{(b)_n}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a)_k(b/a)_{n-k}\frac{(c/b)_k}{(c)_k}\left(\frac{b}{a}\right)^k = \frac{(c/a)_n}{(c)_n} , \tag{5.7}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k(q)_{n-k}} . \tag{5.8}$$

We may think of (5.7) as a linear transformation from the set of rational functions

$$\frac{(x/b)_n}{(x)_n} \qquad (n = 0, 1, 2, \ldots)$$

to the set

$$\frac{(x/a)_n}{(x)_n} \qquad (n = 0, 1, 2, \ldots) .$$

We define the linear transformation

$$(5.9) \qquad T_{a,b}\colon x_n = \frac{1}{(b)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a)_k (b/a)_{n-k} (b/a)^k y^k \qquad (n = 0, 1, 2, \ldots) .$$

We assume that the parameters $a, b, c, \ldots$ are not equal to

$$(5.10) \qquad q^{-t} \ (t = 0, 1, 2, \ldots) .$$

It follows from (5.9) that

$$(5.11) \qquad T_{a,a} = I \ (\text{identity}) .$$

Also, as in the ordinary case, it is easy to show that $T_{a,b} = T_{c,d}$ if and only if

(i) $\qquad a = b , \quad c = d$

or

(ii) $\qquad a = c , \quad b = d .$

Indeed, by (5.9), $T_{a,b} = T_{c,d}$ if and only if

$$\frac{1}{(b)_n} (a)_k (b/a)_{n-k} (b/a)^k = \frac{1}{(d)_n} (c)_k (d_k c)_{n-k} (d/c)^k \qquad (0 \leqslant k \leqslant n) .$$

For $k = n$, this reduces to

$$\frac{(a)_n}{(b)_n} \left(\frac{b}{a}\right)^n = \frac{(c)_n}{(d)_n} \left(\frac{c}{d}\right)^n \qquad (n = 1, 2, 3, \ldots) .$$

Hence

$$\frac{1 - q^n a}{1 - q^n b} \frac{a}{b} = \frac{1 - q^n c}{1 - q^n d} \frac{d}{c} \qquad (n = 1, 2, 3, \ldots) ,$$

so that

$$\frac{1 - ax}{1 - bx} \frac{b}{a} \equiv \frac{1 - cx}{1 - dx} \frac{d}{c} .$$

We shall now show that

$$T_{a,b}T_{b,c} = T_{a,c}\,. \tag{5.12}$$

PROOF. Put

$$y_k = \frac{1}{(c)_k}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}(b)_j(c/b)_{k-j}(c/b)^j z_j \qquad (k=0,1,2,\ldots)\,.$$

Then, by (5.9) and (5.12),

$$\begin{aligned}x_n &= \frac{1}{(b)_n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(a)_k(b/a)_{n-k}(b/a)^k\cdot\frac{1}{(c)_k}\sum_{j=0}^{k}(b)_j(c/b)_{k-j}(c/b)^j z_j = \\ &= \frac{1}{(b)_n}\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}(b)_j(c/b)^j z_j\sum_{k=j}^{n}\begin{bmatrix}n-j\\k-j\end{bmatrix}(a)_k(b/a)_{n-k}(b/a)^k\frac{(c/b)_{k-j}}{(c)_k} = \\ &= \frac{1}{(b)_n}\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}(b)_j(c/b)^j z_j\sum_{k=0}^{n-j}\begin{bmatrix}n-j\\k\end{bmatrix}(a)_{j+k}(b/a)_{n-j-k}(b/a)^{j+k}\frac{(c/b)_k}{(c)_{j+k}} = \\ &= \frac{1}{(b)_n}\sum_{j=0}^{n}\frac{(a)_j(b)_j}{(c)_j}(c/a)^j z_j\sum_{k=0}^{n-j}\begin{bmatrix}n-j\\k\end{bmatrix}(q^j a)_k(b/a)_{n-j-k}(b/a)^k\frac{(c/b)_k}{(q^j c)_k}\,.\end{aligned}$$

By (5.7),

$$\sum_{k=0}^{n-j}\begin{bmatrix}n-j\\k\end{bmatrix}(q^j a)_k(b/a)_{n-j-k}(b/a)^k\frac{(c/b)_k}{(q^j c)_k} = \frac{(c/a)_{n-j}(q^a b)_{n-j}}{(q^j c)_{n-j}}$$

so that

$$\begin{aligned}x_n &= \frac{1}{(b)_n}\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}\frac{(a)_j(b)_j}{(c)_j}(c/a)^j\frac{(c/a)_{n-j}(q^j b)_{n-j}}{(q^j c)_{n-j}}z_j = \\ &= \frac{1}{(c)_n}\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}(a)_j(c/a)_{n-j}(c/a)^j z_j\,.\end{aligned}$$

This proves (5.12).

We show next that

$$T_{b,c}T_{a,b} = T_{a,c}\,. \tag{5.13}$$

PROOF. Put

$$x_n = \frac{1}{(c)_n}\sum_{k=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}(b)_k(c/b)_{n-k}(c/b)^k y_k\,,$$

$$y_k = \frac{1}{(b)_k}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}(a)_j(b/a)_{k-j}(b/a)^j z_j\,.$$

Then

$$x_n = \frac{1}{(c)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (c/b)_{n-k}(c/b)^k \cdot \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (a)_j (b/a)_{k-j} (b/a)^j z_j =$$

$$= \frac{1}{(c)_n} \sum_{n-j}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (a)_j (b/a) z_j \sum_{k=0}^{j=0} \begin{bmatrix} n-j \\ k \end{bmatrix} (c/b)_{n-j-k} (b/a)_k (c/b)^k .$$

It will therefore suffice to show that

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (c/b)_{n-k} (b/a)^k (c/b)^k = (c/a)_n . \tag{5.14}$$

To prove (5.14), we make use of the identity

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} x^n = \prod_{n=0}^{\infty} \frac{1-q^n ax}{1-q^n x} . \tag{5.15}$$

Then

$$\sum_{n=0}^{\infty} \frac{x^n}{(q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (c/b)_{n-k} (b/a)_k (c/b)^k = \sum_{k=0}^{\infty} \frac{(b/a)_k}{(q)_k} (c/b)^k x^k \sum_{n=0}^{\infty} \frac{(c_k b)_n}{(q)_n} x^n =$$

$$= \prod_{k=0}^{\infty} \frac{1-q^n cx/a}{1-q^n cx/b} \cdot \prod_{n=0}^{\infty} \frac{1-q^n cx/b}{1-q^n x} = \prod_{n=0}^{\infty} \frac{1-q^n cx/a}{1-q^n x} = \sum_{n=0}^{\infty} \frac{(c/a)_n}{(q)_n} x^n$$

and (5.14) follows at once.

6. It is evident from (5.12) and (5.13) that

$$T_{a,b} T_{b,c} = T_{b,c} T_{a,b} . \tag{6.1}$$

We shall now prove that

$$T_{a,b} T_{c,d} = T_{c,d} T_{a,b} . \tag{6.2}$$

PROOF. By (5.12) and (5.13),

$$\begin{aligned} T_{a,b} T_{c,d} &= T_{c,b} T_{a,c} T_{c,d} \\ &= T_{c,b} T_{a,d} \\ &= T_{c,b} T_{b,d} T_{a,b} \\ &= T_{c,d} T_{a,b} . \end{aligned}$$

We shall now give another proof of (6.2) that makes use of an explicit formula for the transformation $T_{a,b}T_{c,d}$. Put

$$x_n = \frac{1}{(b)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}(a)_k(b/a)_{n-k}(b/a)^k y_k\,,$$

$$y_k = \frac{1}{(d)_k}\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}(c)_j(d/c)_{k-j}(d/c)^j z_j\,.$$

Then

$$x_n = \frac{1}{(b)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}(a)_k(b/a)_{n-k}(b/a)^k\frac{1}{(d)_k}\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}(c)_j(d/c)_{k-j}(d/c)^j z_j = \tag{6.3}$$

$$= \frac{1}{(b)_n}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}(c)_j(d/c)^j z_j\sum_{k=j}^{n}\begin{bmatrix} n-j \\ k-j \end{bmatrix}(a)_k(b/a)_{n-k}\cdot(d/c)_{k-j}(b/a)^k/(d)_k =$$

$$= \frac{1}{(b)_n}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}\frac{(a)_j(c)_j}{(d)_j}\left(\frac{bd}{ac}\right)^j z_j\cdot\sum_{k=0}^{n-j}\begin{bmatrix} n-j \\ k-j \end{bmatrix}(q^j a)_k(b/a)_{n-j-k}\frac{(d_k c)_k}{(q^j d)_k}(b/a)^k\,.$$

Let

$$S_n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(a)_k(b/a)_{n-k}\frac{(d/c)_k}{(d)_k}(b/a)^k\,, \tag{6.4}$$

$$R_n = (b)_n\sum_{s=0}^{n}(-1)^s\begin{bmatrix} n \\ s \end{bmatrix}\frac{(a)_s(c)_s}{(b)_s(d)_s}q^{\frac{1}{2}s(s-1)}\left(\frac{bd}{ac}\right)^s. \tag{6.5}$$

We shall show that

$$R_n = S_n\,. \tag{6.6}$$

Proof. Clearly (6.6) is equivalent to

$$\sum_{n=0}^{\infty}R_n\frac{x^n}{(q)_n} = \sum_{n=0}^{\infty}S_n\frac{x^n}{(q)_n}\,. \tag{6.7}$$

We rewrite (5.15) in the form

$$\sum_{n=0}^{\infty}\frac{(a)_n}{(q)_n}x^n = \frac{e(a)}{e(ax)}\,, \tag{6.8}$$

where

$$e(x) = \sum_{n=0}^{\infty}\frac{x^n}{(q)_n} = \prod_{n=0}^{\infty}(1-q^n x)^{-1}\,. \tag{6.9}$$

Then

$$\sum_{n=0}^{\infty} S_n \frac{x^n}{(q)_n} = \sum_{k=0}^{\infty} \frac{(a)_k(d/c)_k}{(q)_k(d)_k} (bx/a)^k \sum_{n=0}^{\infty} \frac{(b/a)_n}{(q)_n} x^n =$$

$$= \sum_{k=0}^{\infty} \frac{(a)_k(d/c)_k}{(q)_k(d)_k} (bx/a)^k \frac{e(x)}{e(bx/a)},$$

$$\sum_{n=0}^{\infty} R_n \frac{x^n}{(q)_n} = \sum_{s=0}^{\infty} (-1)^s \frac{(a)_s(c)_s}{(d)_s} q^{\frac{1}{2}s(s-1)} \left(\frac{bdx}{ac}\right)^s \sum_{n=0}^{\infty} \frac{(q^s b)_n}{(q)_n} x^n =$$

$$= \sum_{s=0}^{\infty} (-1)^s \frac{(a)_s(c)_s}{(q)_s(d)_s} q^{\frac{1}{2}s(s-1)} \left(\frac{bdx}{ac}\right)^s \frac{e(x)}{e(q^s bx)}.$$

Thus (6.7) is equivalent to

$$\sum_{k=0}^{\infty} \frac{(a)_k(d/c)_k}{(q)_k(d)_k} (bx/a)^k = \sum_{s=0}^{\infty} (-1)^s \frac{(a)_s(c)_s}{(q)_s(d)_s} q^{\frac{1}{2}s(s-1)} \left(\frac{bdx}{ac}\right)^s \frac{e(bx/a)}{e(q^s bx)}. \tag{6.10}$$

The right hand side of (6.10) is equal to

$$\sum_{s=0}^{\infty} (-1)^s \frac{(a)_s(c)_s}{(q)_s(d)_s} q^{\frac{1}{2}s(s-1)} \left(\frac{bdx}{ac}\right)^s \sum_{j=0}^{\infty} \frac{(q^s a)_j}{(q)_j} (bx/a)^j =$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} \left(\frac{bx}{a}\right)^n \sum_{s=0}^{n} (-1)^s \begin{bmatrix} n \\ s \end{bmatrix} \frac{(c)_s}{(d)_s} q^{\frac{1}{2}s(s-1)} \left(\frac{d}{c}\right)^s =$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} \left(\frac{bx}{a}\right)^n \sum_{s=0}^{n} \frac{(q^{-n})_s(c)_s}{(q)_s(d)_s} \left(\frac{q^n d}{c}\right)^s = \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} \left(\frac{bx}{a}\right)^n \frac{(d/c)_n}{(d)_n},$$

by (5.6). This evidently proves (6.10) and therefore proves (6.6). For an identity containing (6.10) see [2].

Returning to (6.3), we have

$$x_n = \frac{1}{(b)_n} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \frac{(a)_j(c)_j}{(d)_j} \left(\frac{bd}{ac}\right)^s z_j \cdot$$

$$\cdot (q^j b)_{n-j} \sum_{s=0}^{n-j} (-1)^s \begin{bmatrix} n-j \\ s \end{bmatrix} \frac{(q^j a)_s(q^j c)_s}{(q^j b)_s(q^j d)_s} q^{\frac{1}{2}s(s-1)} \left(\frac{bd}{ac}\right)^s.$$

Rearranging and simplifying, we get

$$x_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_k)(c)_k}{(b)_k(d)_k} \left(\frac{bd}{ac}\right)^k \sum_{s=0}^{k} (-1)^s \begin{bmatrix} k \\ s \end{bmatrix} q^{\frac{1}{2}s(s-1)} z_{k-s}. \tag{6.11}$$

Thus we have an explicit formula for the transformation $T_{a,b}T_{c,d}$. Clearly (6.11) implies (6.1).

7. We may rewrite (5.7) in the following form.

$$(x/a)_n = \frac{1}{(b)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(a)_k(b/a)_{n-k}(b/a)^k(x/b)_k(q^kx)_{n-k}\,. \tag{7.1}$$

This suggests the following problem. Let $f(x)$ be a polynomial of degree $\leqslant n$. Assume that $f(x)$ can be written in the form

$$f(x) = \sum_{k=0}^{n} A_{n,k}(x/a)_k(q^kx)_{n-k}\,, \tag{7.2}$$

where a is an arbitrary constant and

$$A_{n,k} = A_{n,k}(a, q)$$

is independent of x. We shall show how to evaluate the coefficients $A_{n,k}$.

Taking $x = a$ in (7.2), we get

$$(a)_nA_{n,0} = f(a)\,. \tag{7.3}$$

Next, for $x = q^{-1}a$, (7.2) reduces to

$$f(q^{-1}a) = (q^{-1}a)_nA_{n,0} - q^{-1}(1-q)(a)_{n-1}A_{n,1}\,,$$

so that

$$(1-q)(a)_nA_{n,1} = q(1-q^{-1}a)f(a) - q(1-q^{n-1}a)f(q^{-1}a)\,. \tag{7.4}$$

For $x = q^{-2}a$, (7.2) becomes

$$f(q^{-2}a) = (q^{-2}a)_nA_{n,0} - q^{-2}(1-q^2)(q^{-1}a)_{n-1}A_{n,1} + q^{-}(1-q^2)(a)_{n-2}A_{n,2}\,.$$

This yields

$$(q)_2(a)_nA_{n,2} = q^2(1-q^{-1}a)(1-a)f(a) -$$
$$- q^2(1+q)(1-q^{-1}a)(1-q^{n-1}a)f(q^{-1}a) + q(1-q^{n-1}a)(1-q^{n-2}a)f(q^{-2}a)\,. \tag{7.5}$$

For $x = q^{-3}a$, we get

$$(q)_3(a)_nA_{n,3} = q^3(q^{-1}a)_3f(a) - q^3(1+q+q^2)(q^{-1}a)_2(q^{n-1}a)f(q^{-1}a) +$$
$$+ q^4(1+q+q^2)(q^{-1}a)_1(q^{n-2}a)_2f(q^{-2}a) - q^6(q^{n-3}a)_3f(q^{-3}a)\,. \tag{7.6}$$

This suggests the general formula

$$(7.7)\qquad (q)_k(a)_k A_{n,k} = q^k \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} (q^{n-j}a)_j q^{\frac{1}{2}j(j-1)} f(q^{-j}a) .$$

PROOF OF (7.7). To begin with, we have

$$(7.8)\qquad (a)_n = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} (x/a)_k (q^k x)_{n-k} a^k .$$

Indeed, since

$$(1-q^n) \dots (1-q^{n-k+1}) = (-1)^k q^{nk-\frac{1}{2}k(k-1)} (q^{-n})_k ,$$

it follows from (5.6) that

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} (x/a)_k (q^k x)_{n-k} a^k =$$

$$= (x)_n \sum_{k=0}^{n} \frac{(q^{-n})_k (x/a)_k}{(q)_k (x)_k} (q^n a)^k = (x)_n \frac{(a)_n}{(x)_n} = (a)_n .$$

This proves (7.8). Now replace x by $q^m x$ in (7.8) and multiply by $(x/a)_m$. Then

$$(a)_n (x/a)_m = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} (x/a)_{k+m} (q^{k+m} x)_{n-k} a^k a^k .$$

Changing the notation slightly, this becomes

$$(7.9)\qquad (x/a)_m = \frac{(a)_{n-m}}{1} \sum_{k=m}^{n} (-1)^{k-m} \begin{bmatrix} n-m \\ k-m \end{bmatrix} q^{\frac{1}{2}(k-m)(k-m-1)} \cdot$$

$$\cdot (x/a)_k (q^k x)_{n-k} a^{k-m} .$$

Next, if $f(x)$ is an arbitrary polynomial of degree $\leqslant n$, we may put

$$(7.10)\qquad f(x) = \sum_{m=0}^{n} C_m (x/a)_m ,$$

where $C_m = C_m(a, q)$ is independent of x. It is easily verified that

$$(7.11)\qquad (q)_k C_k = (qa)^k \delta^k f(a) ,$$

where

$$(7.12)\quad \begin{cases} \delta f(x) = \dfrac{f(x)-f(q^{-1}x)}{x}, \\ \delta^k f(x) = \delta\cdot\delta^{k-1}f(x) = x^{-k}\sum\limits_{j=0}^{k}(-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\frac12 j(j-1)} f(q^{-j}x) . \end{cases}$$

It follows from (7.9) and (7.10) that

$$f(x) = \sum_{m=0}^{n} \frac{C_m}{(a)_{n-m}} \sum_{k=m}^{n} (-1)^{k-m} \begin{bmatrix} n-m \\ k-m \end{bmatrix} q^{\frac12(k-m)(k-m-1)}(x/a)_k (q^k x)_{n-k} a^{k-m} =$$

$$= \sum_{k=0}^{n} (x/a)_k (q^k x)_{n-k} \sum_{m=0}^{k} (-1)^{k-m} \begin{bmatrix} n-m \\ k-m \end{bmatrix} q^{\frac12(k-m)(k-m-1)} a^{k-m} \frac{C_m}{(a)_{n-m}} .$$

By (7.11) and (7.12),

$$(q)_m C_m = q^m \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix} q^{\frac12 j(j-1)} f(q^{-j}a) ,$$

so that

$$\sum_{m=0}^{k} (-1)^{k-m} \begin{bmatrix} n-m \\ k-m \end{bmatrix} q^{\frac12(k-m)(k-m-1)} a^{k-m} \frac{C_m}{(a)_{n-m}} = \sum_{m=0}^{k} (-1)^{k-m} \begin{bmatrix} n-m \\ k-m \end{bmatrix} \cdot$$

$$\cdot q^{\frac12(k-m)(k-m-1)} a^{k-m} \cdot \frac{q^m}{(q)_m (a)_{n-m}} \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix} q^{\frac12 j(j-1)} f(q^{-j}a) =$$

$$= \sum_{j=0}^{k} (-1)^j q^{\frac12 j(j-1)} f(q^{-j}a) \sum_{m=j}^{k} (-1)^{k-m} q^{\frac12(k-m)(k-m-1)+m} a^{k-m} \cdot$$

$$\cdot \frac{(q)_m (a)_{n-m}}{1} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} n-m \\ k-m \end{bmatrix} = \sum_{k-j} (-1)^j q^{\frac12 j(j-1)} f(q^{-j}a) \cdot$$

$$\cdot \sum_{m=0}^{j=0} (-1)^m q^{\frac12 m(m-1)+k-m} a^m \cdot \frac{1}{(q)_{k-m}(a)_{n-k+m}} \begin{bmatrix} k-m \\ j \end{bmatrix} \begin{bmatrix} n-k+m \\ m \end{bmatrix} .$$

The inner sum is equal to

$$\sum_{m=0}^{k-j} (-1)^m q^{\frac12 m(m-1)+k-m} a^m \frac{(q)_{n-k+m}}{(q)_j (q)_{k-j-m} (q)_m (a)_{n-k+m}} =$$

$$= \frac{q^k}{(q)_j (q)_{k-j} (a)_{n-k}} \sum_{m=0}^{k-j} \frac{(a^{-k+j})_m (q^{n-k+1})_m}{(q)_m (q^{n-k}a)_m} q^{(k-j-1)m} a^m =$$

$$= \frac{q^k}{(q)_j (q)_{k-j} (a)_{n-k}} \frac{(q^{-1}a)_{k-j}}{(q^{n-k}a)_{k-j}} = \frac{q^k (q^{-1}a)_{k-j}}{(q)_j (q)_{k-j} (a)_{n-j}} .$$

We have therefore

$$(7.13)\qquad f(x) = \sum_{k=0} (x/a)_k (q^k x)_{n-k} \sum_{j=0}^{k} (-1)^s q^{\frac{1}{2}s(s-1)} f(q^{-j}a) \cdot$$

$$\cdot \frac{q^k (q^{-1}a)_{k-j}}{(q_j)(q)_{k-j}(a)_{n-j}} = \sum_{k=0}^{n} (x_k a)_k (q^k x)_{n-k} \frac{q^k}{(q)_k (a)_n} \cdot$$

$$\cdot \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} (q^{-1}a)_{k-j} (q^{n-j}a)_j q^{\frac{1}{2}j(j-1)} f(q^{-j}a) .$$

It remains to show that the coefficients $A_{n,k}$ in (7.2) are uniquely determined. Otherwise we should have

$$(7.14)\qquad \sum_{k=0}^{n} B_k (x/a)_k (q^k x)_{n-k} = 0 ,$$

where not all the B_k equal zero. For $x = a$, (7.14) implies $B_0 = 0$. Assume that

$$B_0 = \dots = B_{m-1} = 0 , \qquad B_m \neq 0 .$$

For $x = q^{-m}a$, (7.14) reduces to

$$(q^{-m})_m (a)_{n-m} B_m = 0 .$$

Since a is not equal to q^{-k}, $k = 0, 1, 2, \dots$, we get $B_m = 0$.

We have therefore proved the following.

THEOREM B. *Assume $a \neq q^{-k}$, where k is a nonnegative integer. Let $f(x)$ be a polynomial of degree $\leqslant n$. Then the coefficients in*

$$f(x) = \sum_{k=0}^{n} A_{n,k} (x/a)_k (q^k a)_{n-k}$$

are uniquely determined by

$$(q)_k (a)_k A_{n,k} = q^k \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} (q^{-1}a)_{k-j} (q^{n-j}a)_j q^{\frac{1}{2}j(j-1)} f(q^{-j}a) .$$

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