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On the Estension of Lipschitz Functions with Respect to Two Hilbert Norms and Two Lipschitz Conditions.

CHANDAN S. VORA (*)

1. – Introduction.

Kirszbraun theorem [5] and [9] asserts that a Lipschitz function from a finite subset of R^n to R^n can be extended, maintaining the same Lipschitz constant to a larger domain including any arbitrarily chosen point. (The Euclidean norm is essential; see Schönbeck [13], Grünbaum [4]). This theorem was rediscovered by Valentine [14] and many others in a Hilbert space. Minty [8] proved the same fact for a « monotone » function and Grünbaum [5] combined these two theorems into one. A further improvement to Minty's theorem was given by Debrunner and Flor [3], who showed that the desired new functional value could always be chosen in the closure of the convex hull of the given functional values; several different proofs of this fact have now been given (see [9], [1]). Minty in [10] gave a unified method for proving all the above results including the generalisation of the Kirszbraun and Banach theorems. D. G. Figueiredo and F. E. Browder pointed out to Minty that the theorem 1 part (ii) of [10] was actually due to Mickle [7]. The inequality stated by him was proved by Prézis and Fox was (essentially) given by I. J. Schoenberg [12].

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Out of the network theory arose the question of extendability of a Lipschitz mapping with respect to two Hilbert norms C and E say, defined on a subset of a vector space M to the whole of M. Precisely, the question asked by G. Darbo in his seminars is as follows:

Let M be R-vector space with two Hilbert norms C and E say and A a subset of M and $T\colon A \to M$ be a Lipschitz function of constant one in both the norms, that is, for every $x_1, x_2 \in A$,

$$(1) \qquad \|Tx_1 - Tx_2\|_C \leqslant \|x_1 - x_2\|_C \quad \text{and} \quad \|Tx_1 - Tx_2\|_E \leqslant \|x_1 - x_2\|_E .$$

Does there exist an extension $T'\colon M \to M$ of T to the whole M such that it preserves the Lipschitz condition (1)?

In this paper we treat G. Darbo's problem and three other analogous problems which can be formulated as follows with two Hilbert norms C and E.

Let A, M be as before and $T\colon A \to M$ be a Lipschitz function of constant one in both the norms, that is,

(2) for every $x_1, x_2 \in A$,

$$\|Tx_1 - Tx_2\|_C \leqslant \|x_1 - x_2\|_C \quad \text{and} \quad \|Tx_1 - Tx_2\|_C \leqslant \|x_1 - x_2\|_E ,$$

respectively

(3) for every $x_1, x_2 \in A$

$$\|Tx_1 - Tx_2\|_C \leqslant \|x_1 - x_2\|_C \quad \text{and} \quad \|Tx_1 - Tx_2\|_E \leqslant \|x_1 - x_2\|_C ,$$

or

(4) for every $x_1, x_2 \in A$

$$\|Tx_1 - Tx_2\|_C \leqslant \|x_1 - x_2\|_E \quad \text{and} \quad \|Tx_1 - Tx_2\|_E \leqslant \|x_1 - x_2\|_C .$$

Does there exist an extension $T'\colon M \to M$ of T to the whole M such that it preserves both the Lipschitz conditions (2) (respectively, (3) or (4))?

In all four problems, we show with convenient examples that when the two conditions of Lipschitz are independent (that is, one condition of Lipschitz is not a consequence of the other) the answer in general is no.

The other problems corresponding to the position of the norm E in the Lipschitz conditions are analogous to that of the position of the norm C.

Since the condition of Lipschitz could be referred in general as a pair of norms, we take the opportunity of introducing the following definition.

2. DEFINITION. Let C_1, C_2, E_1, E_2 be four Hilbert norms on a vector space M. The pair of norms (E_1, C_1) is said to precede the pair of norms (E_2, C_2) (abbreviated as $(E_1, C_1) \propto (E_2, C_2)$) if there exists a real number $\varrho > 0$ such that for every $x \in M$, the following inequalities hold

$$\varrho \|x\|_{C_2} \leqslant \|x\|_{C_1} \quad \text{and} \quad \|x\|_{E_1} \leqslant \varrho \|x\|_{E_2}$$

3. REMARK. The (E, C)-Lipschitz condition for a function F means

$$\|Fx_1 - Fx_2\|_C \leqslant \|x_1 - x_2\|_E\,, \qquad \text{for every } x_1, x_2 \subseteq \text{domain of } F\,.$$

We observe that if $(E_1, C_1) \propto (E_2, C_2)$ then the Lipschitz condition $E_1, C_1)$ implies in general the Lipschitz condition (E_2, C_2).

4. DEFINITION. The pair of norms (E_1, C_1) is said to be comparable to the pair of norms (E_2, C_2) if and only if $(E_1, C_1) \propto (E_2, C_2)$ or $(E_2, C_2) \propto (E_1, C_1)$.

Now we consider the preliminary lemmas. In the following considerations let M be a vector space with two Hilbert norms C and E.

5. LEMMA. The two pair of norms (C, C) and (E, E) are comparable if and only if the two norms C and E are proportional i.e. for some constant $\varrho > 0$ and for every point x of the vector space M, we have $\|x\|_C = \varrho \|x\|_E$.

6. LEMMA. The following conditions are equivalent

(i) the two norms C and E are comparable, i.e., for every point $x \in M$, $\|x\|_E \leqslant \|x\|_C$ or for every point $x \in M$, $\|x\|_C \leqslant \|x\|_E$:

(ii) the pair of norms (C, C) and (E, C) are comparable;

(iii) the pair of norms (C, C) and (C, E) are comparable;

(iv) the pair of norms (C, E) and (E, C) are comparable.

Proof of Lemma 5. Suppose $(E, E) \propto (C, C)$. This is equivalent to saying that there exist $\varrho > 0$, such that for every $x \in M$, $\varrho\|x\|_C \leqslant \|x\|_E$ and $\|x\|_E \leqslant \varrho\|x\|_C$. But this is equivalent to saying that there exist $\varrho > 0$ such that for every $x \in M$, $\varrho\|x\|_C = \|x\|_E$ i.e. the norms are proportional; similarly for $(C, C) \propto (E, E)$ we get that the norms C and E are proportional.

Reversing the steps of the argument we get the converse.

Proof of Lemma 6. (i) $\Leftrightarrow$ (ii) The two norms are comparable is equivalent to saying that either for every point $x \in M$, $\|x\|_E \leqslant \|x\|_C$ or for every point $x \in M$, $\|x\|_C \leqslant \|x\|_E$. Suppose, for every point $x \in M$, $\|x\|_E \leqslant \|x\|_C$. Then for $\varrho = 1$ and for every point $x \in M$ we have, $\varrho\|x\|_C = \|x\|_C$ and $\|x\|_E \leqslant \varrho\|x\|_C$. But, this implies that $(E, C) \propto (C, C)$. Similarly, for every point $x \in M$, if we have $\|x\|_C \leqslant \|x\|_E$ then by taking $\varrho = 1$ and $\varrho\|x\|_C = \|x\|_C \leqslant \varrho\|x\|_E$ we get $(C, C) \propto (E, C)$. Therefore, we have either $(E, C) \propto (C, C)$ or $(C, C) \propto (E, C)$, i.e. the pair of norms (C, C) and (E, C) are comparable.

Conversely, if the pair of norms are comparable, we have either $(E, C) \propto (C, C)$ or $(C, C) \propto (E, C)$. Suppose, $(E, C) \propto (C, C)$, then there exists $\varrho > 0$ such that for every $x \in M$, $\varrho\|x\|_C \leqslant \|x\|_C$ and $\|x\|_E \leqslant \varrho\|x\|_C$. This implies that for every $x \in M$, $\|x\|_E \leqslant \|x\|_C$. Now assume that $(C, C) \propto (E, C)$; then there exists $\varrho > 0$ such that for every $x \in M$, $\varrho\|x\|_C \leqslant \|x\|_C$ and $\|x\|_C \leqslant \varrho\|x\|_E$. But, this implies, for $\varrho > 0$ and for every $x \in M$, we have $\varrho\|x\|_C \leqslant \varrho\|x\|_E$. Since $\varrho > 0$, we have for every $x \in M$, $\|x\|_C \leqslant \|x\|_E$. Therefore, either, for every $x \in M$ we have $\|x\|_E \leqslant \|x\|_C$ or for every $x \in M$, $\|x\|_C \leqslant \|x\|_E$. Hence, the norms C and E are comparable.

(i) $\Leftrightarrow$ (iii) To show (i) implies (iii) we take $\varrho = 1$ and consider the inequalities $\varrho\|x\|_E \leqslant \|x\|_C$ and $\|x\|_C = \varrho\|x\|_C$ or $\varrho\|x\|_C \leqslant \|x\|_E$ and $\|x\|_C = \varrho\|x\|_C$.

To show the converse, one can easily show that $(C, C) \propto (C, E)$ implies for every $x \in M$, $\|x\|_E \leqslant \|x\|_C$ and $(C, E) \propto (C, C)$ implies for every $x \in M$, $\|x\|_C \leqslant \|x\|_E$. Hence the result follows

(i) $\Leftrightarrow$ (iv) To show (i) implies (iv), we again take $\varrho = 1$ and consider the inequalities $\varrho\|x\|_E \leqslant \|x\|_C$ and $\|x\|_E \leqslant \varrho\|x\|_C$ or $\varrho\|x\|_C \leqslant \|x\|_E$ and $\|x\|_C \leqslant \varrho\|x\|_E$.

Conversely, suppose the pair of norms (C, E) and (E, C) are comparable, i.e. either $(E, C) \propto (C, E)$ or $(C, E) \propto (E, C)$. If $(E, C) \propto$

$\propto (C, E)$, then there exists $\varrho > 0$ such that for every $x \in M$, we have $\varrho \|x\|_E \leqslant \|x\|_C$ and $\|x\|_E \leqslant \varrho \|x\|_C$.

Therefore, for every $x \in M - \{0\}$, $\|x\|_E / \|x\|_C \leqslant \varrho \leqslant \|x\|_C / \|x\|_E$, that is, $\|x\|_E^2 \leqslant \|x\|_C^2$ which implies $\|x\|_E \leqslant \|x\|_C$. Similary, if $(C, E) \propto (E, C)$, then we get for some $\varrho > 0$ and for every $x \in M - \{0\}$, $\|x\|_C / \|x\|_E \leqslant \varrho \leqslant \|x\|_E / \|x\|_C$ which implies $\|x\|_C \leqslant \|x\|_E$.

Therefore we have either for every $x \in M$, $\|x\|_E \leqslant \|x\|_C$ or for every $x \in M$, $\|x\|_C \leqslant \|x\|_E$, i.e., the norms C and E are comparable.

Before stating a theorem which contains the answers to the problems 1, 2, 3, 4, we state a general problem for two Lipschitz conditions and four Hilbert norms as follows:

PROBLEM 5. Let M be a R-vector space with four Hilbert norms C_1, C_2, E_1, E_2, A a subset of M and $T: A \to M$ be a function satisfying (E_1, C_1) and (E_2, C_2)-Lipschitz conditions. Does there exist an extension T': $M \to M$ of T to the whole of M such that T' also satisfies the Lipschitz conditions-(E_1, C_1) and (E_2, C_2)?

Now we state the theorem

7. THEOREM. (*a*) If the pair of norms (E_1, C_1) and (E_2, C_2) are comparable then the desired extension of problem 5 always exists.

(*b*) If the pair of norms (C, C) and (E, E) (respectively, the pairs of norms (C, C) and (E, C); (C, C) and (C, E); and (C, E) and (E, C)) are not comparable then there exists a counterexample.

PROOF. Part (*a*) of the theorem obviously follows from the facts that $(E_1, C_1) \propto (E_2, C_2)$ or $(E_2, C_2) \propto (E_1, C_1)$, Remark (3), (E_1, C_1)- or (E_2, C_2)-Lipschitz extension theorem respectively.

For part (*b*) see 10, 11, 12, 13 of this paper.

In particular, we have for the problems 1, 2, 3 and 4 respectively

8. REMARK. If the pair of norms corresponding to the Lipschitz conditions in problem 1 (respectively 2, 3 and 4) are comparable then the extension problem 1 (respectively 2, 3 and 4) has an affirmative answer.

Now, we show successively for the four problems that if the respective pair of norms are not comparable then there exists a counterexample.

9. COUNTEREXAMPLE FOR PROBLEM 1. Since the pair of norms (C, C) and (E, E) are not comparable, that is, the two norms C and E

are not proportional and hence dimension of $M \geqslant 2$, and therefore there exist two vectors u, v belonging to M such that, u, v are orthonormal with respect to the norm C and orthogonal with respect to the norm E. Let $uR + vR = H \subset M$ be the subspace of M generated by u, v. Then for every $z \in H$ of the type

$$z = \xi u + \eta v\,, \qquad \xi, \eta \in R$$

we take W.L.O.G.

$$\|z\|_C = \{\xi^2 + \eta^2\}^{\frac{1}{2}}\,,$$
$$\|z\|_E = \left\{\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2}\right\}^{\frac{1}{2}}\,,$$

with a, b to be positive constants, $a \neq b$ and we may assume $a > b$.

Let z_1, z_2, z_3 be points belonging to H defined by

$$z_1 = -\frac{1}{\sqrt{2}}u - \frac{1}{\sqrt{2}}v\,,$$
$$z_2 = -\frac{1}{\sqrt{2}}u + \frac{1}{\sqrt{2}}v\,,$$
$$z_3 = \frac{1}{\sqrt{2}}u + \frac{1}{\sqrt{2}}v\,,$$

$$\|z_1\|_C = \|z_2\|_C = \|z_3\|_C = 1\,, \qquad \|z_1\|_E = \|z_2\|_E = \|z_3\|_E = \left\{\frac{1}{2a^2} + \frac{1}{2b^2}\right\}^{\frac{1}{2}}\,,$$

$$\|z_1 - z_2\|_C = \|z_2 - z_3\|_C = \frac{2}{\sqrt{2}}\,, \qquad \|z_2 - z_3\|_E = \frac{\sqrt{2}}{a}\,, \qquad \|z_2 - z_1\|_E = \frac{\sqrt{2}}{b}\,.$$

Let $A = \{0, z_1, z_2\}$ and define $T\colon A \to M$ by $T(0) = 0$, $T(z_1) = z_2$, $T(z_2) = z_3$. Clearly, T is (C, C)-Lipschitz as well as (E, E)-Lipschitz on A. The map T is not extendable to the point $z = (z_1 + z_2)/2$ (in fact, T is not extendable to any point on the line segment $\overline{z_1 z_2}$).

Suppose T were extendable to the point z. Let $T'\colon A \cup \{z\} \to M$ be its extension. Since T preserves distances for the norm C, T' must also preserve distances for the points in the convex hull of A with respect to the norm C. This forces $T'(z) = \big(T(z_1) + T(z_2)\big)/2 = (z_2 + z_3)/2$. But this does not satisfy the (E, E)-Lipschitz condition on $A \cup \{z\}$. Hence, no such extension exists.

10. COUNTEREXAMPLE FOR PROBLEM 2. Since the pair of norms (C, C) and (E, C) are not comparable, that is, the two norms C and E are not comparable, there exists two vectors u, v belonging to M with u, v orthonormal with respect to the norm C and orthogonal with respect to the norm E such that

$$\|v\|_C < \|v\|_E$$

$$\|u\|_E < \|u\|_C$$

Let $H \subset M$ be the subspace of M generated by u, v. Then for every $z \in H$ of the type

$$z = \xi u + \eta v , \qquad \xi, \eta \in R ,$$

$$\|z\|_C = \{\xi^2 + \eta^2\}^{\frac{1}{2}} ,$$

$$\|z\|_E = \left\{\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2}\right\}^{\frac{1}{2}} ,$$

with $a > 1 > b > 0$; (we can assume W.L.O.G. $a > b$ and non comparability of norms gives $a > 1 > b$).

Let z_1 and z_2 be the points

$$z_1 = -x_1 u - y_1 v$$

$$z_2 = -x_1 u + y_1 v$$

where $x_1^2 + y_1^2 = 1 = x_1^2/a^2 + y_1^2/b^2$; $x_1, y_1 > 0$. Let $A = \{0, z_1, z_2\}$ and define $T: A \to M$ by $T(0) = 0$, $T(z_1) = z_1$ and $T(z_2) = z_2$. Clearly,

$$\|z_1\|_C = \|z_2\|_C = \|z_1\|_E = \|z_2\|_E = 1 , \quad \|z_1 - z_2\|_C = 2y_1 , \quad \|z_1 - z_2\|_E = \frac{2y_1}{b}$$

and T is (C, C) and (E, C)-Lipschitz on A. The map T is not extendable to the point $z = (z_1 + z_2)/2$.

Suppose T were extendable to the point z. Let $T'\colon A \cup \{z\} \to M$ be its extension. Since T preserves distances for the norm C, T' must also preserve distances for the points in the convex hull of A with respect to the norm C. This forces $T'(z) = z$. But, this does not satisfy (E, C)-Lipschitz condition on $A \cup \{z\}$. Hence no such extension exists.

11. COUNTEREXAMPLE FOR PROBLEM 3. Since the pair of norms (C, C) and (C, E) are not comparable, that is, the two norms C and E are nor comparable, there exist two vectors u, v belonging to M which generate H and for every $z \in H$, $\|z\|_C$ and $\|z\|_E$ as before in counterexample to problem 2.

Let z_2, z_3 be the points

$$z_2 = -x_1 u + y_1 v$$
$$z_3 = \quad x_1 u + y_1 v$$

where $x_1^2 + y_1^2 = 1 = x_1^2/a^2 + y_1^2/b^2$, $x_1, y_1 > 0$, $a > 1 > b > 0$. Clearly,

$$\|z_2\|_C = \|z_3\|_C = \|z_2\|_E = \|z_3\|_E = 1\,, \quad \|z_2 - z_3\|_C = 2x_1\,, \quad \|z_2 - z_3\|_E = \frac{2x_1}{a}.$$

Let $A = \{0, z_2, z_3\}$ and define $T\colon A \to M$ by $T(0) = 0$, $T(z_2) = z_2$ and $T(z_3) = z_3$. Clearly, the map T is (C, C) and (C, E)-Lipschitz.

By a similar kind of argument as in counterexample for problen 2, it can be shown that T is not extendable to the point $z = (z_2 + z_3)/2$.

12. COUNTEREXAMPLE FOR PROBLEM 4. Since the pair of norms (C, E) and (E, C) are not comparable, that is, the two norms C and E are not comparable, there exist two vectors u, v belonging to M which generate H and for every $z \in H$, $\|z\|_C$ and $\|z\|_E$ as before in counterexample for problem 2.

Let z_1, z_2 and z_1' be the points

$$z_1 = -x_1 u - y_1 v$$
$$z_2 = -x_1 u + y_1 v$$
$$z_1' = -x_1 u + (\varepsilon - y_1) v$$

with $0 < \varepsilon = 2y_1(1 - b)$, $x_1^2 + y_1^2 = 1 = x_1^2/a^2 + y_1^2/b^2$, $x_1, y_1 > 0$, $a > 1 > > b > 0$.

Clearly,

$$\|z_1\|_C = \|z_2\|_C = \|z_1\|_E = \|z_2\|_E = 1\,, \quad \|z_1 - z_2\|_C = 2y_1\,, \quad \|z_1 - z_2\|_E = \frac{2y_1}{b}\,,$$

$$\|z_1'\|_C = x_1^2 + (y_1 - \varepsilon)^2, \quad \|z_1'\|_E = \frac{x_1^2}{a^2} + \frac{(y_1 - \varepsilon)^2}{b^2}\,, \quad \|z_1' - z_2\|_C = 2y_1 - \varepsilon$$

and

$$\|z_1' - z_2\|_E = \frac{2y_1 - \varepsilon}{b} = 2y_1 .$$

Let $A = \{0, z_1, z_2\}$ and define $T: A \to M$ by $T(0) = 0$, $T(z_1) = z_1'$ and $T(z_2) = z_2$. It can easily be verified that T is (C, E) and (E, C) Lipschitz on A.

Now, consider the point $z = (z_1 + z_2)/2$. If $T': A \cup \{z\} \to M$ is an extension of T of the type required in the problem with $T'(z) = u$, then

$$(7)\quad \begin{cases} \|u\|_C \leqslant \|z\|_E = \dfrac{x_1}{a} , \\ \|u - z_1'\|_C \leqslant \|z - z_1\|_E = \dfrac{y_1}{b} , \\ \|u - z_2\|_C \leqslant \|z - z_2\|_E = \dfrac{y_1}{b} , \\ \|u\|_E \leqslant \|z\|_C = x_1 , \end{cases}$$

$$(8)\quad \|u - z_1'\|_E \leqslant \|z - z_1\|_C = y_1 ,$$

$$(9)\quad \|u - z_2\|_E \leqslant \|z - z_2\|_C = y_1 .$$

Now conditions (8), (9), (6) and the choice of ε imply that

$$\|u - z_1'\|_E \leqslant \|z - z_1\|_C = y_1 = \frac{\|z_1 - z_2\|_C}{2} = \frac{\|z_1' - z_2\|_E}{2} ,$$

and

$$\|u - z_2\|_E \leqslant \|z - z_2\|_C = y_1 = \frac{\|z_1 - z_2\|_C}{2} = \frac{\|z_1' - z_2\|_E}{2} .$$

These imply that $u = (z_1' + z_2)/2$. But, u must satisfy condition (1) also. Since $u = -x_1 u + (\varepsilon/2)v$ we obtain

$$\|u\|_C = \left\{x_1^2 + \frac{\varepsilon^2}{4}\right\}^{\frac{1}{2}} \leqslant \frac{x_1}{a} ,$$

i.e.,

$$x_1^2 + \frac{\varepsilon^2}{4} < \frac{x_1^2}{a^2}.$$

But, $a > 1$ and hence $x_1^2/a^2 < x_1^2$, so we get

$$x_1^2 + \frac{\varepsilon^2}{4} < x_1^2,$$

i.e.,

$$\frac{\varepsilon^2}{4} < 0,$$

a contradiction, so no such extension exists.

13. Remark. The author thinks that there exist counterexamples in the general case of four norms when the pair of norms (E_1, C_1) and (E_2, C_2) are not comparable. It can easily be shown that counterexamples if they exist must exist between dimensions two to four.

13.1. Open problem. Let M be a vector space with four Hilbertian norms C_1, C_2, E_1, E_2. If the pair of norms (E_1, C_1) and (E_2, C_2) are not comparable then there exist A, T and $x \in M - A$ such that T is (E_1, C_1) and (E_2, C_2)-Lipschitz on A but there exists no extension $T: A \cup \{x\} \to M$ of T with T to be (E_1, C_1) and (E_2, C_2)-Lipschitz.

13.2. Announcement. In the case of three Hilbert norms C, E_1, E_2 (i.e. $C_1 = C_2$), there are six problems corresponding to the position of the norm C in the Lipschitz inequalities and the other problems corresponding to the positions of the norms E_1 and E_2 in the Lipschitz inequalities are analogous to that of the norm C.

The six Lipschitz conditions corresponding to the six problems are (i) (C, C) and (E_1, E_2)-Lipschitz (ii) (C, C) and (E_2, E_1)-Lipschitz, (iii) (E_1, C) and (E_2, C)-Lipschitz (IV) (C, E_1) and (C, E_2)-Lipschitz (V) (E_2, C) and (C, E_1)-Lipschitz and (VI) (E_1, C) and (C, E_2)-Lipschitz.

When the respective pairs of norms are comparable the extension always exists, see Theorem 7. The author has obtained the counterexamples for all the problems for three norms and two Lipschitz conditions when the respective pairs of norms are not comparable. In short, the work for three norms and two Lipschitz conditions has been completed and the author will try to publish it at a later date.

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