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A Characterization of Algebraic Measures.

PETER ASHLEY LAWRENCE (*)

1. Let \mathcal{G} be a locally compact Hausdorff abelian topological group. Let $M(\mathcal{G})$ be the set of complex measures μ on the Borel sets of \mathcal{G} such that $\|\mu\|$ (defined in the usual way, *vide* [2]) is finite. $M(\mathcal{G})$ is an algebra over the complex numbers C with the convolution operation $*$ (*vide* [4]) as multiplication on $M(\mathcal{G})$.

Cohen [1, 4] completely determined the measures for which

$$\mu * \mu = \mu .$$

Such measures are called *idempotent*. The problem considered and solved in the paper is the characterization of all μ that satisfy an algebraic equation.

More precisely, define

$$\begin{aligned} \mu^0 &= \delta \\ \mu^n &= \mu * \mu^{n-1}, \quad n \geq 1 \end{aligned}$$

where δ is the unit element of $M(\mathcal{G})$, picturesquely described as « unit mass concentrated at the origin ». A complete characterization is given of those measures μ for which there exists a set (in general dependent

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on μ and not unique) of complex numbers z_i , $0 \leq i \leq n$, $z_n \neq 0$, such that

$$\sum_{i=0}^n z_i \mu^i = 0 .$$

Such measures are called *algebraic*. They were first considered Istratescu, who proved that the carrier-group of an algebraic measure is compact [3].

The main result of this paper is the theorem in Section 4 that characterizes an algebraic measure as one such that the partition induced on the dual group by the Fourier-Stieltjes transform of the measure is generated by cosets of the dual group.

2. Let Γ be the dual group of \mathfrak{G} . Let

$$\hat{\mu}: \Gamma \rightarrow \mathbb{C}$$

be the Fourier-Stieltjes transform of μ [4]. Let P be the formal polynomial

$$P(X) = \sum_{i=0}^n c_i X^i$$

where the c_i 's are complex numbers.

$$\begin{aligned} P(\hat{\mu}(\gamma)) &= (P(\hat{\mu}))(\gamma) = \\ &= (P(\mu))^{\wedge}(\gamma) = 0 . \end{aligned}$$

Thus $\hat{\mu}(\gamma)$ must be a root of P in \mathbb{C} .

Conversely if $\hat{\mu}(\gamma)$ is always one of the complex roots of P , for all $\gamma \in \Gamma$, then $P(\hat{\mu})$ vanishes identically on Γ and the uniqueness theorem for Fourier-Stieltjes transforms [4] shows that $P(\mu) = 0$. Since the functions $\mathbb{C} \rightarrow \mathbb{C}$ with finite images are exactly the functions that can be written as polynomials it follows that the algebraic measures are exactly the measures μ such that the image of $\hat{\mu}$ is finite. It is now clear that the sum of algebraic measures is algebraic and so is the product.

3. For our purposes a partition of a set A is any set of pairwise disjoint subsets of A whose union is A . In particular partitions are

allowed to have empty members. Let C be given some linear ordering which will be kept fixed. Then an injective mapping f from algebraic measures to ordered partitions is given by

$$f(\mu) = \langle \{\gamma | \hat{\mu}(\gamma) = z\} | z \in C \rangle .$$

The main result of this paper is the explicit description of the image of f . Clearly any partition may be replaced by an equivalent one by throwing away some or all of the empty members. If the image of $\hat{\mu}$ is a subset of a set K we write

$$f_k(\mu) = \langle \{\gamma | \hat{\mu}(\gamma) = z\} | z \in K \rangle .$$

Recall that a ring of sets is a set of sets stable under the formation of complements and finite unions (and hence under the formation of finite intersections).

In the case of idempotent measures the polynomial involved, *viz.* $P(X) = X^2 - X$ has only two distinct roots in C , 0 and 1. Thus the idempotent measures can be described by considering one member of the partition induced by the polynomial. Let

$$S(\mu) = \{\gamma \in I | \hat{\mu}(\gamma) = 1\}$$

for idempotent μ . Cohen showed that a subset A of I has the form $S(\mu)$ for some idempotent μ if and only if A lies in the ring of sets generated by the cosets of open subgroups of I .

Let A be a set. An ordered m -partition of A is an ordered m -tuple of pairwise disjoint sets whose union is A . Let $\mathfrak{S} = \{\mathfrak{F}_1, \mathfrak{F}_2, \dots, \mathfrak{F}_n\}$ be a finite set of m -partitions of A . Let \mathfrak{F}_{ij} be the j -th member of \mathfrak{F}_i : The ordered m -partition \mathfrak{F} is *primitively generated* by \mathfrak{S} if every finite intersection of form $\bigcap_{i=1}^n \mathfrak{F}_{ij_i}$ is a subset of some member of \mathfrak{F} . Hence every member of \mathfrak{F} must be the union of intersections of this form. Let \mathfrak{C} be a set of ordered m -partitions of A . \mathfrak{C} is an m -partition algebra if \mathfrak{C} contains every ordered m -partition primitively generated by a finite subset of \mathfrak{C} . It is clear that the intersection of a collection of m -partition algebras is itself an m -partition algebra. Thus there is a smallest m -partition algebra containing any set of ordered m -partitions. This algebra is called the m -partition algebra *generated* by the given set of partitions.

A *Type I* partition of Γ is a finite ordered partition of Γ in which at most two members are non-empty and in which one member is a coset of an open subgroup of Γ .

4. We shall now state the theorem that characterizes algebraic measures.

THEOREM. *Let P be a polynomial over C with distinct roots $\{c_1, c_2, \dots, c_m\} = K$ in C . Let $\mu \in M(\mathfrak{G})$. $P(\mu) = 0$ if and only if: (a) for all $\gamma \in \Gamma$, $\hat{\mu}(\gamma) \in K$ and (b) $f_k(\mu)$ belongs to the m -partition algebra generated by the ordered m -partitions of Type I.*

First let us look at Cohen's result in our terminology. The ring of sets generated by a set of sets \mathfrak{U} is precisely the set of sets B that can be written as finite unions of finite intersections

$$B = \cup \cap B_{ij}$$

where every B_{ij} or its complement B'_{ij} belongs to \mathfrak{U} . Thus the ordered partition $\langle B, B' \rangle$ is primitively generated by the set of ordered partitions of form $\langle B_{ij}, B'_{ij} \rangle$. The ring of sets generated by the cosets of open subgroups of Γ thus corresponds to the 2-partition algebra generated by Type I ordered 2-partitions. Cohen's Theorem is seen to be a special case of our theorem.

The remainder of this paper will be devoted to a proof of our theorem above. We shall prove six lemmas and then the theorem.

5. LEMMA 1. *Let $P(\mu) = 0$, $\mu \in M(\mathfrak{G})$. Let c_1, \dots, c_m be the distinct complex roots of P . Let k_1, \dots, k_m be complex numbers, not necessarily distinct. There exists $\nu \in M(\mathfrak{G})$ such that $\hat{\nu}(\gamma) = c_i$ if and only if $\hat{\nu}(\gamma) = k_i$.*

PROOF. There exists a polynomial $P_1(X)$ over C such that $P_1(c_i) = k_i$ for all i . $P_1(\mu) \in M(\mathfrak{G})$. $P_1(\mu)\hat{\nu}(\gamma) = P_1(\hat{\mu}(\gamma))$. Thus ν may be chosen as $P_1(\mu)$.

Two particular cases of this lemma are of special interest. The k_i may be chosen so that each is equal to some c_j . Thus if an algebraic measure μ is given, which induces an ordered partition \mathfrak{F} on C and if an ordered partition \mathfrak{F}' is given such that every member of \mathfrak{F}' is equal to the union of members of \mathfrak{F} , there exists a measure ν that induces \mathfrak{F}' on C . Another special case of the lemma is obtained when the c_i are all non-zero. Then a measure ν exists such that $\hat{\nu}(\gamma) = (\hat{\mu}(\gamma))^{-1}$ for all $\gamma \in \Gamma$. Then ν is the convolution inverse of μ . An algebraic measure μ is therefore invertible if and only if $\hat{\mu}(\gamma)$ is never 0.

LEMMA 2. *Let A be an open subgroup of Γ . Let c_1 and c_2 be two complex roots of P . There exists $\mu \in M(\mathfrak{G})$ such that*

$$P(\mu) = 0; \quad \hat{\mu}(\gamma) = c_1, \quad \gamma \in A; \quad \hat{\mu}(\gamma) = c_2, \quad \gamma \notin A.$$

PROOF. Let H be the annihilator of A . H is compact and isomorphic to the dual of Γ/A . Let m be Haar measure on H with $m(H) = 1$. m defines a measure m_1 in $M(\mathfrak{G})$ by

$$m_1(B) = m(B \cap H)$$

for all Borel sets B in \mathfrak{G} . $\hat{m}_1(\mu)$ is 1 if $\gamma \in A$ and 0 otherwise. The previous lemma shows the existence of a measure with the desired properties.

LEMMA 3. *Let A be an open subgroup of Γ . Let $\gamma_0 \in \Gamma$. Let c_1 and c_2 be two roots of P . There exists $\mu \in M(\mathfrak{G})$ with $P(\mu) = 0$, $\hat{\mu}(\gamma) = c_1$ if $\gamma \in \gamma_0 + A$, $\hat{\mu}(\gamma) = c_2$ otherwise.*

PROOF. By the previous lemma there exists μ_1 such that:

$$\begin{aligned} \hat{\mu}_1(\gamma) &= c_1 \text{ if } \gamma \in A \\ \hat{\mu}_1(\gamma) &= c_2 \text{ otherwise.} \end{aligned}$$

Let

$$d\mu = \gamma_0 d\mu_1$$

then

$$\begin{aligned} \hat{\mu}(\gamma) &= \hat{\mu}_1(\gamma - \gamma_0) \\ \hat{\mu}(\gamma) &= c_1 \quad \text{if } \gamma \in \gamma_0 + A \\ \hat{\mu}(\gamma) &= c_2 \quad \text{if } \gamma \notin \gamma_0 + A. \end{aligned}$$

LEMMA 4. *Let $P(\mu_i) = 0$, $\mu_i \in M(\mathfrak{G})$, for $1 \leq i \leq n$. Let \mathfrak{F}_i be the ordered partition of Γ induced by μ_i . Let \mathfrak{R} be an ordered partition of Γ primitively generated by the \mathfrak{F}_i . There exists a measure $\mu \in M(\mathfrak{G})$ whose induced partition is \mathfrak{R} .*

PROOF. Let $\mathfrak{F}_{ij} = \{\gamma \in \Gamma | \hat{\mu}_i(\gamma) = c_j\}$.

To every intersection of form $\bigcap_{i=1}^n \mathfrak{F}_{ij_i}$ we associate the point $(c_{j_1}, \dots, c_{j_n})$. Thus partitions of Γ primitively generated by the \mathfrak{F}_i

correspond to partitions of the finite set

$$\{(\hat{\mu}_1(\gamma), \dots, \mu_n(\gamma)) | \gamma \in \Gamma\} = S.$$

Let S_i be the subset of S corresponding to \mathcal{R}_i . Let Q be a polynomial in $X_1, \dots, X_n, \bar{X}_n, \dots, \bar{X}_1$ over C such that $Q(S_i) = c_i$ for all i . Then $Q(\mu_1, \dots, \mu_n, \bar{\mu}_1, \dots, \bar{\mu}_n)(\gamma) = Q(\mu_1(\gamma), \dots, \mu_n(\gamma), \bar{\mu}_1(\gamma), \dots, \bar{\mu}_n(\gamma))$. Thus $\mu = Q(\mu_1, \dots, \bar{\mu}_1, \dots, \bar{\mu}_n)$ is the desired measure.

LEMMA 5. *Let \mathfrak{S} be a member of the algebra of ordered m -partitions generated by the set of all m -partitions of Type I. Let \mathfrak{P} be a polynomial with distinct complex roots c_1, \dots, c_m . Then there exists $\mu \in \mathbf{M}(\mathfrak{S})$ such that $P(\mu) = 0$ and $\mathfrak{F}_i = \{\gamma \in \Gamma | \hat{\mu}(\gamma) = c_i\}$.*

PROOF. By Lemma 4 the set \mathfrak{T} of all ordered m -partitions which arise from some μ such that $P(\mu) = 0$ is an algebra. By Lemma 3, \mathfrak{T} contains all the partitions of Type I. Thus \mathfrak{T} contains the algebra generated by the partitions of Type I.

Our next test is to show that \mathfrak{T} is the algebra generated by the partitions of Type I.

LEMMA 6. *If $P(\mu) = 0$ there exist $c_i \in C$, and idempotent $\mu_i \in \mathbf{M}(\mathfrak{S})$, $1 \leq i \leq m$ such that*

$$\mu = \sum_{i=1}^m c_i \mu_i.$$

PROOF. Let c_i be the distinct roots of P in C . There exist polynomials P_i over C such that $P_i(c_j) = \delta_{ij}$. Let $\mu_i = P_i(\mu)$. Clearly $\hat{\mu}_i(\gamma) = 1$ if $\hat{\mu}(\gamma) = c_i$ and $\hat{\mu}_i(\gamma) = 0$ otherwise. Thus every μ_i is idempotent. It is also clear that $\mu = \sum_{i=1}^m c_i \mu_i$.

We shall now prove the theorem stated in Section 4.

PROOF. Let $P(\mu) = 0$. By Lemma 6 there exist $c_i \in C$ and μ_i idempotent, $1 \leq i \leq m$, such that $\mu = \sum_{i=1}^m c_i \mu_i$.

Each idempotent μ_i induces an m -partition \mathfrak{F}_i of Γ with

$$\begin{aligned} \mathfrak{F}_{i1} &= \hat{\mu}_i^{-1}(0) \\ \mathfrak{F}_{i2} &= \hat{\mu}_i^{-1}(1) \\ \mathfrak{F}_{ij} &= \emptyset \quad \text{otherwise.} \end{aligned}$$

Then in the partition \mathfrak{F}^* of Γ induced by μ , \mathfrak{F}_j^* is exactly the union of all intersections of form $\mathfrak{F}_{1i_1} \cap \mathfrak{F}_{2i_2} \cap \dots \cap \mathfrak{F}_{mi_m}$ ($i_k = 1$ or 2) such that $\sum_1^m c_k \delta_{1, i_k-1} = c_j$.

By Cohen's Theorem each P_{ki_k} lies in the ring of sets generated by the open cosets in Γ . In our terminology each partition \mathfrak{F}_i is primitively generated by Type I partitions. But the partition \mathfrak{F}^* is primitively generated by the partitions \mathfrak{F}_i . Thus \mathfrak{F}^* lies in the algebra generated by the Type I partitions.

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