Rendiconti del Seminario Matematico della Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 50 (1973), p. 119-125

<http://www.numdam.org/item?id=RSMUP_1973__50__119_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 50 (1973)

A Characterization of Algebraic Measures.

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1. Let $\mathcal{G}$ be a locally compact Hausdorff abelian topological group. Let $M(\mathcal{G})$ be the set of complex measures μ on the Borel sets of $\mathcal{G}$ such that $\|\mu\|$ (defined in the usual way, *vide* [2]) is finite. $M(\mathcal{G})$ is an algebra over the complex numbers C with the convolution operation $*$ (*vide* [4]) as multiplication on $M(\mathcal{G})$.

Cohen [1, 4] completely determined the measures for which

$$\mu * \mu = \mu .$$

Such measures are called *idempotent.* The problem considered and solved in the paper is the characterization of all μ that satisfy an algebraic equation.

More presisely, define

$$\mu^0 = \delta$$
$$\mu^n = \mu * \mu^{n-1}, \qquad n \geqslant 1$$

where δ is the unit element of $M(\mathcal{G})$, picturesquely described as « unit mass concentrated at the origin ». A complete characterization is given of those measures μ for which there exists a set (in general dependent

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The author gratefully acknowledges the financial support of the National Research Council of Canada, which gave the author a Post-Doctoral Fellowship to Catania. The provision of generous facilities by the Seminario Matematico of the Università di Catania was greatly appreciated.

on μ and not unique) of complex numbers z_i, $0 \leqslant i \leqslant n$, $z_n \neq 0$, such that

$$\sum_{i=0}^{n} z_i \mu^i = 0 .$$

Such measures are called *algebraic*. They were first considered Istratescu, who proved that the carrier-group of an algebraic measure is compact [3].

The main result of this paper is the theorem in Section 4 that characterizes an algebraic measure as one such that the partition induced on the dual group by the Fourier-Stieltjes transform of the measure is generated by cosets of the dual group.

2. Let Γ be the dual group of $\mathcal{G}$. Let

$$\hat{\mu}\colon \Gamma \to C$$

be the Fourier-Stieltjes transform of μ [4]. Let P be the formal polynomial

$$P(X) = \sum_{i=0}^{n} c_i X^i$$

where the c_i's are complex numbers.

$$P(\hat{\mu}(\gamma)) = (P(\hat{\mu}))(\gamma) = \\ = (P(\mu))\hat{}\,(\gamma) = 0 .$$

Thus $\hat{\mu}(\gamma)$ must be a root of P in C.

Conversely if $\hat{\mu}(\gamma)$ is always one of the complex roots of P, for all $\gamma \in \Gamma$, then $P(\hat{\mu})$ vanishes identically on Γ and the uniqueness theorem for Fourier-Stieltjes transforms [4] shows that $P(\mu) = 0$. Since the functions $C \to C$ with finite images are exactly the functions that can be written as polynomials it follows that the algebraic measures are exactly the measures μ such that the image of $\hat{\mu}$ is finite. It is now clear that the sum of algebraic measures is algebraic and so is the product.

3. For our purposes a partition of a set A is any set of pairwise disjoint subsets of A whose union is A. In particular partitions are allowed to have empty members. Let C be given some linear ordering which will be kept fixed. Then an injective mapping f from algebraic measures to ordered partitions is given by

$$f(\mu) = \langle \{\gamma | \hat{\mu}(\gamma) = z\} | z \in C \rangle .$$

The main result of this paper is the explicit description of the image of f. Clearly any partition may be replaced by an equivalent one by throwing away some or all of the empty members. If the image of $\hat{\mu}$ is a subset of a set K we write

$$f_k(\mu) = \langle \{\gamma | \hat{\mu}(\gamma) = z\} | z \in K \rangle .$$

Recall that a ring of sets is a set of sets stable under the formation of complements and finite unions (and hence under the formation of finite intersections).

In the case of idempotent measures the polynomial involved, *viz.* $P(X) = X^2 - X$ has only two distinct roots in C, 0 and 1. Thus the idempotent measures can be described by considering one member of the partition induced by the polynomial. Let

$$S(\mu) = \{\gamma \in \Gamma | \hat{\mu}(\gamma) = 1\}$$

for idempotent μ. Cohen showed that a subset Λ of Γ has the form $S(\mu)$ for some idempotent μ if and only if Λ lies in the ring of sets generated by the cosets of open subgroups of Γ.

Let A be a set. An ordered m-partition of A is an ordered m-tuple of pairwise disjoint sets whose union is A. Let $\mathcal{S} = \{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n\}$ be a finite set of m-partitions of A. Let $\mathcal{T}_{ij}$ be the j-th member of $\mathcal{T}_i$: The ordered m-partition $\mathcal{T}$ is *primitively generated* by $\mathcal{S}$ if every finite intersection of form $\bigcap_{i=1}^{n} \mathcal{T}_{ij_i}$ is a subset of some member of $\mathcal{T}$. Hence every member of $\mathcal{T}$ must be the union of intersections of this form. Let $\mathcal{C}$ be a set of ordered m-partitions of A. $\mathcal{C}$ is an m-partition algebra if $\mathcal{C}$ contains every ordered m-partition primitively generated by a finite subset of $\mathcal{C}$. It is clear that the intersection of a collection of m-partition algebras is itself an m-partition algebra. Thus there is a smallest m-partition algebra containing any set of ordered m-partitions. This algebra is called the m-partition algebra *generated* by the given set of partitions.

A *Type I* partition of Γ is a finite ordered partition of Γ in which at most two members are non-empty and in which one member is a coset of an open subgroup of Γ.

4. We shall now state the theorem that characterizes algebraic measures.

THEOREM. *Let P be a polynomial over C with distinct roots* $\{c_1, c_2, \ldots, c_m\} = K$ in C. *Let $\mu \in M(\mathcal{G})$. $P(\mu) = 0$ if and only if: (a) for all $\gamma \in \Gamma$, $\hat{\mu}(\gamma) \in K$ and (b) $f_k(\mu)$ belongs to the m-partition algebra generated by the ordered m-partitions of Type I.*

First let us look at Cohen's result in our terminology. The ring of sets generated by a set of sets $\mathcal{U}$ is precisely the set of sets B that can be written as finite unions of finite intersections

$$B = \cup \cap B_{ij}$$

where every B_{ij} or its complement B'_{ij} belongs to $\mathcal{U}$. Thus the ordered partition $\langle B, B' \rangle$ is primitively generated by the set of ordered partitions of form $\langle B_{ij}, B'_{ij} \rangle$. The ring of sets generated by the cosets of open subgroups of Γ thus corresponds to the 2-partition algebra generated by Type I ordered 2-partitions. Cohen's Theorem is seen to be a special case of our theorem.

The remainder of this paper will be devoted to a proof of our theorem above. We shall prove six lemmas and then the theorem.

5. LEMMA 1. Let $P(\mu) = 0$, $\mu \in M(\mathcal{G})$. *Let $c_1, \ldots, c_m$ be the distinct complex roots of P. Let $k_1, \ldots, k_m$ be complex numbers, not necessarily distinct. There exists $\nu \in M(\mathcal{G})$ such that $\hat{\mu}(\gamma) = c_i$ if and only if $\hat{\nu}(\gamma) = k_i$.*

PROOF. There exists a polynomial $P_1(X)$ over C such that $P_1(c_i) = k_i$ for all i. $P_1(\mu) \in M(\mathcal{G})$. $P_1(\mu)\hat{}(\gamma) = P_1(\hat{\mu}(\gamma))$. Thus ν may be chosen as $P_1(\mu)$.

Two particular cases of this lemma are of special interest. The k_i may be chosen so that each is equal to some c_j. Thus if an algebraic measure μ is given, which induces an ordered partition $\mathcal{T}$ on C and if an ordered partition $\mathcal{T}'$ is given such that every member of $\mathcal{T}'$ is equal to the union of members of $\mathcal{T}$, there exists a measure ν that induces $\mathcal{T}'$ on C. Another special case of the lemma is obtained when the c_i are all non-zero. Then a measure ν exists such that $\hat{\nu}(\gamma) = (\hat{\mu}(\gamma))^{-1}$ for all $\gamma \in \Gamma$. Then ν is the convolution inverse of μ. An algebraic measure μ is therefore invertible if and only if $\hat{\mu}(\gamma)$ is never 0.

LEMMA 2. *Let Λ be an open subgroup of Γ. Let c_1 and c_2 be two complex roots of P. There exists $\mu \in M(\mathcal{G})$ such that*

$$P(\mu) = 0 ; \quad \hat{\mu}(\gamma) = c_1 , \quad \gamma \in \Lambda ; \quad \hat{\mu}(\gamma) = c_2 , \quad \gamma \notin \Lambda .$$

PROOF. Let H be the annihilator of Λ. H is compact and isomorphic to the dual of Γ/Λ. Let m be Haar measure on H with $m(H) = 1$. m defines a measure m_1 in $M(\mathcal{G})$ by

$$m_1(B) = m(B \cap H)$$

for all Borel sets B in $\mathcal{G}$. $\hat{m}_1(\mu)$ is 1 if $\gamma \in \Lambda$ and 0 otherwise. The previous lemma shows the existence of a measure with the desired properties.

LEMMA 3. *Let Λ be an open subgroup of Γ. Let $\gamma_0 \in \Gamma$. Let c_1 and c_2 be two roots of P. There exists $\mu \in M(\mathcal{G})$ with $P(\mu) = 0$, $\hat{\mu}(\gamma) = c_1$ if $\gamma \in \gamma_0 + \Lambda$, $\hat{\mu}(\gamma) = c_2$ otherwise.*

PROOF. By the previous lemma there exists μ_1 such that:

$$\hat{\mu}_1(\gamma) = c_1 \text{ if } \gamma \in \Lambda$$

$$\hat{\mu}_1(\gamma) = c_2 \text{ otherwise} .$$

Let

$$d\mu = \gamma_0 d\mu_1$$

then

$$\hat{\mu}(\gamma) = \hat{\mu}_1(\gamma - \gamma_0)$$

$$\hat{\mu}(\gamma) = c_1 \quad \text{if } \gamma \in \gamma_0 + \Lambda$$

$$\hat{\mu}(\gamma) = c_2 \quad \text{if } \gamma \notin \gamma_0 + \Lambda .$$

LEMMA 4. *Let $P(\mu_i) = 0$, $\mu_i \in M(\mathcal{G})$, for $1 \leqslant i \leqslant n$. Let $\mathcal{T}_i$ be the ordered partition of Γ induced by μ_i. Let $\mathfrak{R}$ be an ordered partition of Γ primitively generated by the $\mathcal{T}_i$. There exists a measure $\mu_i \in M(\mathcal{G})$ whose induced partition is $\mathfrak{R}$.*

PROOF. Let $\mathcal{T}_{ij} = \{\gamma \in \Gamma | \hat{\mu}_i(\gamma) = c_j\}$.

To every intersection of form $\bigcap_{i=1}^{n} \mathcal{T}_{ij_i}$ we associate the point $(c_{j_1}, \ldots, c_{j_n})$. Thus partitions of Γ primitively generated by the $\mathcal{T}_i$

correspond to partitions of the finite set

$$\{(\hat{\mu}_1(\gamma), \ldots, \mu_n(\gamma)) | \gamma \in \Gamma\} = S .$$

Let S_i be the subset of S corresponding to $\mathfrak{R}_i$. Let Q be a polynomial in $X_1, \ldots, X_n, \bar{X}_n, \ldots, \bar{X}_1$ over C such that $Q(S_i) = c_i$ for all i. Then $Q(\mu_1, \ldots, \mu_n, \tilde{\mu}_1, \ldots, \tilde{\mu}_n)(\gamma) = Q(\mu_1(\gamma), \ldots, \mu_n(\gamma), \bar{\mu}_1(\gamma), \ldots, \bar{\mu}_n(\gamma))$. Thus $\mu = Q(\mu_1, \ldots, \tilde{\mu}_1, \ldots, \tilde{\mu}_n)$ is the desired measure.

LEMMA 5. *Let $\mathfrak{T}$ be a member of the algebra of ordered m-partitions generated by the set of all m-partitions of Type I. Let $\mathfrak{T}$ be a polynomial with distinct complex roots $c_1, \ldots, c_m$. Then there exists $\mu \in M(\mathfrak{G})$ such that $P(\mu) = 0$ and $\mathfrak{T}_i = \{\gamma \in \Gamma | \hat{\mu}(\gamma) = c_i\}$.*

PROOF. By Lemma 4 the set $\mathfrak{C}$ of all ordered m-partitions which arise from some μ such that $P(\mu) = 0$ is an algebra. By Lemma 3, $\mathfrak{C}$ contains all the partitions of Type I. Thus $\mathfrak{C}$ contains the algebra generated by the partitions of Type I.

Our next test is to show that $\mathfrak{C}$ is the algebra generated by the partitions of Type *I*.

LEMMA 6. *If $P(\mu) = 0$ there exist $c_i \in C$, and idempotent $\mu_i \in M(\mathfrak{G})$, $1 \leqslant i \leqslant m$ such that*

$$\mu = \sum_{i=1}^{m} c_i \mu_i .$$

PROOF. Let c_i be the distinct roots of P in C. There exist polynomials P_i over C such that $P_i(c_j) = \delta_{ij}$. Let $\mu_i = P_i(\mu)$. Clearly $\hat{\mu}_i(\gamma) = 1$ if $\hat{\mu}(\gamma) = c_i$ and $\hat{\mu}_i(\gamma) = 0$ otherwise. Thus every μ_i is idempotent. It is also clear that $\mu = \sum_{i=1}^{m} c_i \mu_i$.

We shall now prove the theorem stated in Section 4.

PROOF. Let $P(\mu) = 0$. By Lemma 6 there exist $c_i \in C$ and μ_i idempotent, $1 \leqslant i \leqslant m$, such that $\mu = \sum_{i=1}^{m} c_i \mu_i$.

Each idempotent μ_i induces an m-partition $\mathfrak{T}_i$ of Γ with

$$\mathfrak{T}_{i1} = \hat{\mu}_i^{-1}(0)$$

$$\mathfrak{T}_{i2} = \hat{\mu}_i^{-1}(1)$$

$$\mathfrak{T}_{ij} = \emptyset \qquad \text{otherwise} .$$

Then in the partition $\mathcal{T}^*$ of Γ induced by μ, $\mathcal{T}_j^*$ is exactly the union of all intersections of form $\mathcal{T}_{1i_1} \cap \mathcal{T}_{2i_2} \cap \ldots \cap \mathcal{T}_{mi_m}$ ($i_k = 1$ or 2) such that $\sum_1^m {}_k c_k \delta_{1,i_k-1} = c_j$.

By Cohen's Theorem each P_{ki_k} lies in the ring of sets generated by the open cosets in Γ. In our terminology each partition $\mathcal{T}_i$ is primitively generated by Type I partitions. But the partition $\mathcal{T}^*$ is primitively generated by the partitions $\mathcal{T}_i$. Thus $\mathcal{T}^*$ lies in the algebra generated by the Type I partitions.

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Manoscritto pervenuto in redazione il 23 giugno 1972.