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Binary Multiples of Combinatorial Geometries - II (*).

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1. Introduction.

In [2] we considered the sequence of multiples $G, 2G, ..., mG$ of a projective geometry, where m is the smallest integer such that mG is a Boolean algebra. We showed that the only multiples that were binary were possibly G itself, mG, and $(m-1)G$ if $(n^{q-1}+n^{q-2}+...+n)/q$ is an integer (where G has order n and rank q). In this paper we prove similar results about affine geometries. All definitions not given here can be found in [2] and we will assume the reader is familiar with that paper.

2. Affine geometries.

Given an affine geometry on the set of points S, the lattice of flats (points, lines, etc.) is a geometric lattice, and thus defines a combinatorial geometry. We will refer to these combinatorial geometries as *affine geometries*. Recall that if f is a submodular function on a set S, then it defines a pregeometry on S as follows: $A \subseteq S$ is independent if and only if $f(A') \geqslant |A'|$ for all non-empty $A' \subseteq A$. [1, Prop. 7.3]. (Here $|A'|$ denotes the cardinality of A'.) If $G(S)$ is a combinatorial geometry with rank function r, and k is a positive integer, then kr is submodular and thus defines a new combinatorial geometry, de-

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noted kG. If G is an affine geometry, when are the multiples of G binary geometries? We show that kG is binary only when $k=1$ and G is binary, or when kG is a Boolean truncation of rank one less than the cardinality of S, or when kG is a Boolean algebra. We prove this result first for affine planes and then use induction for the general case. We will make use of the following well-known results on affine geometries.

THEOREM 1. *Suppose $G(S)$ is an affine geometry. There exists an integer $n\geqslant 2$ such that:*

(i) *Every flat of rank q contains exactly n^{q-1} points;*

(ii) *Every coline of G is covered by exactly $n+1$ copoints;*

(iii) *There is a set of n copoints of G which partition S.*

The number n is the order of G. Notice that the rank of a flat is one more than its dimension in the affine geometry. (Thus an affine plane has rank 3.)

THEOREM 2. *If G is an affine geometry of order n and rank $q\geqslant 4$, and C is a copoint of G, then the subgeometry of G on the set C is an affine geometry of order n and rank $q-1$.*

We now prove the theorem for affine planes.

THEOREM 3. *Let $G(S)$ be an affine plane of order n. kG is binary if and only if $k\geqslant(n^2-1)/3$.*

PROOF. We first consider the case $n=2$. By Theorem 1 and [2; Theorem 1], this is the only case in which G is binary. kG, $k\geqslant 2$, is the Boolean algebra on S, and so kG is binary if and only if $k\geqslant(n^2-1)/3=1$. Now let $n\geqslant 3$. Since any line of G is an n-set (i.e., a set of cardinality n), kG is a Boolean truncation if $k\geqslant n/2$. By [2; Lemma 1] kG is binary if and only if every set of $|S|-1=$ $=n^2-1$ elements is k-independent. This is true if and only if $k\geqslant(n^2-1)/3$. Suppose $1<k<n/2$. If $A\subseteq S$ is a $3k$-set no $(2k+1)$-subset of which is contained in a line of G, then A is a basis for kG. We construct a $(3k+2)$-set, every $3k$-subset of which is a basis for kG. By [2; Lemma 2] and [2; Theorem 1], this will show kG is not binary. Let C_1, C_2 be two non-intersecting lines of G. Let A consist of any $2k$ elements of C_1 and any $k+2$ elements of C_2. This is possible as $k+2\leqslant 2k<n$. To show any $3k$-subset of A is a basis for kG, let $B\subseteq A$ be a $(2k+1)$-set. We must show that B is not contained in any line of G. If B were contained in a line of G, then $|B\cap C_1|\leqslant 1$.

For if $|B \cap C_1| > 1$, we would have two lines intersecting in more than one point. Similarly, $|B \cap C_2| \leqslant 1$. This means $|B| \leqslant 2$. But $2k+1 \geqslant 3$, and so it is impossible for B to be contained in a line. This completes the proof.

LEMMA 1. *If n and q are positive integers such that $n \geqslant 2$ and $q \geqslant 4$, then*

$$\frac{n^{q-2}-1}{q-1} \geqslant \frac{n^{q-3}}{q-2}$$

and

$$\frac{n^{q-2}}{q-1} > \frac{n^{q-3}}{q-2}.$$

THEOREM 4. *Let $G(S)$ be an affine geometry of order n and rank q. kG is binary if and only if $k \geqslant (n^{q-1}-1)/q$ or both $k=1$ and $n=2$.*

PROOF. G is binary of and only if $n=2$, thus we may assume $k>1$. Because of the arithmetic details, we first note that the theorem holds where $n=2$, $q=4$. It is easy to see that kG, $k \geqslant 2$, is a Boolean algebra; and so kG is binary if and only if $k \geqslant (n^{q-1}-1)/q = 1\frac{3}{4}$ or $k=1$. We now prove the theorem by induction on q. If $n \neq 2$, $q=3$ is the first step of the induction. The result is then that of Theorem 3. If $n=2$, then the case of $q=3$ is given by Theorem 3, and we let $q=4$ be the first step of the induction. Assume the result holds for affine geometries of order n and rank $q-1$, $q \geqslant 4$ ($q \geqslant 5$ if $n=2$). We show it holds for G. Let G' be the affine geometry of order n, rank $q-1$ on a copoint of G (Theorem 2). kG' is a subgeometry of kG by [2; Lemma 5] and so since kG is binary we conclude that kG' is binary. By the induction hypothesis, $k \geqslant (n^{q-2}-1)/q-1$. Now if $k \geqslant n^{q-2}/(q-1)$, then every set of n^{q-2} elements is k-independent. For if $|A| = n^{q-2}$, then $r(A) \geqslant q-1$, and so $kr(A) \geqslant k(q-1) \geqslant |A|$. If $B \subseteq A$, $kr(B) \geqslant |B|$, for

$$\frac{|B|}{r(B)} \leqslant \frac{|\bar{B}|}{r(\bar{B})} \leqslant \frac{n^m}{m+1} \leqslant \frac{n^{q-2}}{q-1} \leqslant k$$

(for $m = r(\bar{B})-1$) by Lemma 1. This means copoints of G are k-independent (i.e., independent in kG), so kG is a Boolean truncation. Thus, by [2; Lemma 1], kG is binary if and only if every $(n^{q-1}-1)$-set is k-independent, that is, if and only if $k \geqslant (n^{q-1}-1)/q$. The only case left to consider is $k = (n^{q-2}-1)/(q-1)$. In this case, $k > n^{q-3}/(q-2)$ by Lemma 1, and so by an argument like that above, any (n^{q-3})-set is k-independent. In fact, any $[(q-1)k]$-set is k-independent; for if A is a subset of such a set, and if $n^{q-3} < |A| \leqslant k(q-1)$, then $r(A) \geqslant q-1$ and so $kr(A) \geqslant k(q-1) = n^{q-2} - 1 \geqslant |A|$. A copoint (which is a $[(q-1)k+1]$-set) however is not k-independent. We conclude that if A is a qk-set no $[(q-1)k+1]$-subset of which is contained in a copoint of G, then A is a basis for kG. We construct a $(qk+2)$-set every qk-subset of which is a basis for kG. This, by [2; Theorem 1 and Lemma 2], will show that kG is not binary. Let C_1, C_2 be any two non-intersecting copoints of G (Theorem 1). Let A consist of any $(q-1)k$ points of C_1 and any $k+2$ points of C_2. This is possible, as $k+2 \leqslant (q-1)k < n^{q-2}$. $|A| = qk+2$. To show every qk-subset of A is a basis of kG, let $B \subseteq A$ be a $[(q-1)k+1]$-set. We show B is contained in no copoint of G. If B is contained in a copoint, then $|B \cap C_1| \leqslant n^{q-3}$. Otherwise we would have two copoints which intersect in more than n^{q-3} points. This is impossible by Theorem 1. Similarly $|B \cap C_2| \leqslant n^{q-3}$. Thus $|B| \leqslant 2n^{q-3} < n^{q-2}$ if $n \neq 2$. Thus it is impossible to find B unless $n = 2$. If $n = 2$, we must pick B so that $|B \cap C_1| = |B \cap C_2| = n^{q-3}$. But $|A \cap C_2| = k+2$, and it is easy to show $n^{q-3} > k+2$ for $n = 2$, $q \geqslant 5$. Since $B \cap C_2 \subseteq A \cap C_2$, this show it is impossible to find a B in this case also. Thus the desired set A does exist, and the proof is complete.

Let G be an affine geometry with sequence of multiples $G, 2G, \ldots, mG$, m the smallest integer such that mG is a Boolean algebra. mG is always binary. G is binary; its order is 2. Just as for projective geometries, we have shown that the only other multiple of an affine geometry that may be binary is $(m-1)G$. $(m-1)G$ is actually binary if and only if $k = (n^{q-1} - 1)/q$ is an integer. In this case $m - 1 = k$. If n and q are relatively prime and q is a prime, it is well-known that $n^{q-1} \equiv 1 \pmod{q}$. Thus in this case $(m-1)G$ is binary.

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