RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 49 (1973), p. 217-223

<http://www.numdam.org/item?id=RSMUP_1973__49__217_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 49 (1973)

Some Remarks on L^2-Valued Functions.

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Introduction.

Let us consider polynomials $P(s) = a_0 + a_1 s + \dots a_q s^q$ with real coefficients, and then, for arbitrarily given functions $\varphi(s) \in L^2(-\infty, \infty)$, consider the L^2-valued functions defined on $-\infty < t < +\infty$, through the formula:

$$\psi(s, t) = \exp[iP(s)t]\varphi(s) .$$

This class of functions arises naturally when we solve, through Fourier-Plancherel transform, the Cauchy problem for a class of partial differential equations of the form:

$$u_t(x, t) = \sum_{k=0}^{q} \alpha_k \frac{\partial^k u}{\partial x^k}(x, t) ,$$

where α_k are convenient complex numbers, and $u(x, t) \in L^2(R^1)$ for any real t.

Let us consider also polynomials as above, such that $P(s) \leqslant 0$ for any real s (henceforth, necessarily of even degree); thereafter, for a given function $F(s) \in L^2(-\infty, +\infty)$ and for arbitrary $\varphi(s) \in L^2(-\infty, \infty)$ consider the class of functions:

$$U(s, t) = \exp[tP(s)]\varphi(s) + \int_0^t \exp[P(s)(t-\sigma]F(s)\,d\sigma ,$$

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This research is supported through a grant of the National Research Council of Canada at the Université de Montréal.

defined for $t \geqslant 0$ and belonging to $L^2(-\infty, +\infty)$ as easily seen below.

This second class arises when we solve by the same method, partial differential equations of the form:

$$u_t(x, t) = \sum_{k=0}^{q} \beta_k \frac{\partial^k u}{\partial x^k}(x, t) + f(x)$$

where the β_k are again convenient complex numbers, $u(x, t)$ belongs to $L^2(-\infty, +\infty)$ for any $t \geqslant 0$, and $f(x)$ is given in $L^2(-\infty, +\infty)$.

Now, the inverse Fourier-transform of the functions $\psi(s, t)$ corresponding to polynomials of degree $q \geqslant 1$ will give us a class of L^2-valued, L^2-bounded solutions of partial differential equations with constant coefficients which are not L^2-almost-periodic (see the § 1 of our paper [2] and also the monograph [1] for the necessary definitions). On the other hand we shall see that, for polynomials $P(s) \leqslant 0$ with real coefficients and some real roots, one can choose $F(s)$ in order that $\lim_{t \to \infty} \| U(s, t) \|_{L^2} = \infty$. This generalizes to a larger extent the example of an L^2-unbounded solution for the inhomogeneous heat equation which is given in our paper [3]-§ 3.

§ 1. Let us consider, to begin, a slightly more general setting.

Let X be a Banach space, and $x = f(t)$ be a continuous function defined on $-\infty < t < +\infty$ with values in X. When t varies over the real line, the point $x = f(t)$ describes, in the X-space, a set which is called the range of $f(t)$ and is denoted by $\mathfrak{R}_f$. It is known (see [1], pag. 5) that if $f(t)$ is strongly almost-periodic, than $\mathfrak{R}_f$ is a relatively compact set in X; consequently, if $\mathfrak{R}_f$ is not relatively compact in X, the function $f(t)$ is not almost-periodic.

Let us consider now the complex Hilbert space $L^2(-\infty, +\infty)$ of square-integrable complex-valued functions $\varphi(s)$ defined for $-\infty < s < +\infty$, and for an arbitrary polynomial $P(s) = \sum_{j=0}^{q} a_j s^j$ with real coefficients, consider the L^2-valued function

$$(1.1) \qquad \psi(s, t) = \exp[iP(s)t]\varphi(s), \qquad -\infty < s < \infty, \qquad -\infty < t < \infty.$$

We see that the equality

$$(1.2) \qquad \int_{-\infty}^{\infty} |\psi(s, t)|^2 ds = \int_{-\infty}^{\infty} |\varphi(s)|^2 ds, \qquad -\infty < t < \infty$$

is verified and consequently the range $\mathfrak{R}_{\psi(\cdot,t)}$ is located on the sphere in L^2 with center the origin and radius $= \|\varphi\|_{L^2}$. Furthermore, using Lebesgue's theorem on dominated convergence in the expression

$$(1.3) \qquad \int_{-\infty}^{\infty} |\exp[i(t+\delta)P(s)]\varphi(s) - \exp[itP(s)]\varphi(s)|^2 ds$$

we get also that $\psi(s, t)$ is strongly continuous, $-\infty < t < \infty \to \to L^2(-\infty, \infty)$. Let us consider now the simplest case where the polynomial $P(s)$ has degree $0: P(s) \equiv a_0, \; -\infty < s < +\infty$.

Then $\psi(s, t) = \exp(ia_0 t)\varphi(s)$ which is a continuous periodic function, $-\infty < t < \infty \to L^2$; hence $\mathfrak{R}_\psi$ is relatively compact in L^2 (see [1], pag 14). Even more generally, if x belongs to the Banach space X, and $\lambda(t)$, $-\infty < t < +\infty \to C$ is a complex-valued bounded function, then the X-valued function $y(t) = \lambda(t)x$ has relatively compact range.

§ 2. We shall give below the proof of the following.

THEOREM 1. *Let* $P(s) = a_0 + a_1 s + \ldots a_q s^q$, $a_q \neq 0$, $q \geqslant 1$ *be a polynomial with real coefficients, and let* $\varphi_A(s) = 1$ *for* $A \leqslant s \leqslant A + 1$, *and* $\varphi_A(s) = 0$ *for other real* s, *where* A *is a large enough number. Then, for at least a sequence of real numbers* $(t_n)_1^\infty$, *the* $L^2(-\infty, +\infty)$*-valued sequence*: $\{\exp[iP(s)t_n]\varphi_A(s)\}_{n=1}^\infty$ *is not relatively compact in* L^2.

PROOF. Let us remember the identity: $|\exp[i\lambda_1] - \exp[i\lambda_2]|^2 = = 2 - 2\cos(\lambda_1 - \lambda_2)$. Then, for an arbitrary polynomial $Q(s)$ with real coefficients and for any $\varphi(s) \in L^2(-\infty, +\infty)$, we have, for any pair of real numbers t_1, t_2, the relation

$$(2.1) \qquad \int_{-\infty}^{\infty} |\exp[iQ(s)t_1] - \exp[iQ(s)t_2|^2 |\varphi(s)|^2 ds =$$
$$= 2\int_{-\infty}^{\infty} |\varphi(s)|^2 ds - 2\int_{-\infty}^{\infty} \cos[Q(s)(t_1 - t_2)] |\varphi(s)|^2 ds .$$

Let us consider now our given polynomial $P(s)$, of degree $q \geqslant 1$. Its derivative $P'(s)$ is a polynomial of degree $q - 1 \geqslant 0$, hence it has a constant sign (sign of a_q) for large enough s (say, for $s \geqslant s_0$). Hence, the polynomial $P(s)$ is a strictly monotonical function for $s \geqslant s_0$. Let us take now $A > s_0$ and we get from 2.1) the relation

$$(2.2)\qquad \int_{-\infty}^{\infty} |\exp[iP(s)t_1] - \exp[iP(s)t_2]|^2 |\varphi_A(s)|^2 ds = = 2 - 2\int_A^{A+1} \cos[P(s)(t_1 - t_2)]\, ds .$$

Let us consider now the monotonical function $\sigma = P(s)$, $s \geqslant A > s_0$. This will have a regular inverse, $s = P^{-1}(\sigma) = R(\sigma)$, where

$$R'(\sigma) = \frac{1}{P'(R(\sigma))}$$

and

$$R''(\sigma) = -\frac{1}{P'^3(R(\sigma))} P''(R(\sigma)) .$$

We can effect the substitution $P(s) = \sigma$, and obtain the relation

$$(2.3)\qquad \int_A^{A+1} \cos[P(s)(t_1 - t_2)]\, ds = \int_{P(A)}^{P(A+1)} \cos[\sigma(t_1 - t_2)] R'(\sigma)\, d\sigma .$$

In the last integral we use an integration by parts and obtain

$$(2.4)\qquad \int_{P(A)}^{P(A+1)} \cos[\sigma(t_1 - t_2)] R'(\sigma)\, d\sigma =$$

$$= \frac{1}{t_1 - t_2} \{R'(P(A+1)) \sin[P(A+1(t_1 - t_2)] -$$

$$- R'(P(A)) \sin[P(A)(t_1 - t_2)]\} - \frac{1}{t_1 - t_2} \int_{P(A)}^{P(A+1)} \sin[\sigma(t_1 - t_2)] R''(\sigma)\, d\sigma .$$

We can estimate henceforth as follows:

$$(2.5)\qquad \left| \int_{P(A)}^{P(A+1)} \cos[\sigma(t_1 - t_2)] R'(\sigma)\, d\sigma \right| \leqslant$$

$$\leqslant \frac{1}{|t_1 - t_2|} [2C_A + \sup_{A \leqslant s \leqslant A+1} |R''(\sigma)|\, |P(A+1) - P(A)|] \leqslant L_A(|t_1 - t_2|)^{-1}$$

where L_A is a positive constant. We have consequently the estimate

$$(2.6)\qquad \int_{-\infty}^{\infty} |\exp[iP(s)t_1] - \exp[iP(s)t_2]|^2 |\varphi_A(s)|^2 ds \geqslant 2 - 2L_A(|t_1 - t_2|)^{-1}.$$

(Remark [1] that for $q = 1$ we can take for A an arbitrary real, and we can have $L_A = L = 2|a_1|^{-1}$).

Let us consider now the sequence $(t_n)_1^\infty$ where $t_p = 1 + 2 + \dots p$. We have, for $m \neq n$, the inequality $|t_m - t_n| \geqslant \max(m, n)$ and consequently

$$(2.7)\qquad \|[\exp[iP(s)t_n] - \exp[iP(s)t_m]]\varphi_A(s)\|_{L^2}^2 \geqslant 2 - 2L_A(\max(m, n))^{-1} \geqslant 1$$

for $\max(m, n) \geqslant n_0$.

This is contrary to relative compactness in $L^2(-\infty, +\infty)$ of the sequence

$$(2.8)\qquad \{\exp[iP(s)t_n]\varphi_A(s)\}_{n=n_0}^{\infty},$$

which proves the theorem.

§ 3. Let us consider in this § a polynomial $P(s) = \sum_{j=0}^{q} a_j s^j$ with real coefficients, and let us assume that $P(s) \leqslant 0$ for any real s.

Take then $F(s) \in L^2(-\infty, \infty)$ and consider the class Q_F of functions $U(s, t)$ of the form

$$(3.1)\qquad U(s, t) = \exp[tP(s)]\varphi(s) + \int_0^t \exp[P(s)(t-\sigma)]F(s)\,d\sigma$$

where $\varphi(s)$ is an arbitrary function in $L^2(-\infty, +\infty)$ (the particular case of $P(s) = -s^2$ was considered in our paper [3]). We consider the following problem: When we have

$$(3.2)\qquad \lim_{t\to\infty} \|U(s, t)\|_{L^2} = +\infty?$$

But we see that $\|\exp[tP(s)]\varphi(s)\|_{L^2} \leqslant \|\varphi\|_{L^2}$, as $t \geqslant 0$ and $P(s) \leqslant 0$. Hence (3.2) holds if and only if

$$(3.4)\qquad \lim_{t\to\infty} \left\| \int_0^t \exp[P(s)(t-\sigma)]F(s)\,d\sigma \right\|_{L^2} = +\infty.$$

(1) Using a more direct computation, one gets value of second integral in (2.3) without use of partial integration.

Actually we see that

$$(3.5)\qquad F(s)\int_0^t \exp[P(s)(t-\sigma)]\,d\sigma = F(s)(P(s))^{-1}\ (\exp[P(s)t]-1)$$

when $P(s)<0$ and $= tF(s)$, when $P(s)=0$.

But $P(s)=0$ in a finite number of s_j only; furthermore our function is continuous of s in these points because $\lim_{s\to s_j}(P(s))^{-1}\cdot$ $\cdot(\exp[tP(s)]-1)=t$; hence it is continuous on the real axis.

On the other hand we have the estimate $|(P(s))^{-1}|\exp[tP(s)]-1|\leqslant t$, $t\geqslant 0$ for any real s; it follows that the function (3.5) belongs to $L^2(-\infty,\infty)$ for $t\geqslant 0$, and

$$\left\|\int_0^t \exp[P(s)(t-\sigma)]F(s)\,d\sigma\right\|_{L^2}\leqslant t\|F\|_{L^2}.$$

We can give now the following.

THEOREM 2. *Let $P(s)\leqslant 0$ be a polynomial with real coefficients and let us assume that it has at least one real root s_0. Take $F(s)=1$ for $\bar{s}\leqslant s\leqslant s_0$, $F(s)=0$ for other s, where $\bar{s}$ is «near» to s_0. Then, all the functions* (3.1) *are L^2-unbounded as $t\to\infty$.*

PROOF. In view of the above remarks it is enough to consider the (Lebesgue) integral (for $t\geqslant 0$)

$$(3.6)\qquad I_t=\int_{\bar{s}}^{s_0}\frac{1}{P^2(s)}(\exp[P(s)t]-1)^2\,ds$$

for a certain $\bar{s}<s_0$ and near to s_0, and we shall see that it tends to ∞ as $t\to\infty$.

Remark that being $P(s)\leqslant 0$, $P(s)$ will have a local maximum for $s=s_0$; hence $P'(s_0)=0$ too. Furthermore, for $s<s_0$ near to s_0, $P'(s)>0$; if s_1 is the first zero for $P'(s)$ left of s_0, we get, say, $P'(s)>0$ strictly for $\bar{s}\leqslant s<s_0$ where $\bar{s}>s_1$. Hence $P(s)$ is strictly increasing on the interval $\bar{s}\leqslant s\leqslant s_0$. Let also $0<M=\sup_{\bar{s}\leqslant s\leqslant s_0}P'(s)$; hence we have $0<P'(s)\leqslant M$ for $\bar{s}\leqslant s\leqslant s_0$, and we get

$$(3.7)\qquad (P'(s))^{-1}\geqslant M^{-1}>0,\qquad \bar{s}\leqslant s\leqslant s_0.$$

Now, in the integral (3.6) we shall effectuate the substitution: $\sigma = P(s)$, $s = P^{-1}(\sigma) = R(\sigma)$; here $\bar{s} \leqslant s \leqslant s_0$ and $P(\bar{s}) \leqslant \sigma \leqslant 0$; hence $R'(\sigma) = (P'(s))^{-1} \geqslant M^{-1}$ for $P(\bar{s}) \leqslant \sigma \leqslant 0$. We obtain this way

$$(3.8) \qquad I_t = \int_{P(\bar{s})}^{0} \frac{1}{\sigma^2} (\exp(\sigma t) - 1)^2 R'(\sigma)\, d\sigma \geqslant \frac{1}{M} \int_{P(\bar{s})}^{0} \frac{1}{\sigma^2} (\exp(\sigma t) - 1)^2\, d\sigma ;$$

here we effectuate again the substitution $\sigma t = \xi$; hence $P(\bar{s})t \leqslant \xi \leqslant 0$, $t > 0$ and we have

$$(3.9) \qquad I_t \geqslant M^{-1} \int_{P(\bar{s})t}^{0} \frac{t^2}{\xi^2} (e^\xi - 1)^2 \frac{d\xi}{t} = \frac{t}{M} \int_{P(\bar{s})t}^{0} \frac{(e^\xi - 1)^2}{\xi^2}\, d\xi \geqslant \frac{t}{M} \int_{-1}^{0} \frac{(e^\xi - 1)^2}{\xi^2}\, d\xi \quad \text{for } t \geqslant t_0$$

which proves the theorem.

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Manoscritto pervenuto in redazione il 1° settembre 1972.