RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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The interpreted type-free modal calculus MC^{∞}

Rendiconti del Seminario Matematico della Università di Padova, tome 49 (1973), p. 157-194

<http://www.numdam.org/item?id=RSMUP_1973__49__157_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 49 (1973)

The Interpreted Type-Free Modal Calculus MC^∞.

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PART 1

The Type Free Extensional Calculus EC^∞ Involving Individuals, and the Interpreted Language ML^∞ on which MC^∞ is Based.

CHAPTER 1

THE CALCULUS EC^∞

1. Introduction (**).

First we describe quickly the whole work. Then we give a more detailed account of the content of Part 1. The analogue for the other parts of the work is done in the respective introductory sections.

In the present work an interpreted type-free modal calculus, MC^∞, whith identity and descriptions is constructed. It is both a type free analogue of the modal interpreted calculus MC^ν which has types of all finite levels and is constructed in [GIMC], i.e. [1], and a modal analogue

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Lavoro eseguito nell'ambito dell'attività dei gruppi di ricerca matematica del C.N.R.

(**) This publication was worked out in the sphere of activity of the groups for mathematical research of the Consiglio Nazionale delle Ricerche in the academic year 1971-72.

of an extension to the case where individuals exist, of the extensional type-free calculus NBGM according to Neumann, Bernays, Gödel, and Morse, on which [IST], i.e. [4], is based (1). The extension referred to above is the extensional interpreted calculus EC^∞ to be considered in Chap. 1 as a step towards the construction of MC^∞, in that the analogues for MC^∞ of many theorems in [IST] are direct analogues of only the corresponding theorems in EC^∞; furthermore, to state the transfinite semantical rules for the language ML^∞ on which MC^∞ is based, we need the use of an extension of the set theory in [IST] such as EC^∞.

As to the calculus MC^ν of which MC^∞ is a generalization, it is based on a type system containing types for properties, relations, and functions of all finite levels. Function types can be eliminated by defining functions to be certain modally constant relations—cf. [GIMC, n. 14]. The problem of reducing relations to properties in ML^∞ can be solved in a way similar to some procedures that are well known in extensional logic. An aim reached by MC^∞ is the complete elimination of types and the achievement of the possibility (lacking in MC^ν) of dealing with sets whose elements have different and even transfinite ranks. (MC^ν or in particular MC^1 can be considered as a subtheory of MC^∞).

In Part 1, after introducing EC^∞, we state semantical rules for ML^∞, that assign designators quasi intensions, briefly QIs, and are based on elementary possible cases, briefly Γ-cases, and other entities. In Part 2 we state the axioms of MC^∞ (obviously valid in ML^∞), and some basic theorems on classes and sets. Some among them are briefly hinted at because they are similar with their analogues in [IST], up to some changes of certain standard types; other theorems are essentially modal and in part can be derived by means of the analogues of Γ-cases defined within MC^∞ itself. These analogues can be dealt with quickly by a straightforward extension to MC^∞ of nn. 47-49 in [GIMC]. In Part 2 relations and functions in MC^∞ are also considered. Part 3 deals with ordinals, transfinite induction, ordinal arithmetics, and related topics.

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(1) As is customay, we call *set* any class that is an element of another class, and we call *proper class* a class that is not a set. [IST], i.e. [4], deals both with sets and proper classes, while e.g. the Zermelo-Skolem-Fraenkel theory, briefly ZFS, Mostowski [5], and Suppes [7] deal (axiomatically) with sets only.

One of the main aims of Parts 2 and 3 is to show that on the basis of certains modal preliminaries mostly stated in Part 2, the set theory presented in [IST] can be carried over to MC^{∞} rather simply by means of certain changes of standard kinds. This is true especially in connection with pure number theory where only (transfinite) numbers and classes of them are considered. Of course the definitions of some basic notions such as ordinal class (Ord) also have some important modal features. However after they have been stated in MC^{∞} several sections of [IST] practically need only be hinted at.

The essentially modal theorems in Part 2 are numerous and basilar for the whole work, while those in Part 3 are less frequent and in part are only important in limited fields. However several of them are important to apply pure number theory to arbitrary classes.

* * *

Now let us describe the content of Part 1 with more detail. In [5] Mostowski pointed out a simple way of extending Neumann's axioms for set theory to the case where individuals (i.e. non-classes) exist —cf. [3, p. 160]. However, as Monk remarks in [4, p, 14] on the basis of [7], the inclusion of individuals complicates the development of set theory considerably. Therefore in Chap. 1 we explicitly sketch the aforementioned extension EC^{∞} of NBGM to the case where individuals are present, and we show a way based on the introduction of certain restricted variables, by which the whole theory developed in [IST] can practically be extended to EC^{∞} in a rather straight-forward way [n. 5]. Only a few theorems on non-pure number theory differ from their correspondents in [IST] in such a way that explicit enunciates of them in EC^{∞} and the explicit presentation of some steps of their proofs are wanted. Among them are some theorems on universes [n. 6], and in particular Theor. 6.2 on the notion of a partial universe, which generalizes the one of universe.

Incidentally EC^{∞} also differs from [IST], the theory [5] of Mostowki, and Suppes's book [7] in that in EL^{∞} the description operator $\imath$ is included and identity is defined. EC^{∞} differs from [5] and [7] in that, as well as [IST], it deals (axiomatically) with proper classes. EC^{∞} differs from [IST] in that, as well as [7], it contains a primitive constant that expresses the empty set Λ and is useful to deal with individuals (2).

(2) More precisely two versions of EC^{∞} are considered in Chap. 1. In the first we need no additional symbol such as Mostowski's predicate of being

Our tratment of EC^{∞} [Chap. 1] differs from [7] in that, among other things, only in Chap. 1 certain topics such as strongly inaccessible cardinals and universes are considered. We mention them because they are affected by the assumption that individuals exist.

Let us add that in MC^{∞} identity cannot be defined unlike in EC^{∞}—cf. [GIMC].

According to Fraenkel, Skolem, and [IST], set theory must not include individuals, so that Λ is the only elementless entity (and it is an element). Such theories were well accepted by many logicians and mathematicians after Fraenkel criticized Zermelo's theory by its referring to entities whose origin is non-mathematical or even non-conceptual, in that this characteristic is unuseful for the construction of mathematics according to Fraenkel. We are aiming at a logical theory fit for axiomatization of physics or other unspecified natural sciences, where non-classes such as mass-points or cows are referred to, so that non-classes are also to be considered in our extensional semantical metalanguage. In spite of this we are doing pure logic and are using pure mathematics because our individuals are unspecified. In not accepting the aforementioned Fraenkel's criticism Zermelo proved to prefer a more general point of view on mathematics.

In Chap. 2 we introduce the modal type-free language ML^{∞} with identity and descriptions [n. 8] and we state transfinite semantical rules for it [n. 8]. The notions of *modally separated* and *absolute* n-ary attributes ($MSep_n$, Abs_n), introduced in [GIMC], appear basilar also for ML^{∞} and need some slight changes. The (new) notion of inner identity of n-tuples ($=_n$) [n. 12] also is basilar.

We have two kinds of functions, which become equivalent in connection with absolute classes, and in particular as far as pure number theory is concerned [n. 13]. Likewise we have two (main) notions of equipotence: intensional and extensional equipotence ($\approx$, $\approx^{(e)}$). The first is useful to carry over to MC^{∞} most theorems in [IST] on pure number theory, the second is basilar for applying this theory to arbitrary classes [n. 14].

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an individual—cf. [5]—in that $(\imath a)a \neq a$ is identified with Λ; the second (and main) version, as well as [7], contains Λ as a primitive constant.

Let us add that [7] will be followed in part to prove the $(\forall \mathscr{A})\mathscr{A} \notin \Lambda$ [Theor. 4.1] and that, among other things, Chap. 1 differs from [7] in that it includes explicit semantical rules for the language (EL^{∞}) on which the calculus being considered is based.

Lastly the notions of an ordinal class (*Ord*) and an ordinal number (Ord) are defined within ML^∞ [n. 14]; furthermore the semantical designation rules for EL^∞ are explicitly stated [n. 7].

2. The extensional type-free language EL^∞. Definition of $=$. First axioms.

The primitive symbols of EL^∞ are the variables $V_1, V_2, \ldots$, the constants $c_1, c_2, \ldots$ with $V_i \neq V_j \neq c_j \neq c_i$ for $i \neq j$, and the additional seven symbols «$\sim$», «$\wedge$», «$\forall$», «$\imath$», «$\in$», «(» and «)». Let us define the *designators*, or wffs of EL^∞, i.e. its *terms* and *matrices* recursively by the following conditions (1) to (4):

(1) *c_i and V_i are terms.*

(2) *If Δ and Δ_1 are terms, then $(\Delta \in \Delta_1)$ is matrix.*

(3) *If p and q are matrices, then such are $\sim(p)$, $(p \wedge q)$, and $(\forall V_i)(p)$.*

(4) *If p is a matrix, then $(\imath V_i)(p)$ is a term.*

We consider a set D_E of (extensional) individuals, and the sets of any transfinite rank α based on D_E, as well as in [IST] except that in that book D_E is assumed to be Λ. I think of EL^∞ as speaking about those among the entities above for which $0 \leqslant \alpha \leqslant \vartheta$, where ϑ is a given strongly inaccessible cardinal—cf. [IST, p. 159]. We have $\alpha = 0$ for individuals and $\alpha = \vartheta$ for proper classes for EL^∞.

We understand that $\vee$, $\supset$, $\equiv$, $\notin$ (is not), $\exists$ (there is), $\exists^{(1)}$ (there is at most one), and $\exists_1$ (there is exactly one) are defined in the usual way. Furthermore let $\sim$, $\forall$, $\exists$, $\wedge$, $\vee$, $\supset$, $\equiv$, and $\equiv_D$ have decreasing cohesive powers (as in [GIMC]).

We shall write e.g. «$\forall_x$» for «$(\forall x)$» and $(\exists x, y)$ or $\exists_{x,y}$ for «$(\exists x)(\exists y)$». Furthermore we shall use «$\mathscr{U}$», «$\mathscr{V}$», and «$\mathscr{W}$», possibly with subscripts, as unrestricted variables (i.e. $V_1, V_2, \ldots$).

Now let us begin the construction of the type-free extensional calculus EC^∞. We can use any set of axioms for propositional calculus and quantification. Let us accept the analogues for EL^∞ of axiom schemes 1 to 6 in [6, p. 212], i.e. those of AS 12.1-6 in [GIMC].

Syntactical notions and symbols, such as deduction and «$\vdash$» are understood to be defined for EC^∞ as in [6], i.e. as for EC^ν—cf. [GIMC,

n. 29]—so that modus ponens is the only inference rule in EC^∞ and for every axiom A, $(\forall V_i)A$ also is an axiom. The theorems of deduction, generalization, and rule C (which is the formal analogue of an act of choice) obviously hold for EL^∞.

We introduce metalinguistically the predicate El of being an element, its opposite $\overline{El}$ being the predicate of being a proper class (3).

$$\text{D2.1} \qquad (ML^\infty) \qquad \Delta \in El \equiv_D (\exists V_i)\, \Delta \in V_i$$

(where V_i is the first variable that does not occur free in Δ).

After the axioms for classes are laid down, it will appear useful to define the equality $\Delta = \Delta_1$ by the condition that Δ and Δ_1 should have the same elements [properties] in case either Δ or Δ_1 has an element [Δ and Δ_1 are elementless]:

$$\text{D2.2} \quad \Delta = \Delta_1 \equiv_D \exists_{\mathscr{A}}(\mathscr{A} \in \Delta \vee \mathscr{A} \in \Delta_1)\, \forall_{\mathscr{W}}(\mathscr{W} \in \Delta \equiv \mathscr{W} \in \Delta_1) \wedge$$
$$\wedge \sim\exists_{\mathscr{A}}(\mathscr{A} \in \Delta \vee \mathscr{A} \in \Delta_1)\, \forall_{\mathscr{W}}(\Delta \in \mathscr{W} \equiv \Delta_1 \in \mathscr{W})$$

where the variables $\mathscr{A}$ and $\mathscr{W}$ are chosen in the obvious way (they are the first variables that do not occur in Δ or Δ_1).

It is easy to deduce from DD2.1,2 that *identity* ($=$) *is reflexive, symmetric, and transitive.*

Following [IST] we identify *zero* with the empty set Λ. Since we want to take individuals into account, we consider two alternatives for Λ. Either Λ is a primitive notion, expressed in EL^∞ by the constant c_1, or it is identified with the « non-existing object »

$$\text{D2.3-5} \qquad (ML^\infty) \qquad a^* =_D (\imath V_1)\, V_1 \neq V_1\,, \qquad 0 =_D \Lambda \qquad [\Lambda =_D a^*]\,.$$

In both cases our axioms in n. 4 will yield the theorem

$$(2.1) \qquad (MC^\infty) \qquad \vdash (N)\, V_1 \notin \Lambda\,,$$

where, as well as in any formula, we understand that « ( ) » is any string of quantifiers, that its scope is the whole part of the formula at its

(3) E.g. the sign « (ML^∞) » in D2.1, DD3.2-6, or Convention 3.1 below means that D2.1, DD3.2-6, or Covention 3.1 is also understood to hold for the modal language ML^∞ to be introduced in n. 8. We shall use « (MC^∞) » with the obvious analogous aim.

own left, and that it includes the restricted universal quantifiers formed with the restricted variables in the formula.

In the present section we can consider (2.1) as an axiom for the sake of simplicity, or as a tacit assumption in theorems (2.2) and (3. 2-4, 8-10), only until Theor. 4.1 will be proved.

In the first alternative, i.e. when D2.5 is rejected, we accept e.g. the first of the axioms

A2.1 $$a^* \in El$$

or

A2.1′ $$(\ ) V_1 \notin a^* .$$

We consider this alternative as the main one because D2.5, together with D2.4 may turn out to cause confusion (4). We mentioned the second alternative [D2.5] only because it allows is to reduce the number of primitive constants.

The only axiom for ι—cf. A.12.18 in [GIMC]—is

A2.2 $$a)\quad (\ )(\exists_1 V_i)p \supset (\forall V_i)[p \equiv V_i = (\iota V_i)p] ,$$

$$b)\quad (\ ) \sim (\exists_1 V_i)p \supset (\iota V_i)p = a^* .$$

Now we introduce the notions (or predicates) of class (Cl), individual (In), and set (St), considering $\mathscr{W}$ as a suitably chosen variable:

D2.6 (ML^∞) $$\Delta \in Cl \equiv_D \Delta = \Lambda \vee \exists_{\mathscr{W}} \mathscr{W} \in \Delta ,$$

DD2.7,8 (ML^∞) $$\Delta \in In \equiv \Delta \notin Cl , \qquad \Delta \in St \equiv_D \Delta \in Cl \wedge \Delta \in El .$$

From A2.2 the usual properties of ι follow; e.g.

(2.2) $$\vdash p \equiv q \supset (\iota \mathscr{V})p = (\iota \mathscr{V})q .$$

3. On the lambda operator and class operations in EC^∞. Conventions on italicized symbols, non-italicized symbols, and restricted variables.

Finite conjunction ($\bigwedge_{i=1}^{n}$), finite disjunction ($\bigvee_{i=1}^{n}$), lambda expressions ($(\lambda \mathscr{A})p$), and collections ($\{\Delta_1, \ldots, \Delta_n\}$) are understood to be

(4) E.g. according to DD2.4,5 the condition that $f(x)$ is not defined for $x = 1$ reads $f(1) = 0$.

defined in EL^∞ as usually. For instance

$$\text{D3.1} \quad (ML^\infty) \quad (\lambda\mathscr{V})(p) =_D (\imath\mathscr{U})[\mathscr{U} \in Cl\, \forall_{\mathscr{V}} (\mathscr{V} \in \mathscr{U} \equiv \mathscr{V} \in El \wedge p]\,,$$

where $\mathscr{U}$ is the first variable different from $\mathscr{V}$ that does not occur free in p. Furthermore it is useful to define « $^{-}$ » (complement), « $\cap$ », « _ » (difference), « $\cup$ », « V », « $\subseteq$ », and « $\subset$ » as usually in connection with any terms Δ and Δ_1:

$$\text{D3.2} \quad (ML^\infty) \quad \bar{\Delta} =_D (\lambda V_i)\, V_i \notin \Delta\,,$$

$$\text{D3.3} \quad (ML^\infty) \quad \Delta \cap \Delta_1 =_D (\lambda V_j)(V_j \in \Delta \wedge V_j \in \Delta_1)$$

$$\text{DD3.4,5} \quad (ML^\infty) \quad \Delta - \Delta_1 =_D \Delta \cap \bar{\Delta}_1\,, \qquad \Delta \cup \Delta_1 =_D \overline{\bar{\Delta} \cap \bar{\Delta}_1}\,,$$

$$\text{D3.6} \quad (ML^\infty) \quad V =_D (\lambda\mathscr{U})\mathscr{U} = \mathscr{U},$$

$$\text{D3.7} \quad (ML^\infty) \quad \Delta \subseteq \Delta_1 \equiv_D \Delta - \Delta_1 = \Lambda \wedge \Delta,\, \Delta_1 \in Cl\,,$$

$$\text{D3.8} \quad (ML^\infty) \quad \Delta \subset \Delta_1 \equiv_D \Delta \subseteq \Delta_1 \neq \Delta$$

where V_i $[V_j]$ is the first variable that does not occur free in Δ $[\Delta = \Delta_1]$.

The extension of *El* or *St* [of *In*]—cf. DD2.1,8,7— is a mathematical object, and precisely a class [a set],so that it can be denoted by the corresponding lambda expression. E.g. the extension of *El* is $(\lambda V_1)\, V_1 \in El$ [D3.1]. The analogue does not hold for italicized notions such as *Cl*, to be considered as properties holding for some proper classes. Indeed, for instance, by D3.1 and D2.8 $\vdash V_1 \in (\lambda V_1)(V_1 \in Cl) \equiv V_1 \in St$. In spite of this, for the sake of uniformity we state the following

CONVENTION 3.1. (ML^∞) *The non-italicized symbol, say* Sym, *corresponding to any italicized symbol, Sym, is understood to be defined by the corresponding lambda expression*: $\text{Sym} =_D (\lambda V_1)\, V_1 \in Sym$.

Another useful convention is the following:

CONVENTION 3.2. (ML^∞) *We may combine the aforementioned italicized symbols with one another and with terms of EL^∞—or ML^∞* [n. 8]*— using « = » and the symbols « $^{-}$ » (complement), « $\cap$ », ..., « $\subset$ »* DD3.2-5,7,8. *E.g.* $\overline{In} \cap El = St$ *stands for* $(\forall V_1)(V_1 \notin In \wedge V_1 \in El \equiv \equiv V_1 \in St)$ *and* $St \subseteq El$ *stands simply for* $(\forall V_i)(V_i \in St \supset V_i \in El)$.

To present some applications of the conventions above we write the definitions

$$(3.1)\quad (ML^\infty)\qquad \mathrm{El} =_D (\lambda\mathscr{V})\,\mathscr{V} \in El\,,\qquad \mathrm{Cl} =_D (\lambda\mathscr{V})\,\mathscr{V} \in Cl$$

and the theorems

$$(3.2)\quad (ML^\infty)\qquad \mathrm{El} = El\,,\qquad \mathrm{Cl} = St \neq Cl\,.$$

Furthermore, on the basis of D2.6 we turn DD2.7,8 into

$$(3.3)\quad (ML^\infty)\qquad In =_D \overline{Cl}\,,\qquad St =_D Cl \cap El$$

and from DD2.6,7 and (2.1) we deduce in (EC^∞)

$$(3.4)\qquad \vdash \mathscr{U} \in In \vee \mathscr{U} = \Lambda \equiv \forall_{\mathscr{V}}\, \mathscr{V} \notin \mathscr{U}$$

where we need not write « ( ) » by the generalization theorem.

CONVENTION 3.3. *(ML^∞) (a) We use the symbols Δ, Δ_1, Δ_2, ... for any designators, the script capital letters $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{U}$, $\mathscr{V}$, $\mathscr{W}$ (possibly with subscripts) as unrestricted variables, non-script capital letters such as A, B, C, X, Y, Z, U, V, W ... as class variables, the lower case Latin letters a, b, c, d (possibly with subscripts) as element variables, and the lower case Latin letters x, y, w, (and $x_1, x_2, \ldots$) as set variables; hence in EL^∞ we can write*

$$(3.5)\qquad \begin{aligned} \forall_X \Phi(X) &\equiv_D (X V_i)[V_i \in Cl \supset \Phi(V_i)]\,,\\ (\imath X)\,\Phi(X) &=_D (\imath V_i)[V_i \in Cl \wedge \Phi(V_i)]\,,\end{aligned}$$

$$(3.6)\qquad \begin{aligned} \forall_a \Phi(a) &\equiv_D (\forall V_i)[V_i \in El \supset \Phi(V_i)]\,,\\ (\imath a)\,\Phi(a) &=_D (\imath V_i)[V_i \in El \wedge \Phi(V_i)]\,,\end{aligned}$$

$$(3.7)\qquad \begin{aligned} \forall_x \Phi(x) &\equiv_D (\forall V_i)[V_i \in St \supset \Phi(V_i)]\,,\\ (\imath x)\,\Phi(x) &=_D (\imath V_i)[V_i \in St \wedge \Phi(V_i)]\end{aligned}$$

where V_i is the first variables without free occurrences in $\Phi(X)$, $\Phi(a)$, or $\Phi(x)$ respectively.

(*b*) *Roughly speaking e.g.* $a \in A$ *stands for* $a \in El \supset a \in A$; *and if* p *contains restricted variables, then* p *stands for* $r \supset p$, *where* r *expresses the restrictions on those restricted variables* (*E.g.* r *is* $X, Y \in Cl \cap a \in El$ *in case* p *is* $a \in X \in Y$.)

In contrast to usual properties of $\cap$, $\cup$, and $^-$, we have e.g.

$$(3.8) \quad \begin{cases} \vdash In \neq \Lambda \supset \exists_{a,b}(a \neq b \wedge \bar{a} = \bar{b}) \,, & \vdash a \cup X = X \equiv a \in In \cup \{\Lambda\} \cup X \,, \\ \vdash \forall_X(a \cap X = \Lambda) \equiv a \in In \cup \{\Lambda\} \,, & \vdash a \cap X = \Lambda \equiv a \subseteq \bar{X} \vee a \in In \,. \end{cases}$$

Of course the usual formulas involving $\cap$ to $^-$ hold if only class (or set) variables are used.

Let us define the *power class* or *subsect class* ($S\Delta$), *union class* ($\bigcup \Delta$), and *intersection class* ($\bigcap \Delta$) of Δ as follows ([5])

D3.9.10 $\quad (ML^\infty) \quad S\Delta =_D (\lambda X) X \subseteq \Delta \,, \quad \bigcup \Delta =_D (\lambda a) \exists_X (a \in X \in \Delta) \,,$

D3.11 $\quad (ML^\infty) \quad \bigcap \Delta =_D (\lambda a) \forall_X (X \in \Delta \supset a \in X) \,.$

CONVENTION 3.4. (ML^∞) *If* $V_1, \ldots, V_n$ *are distinct variables and* $\Delta_1, \ldots, \Delta_n$, Δ *are expressions in* EL^∞ (*or* ML^∞), *then we denote the expression obtained from* Δ *by simultaneously substituting* Δ_i *for* V_i ($i = 1, \ldots, n$) *by* $\begin{pmatrix} V_1 \ldots V_n \\ \Delta_1 \ldots \Delta_n \end{pmatrix} \Delta$.

Remark that by Convention 3.3 and D3.10

$$(3.9) \quad \vdash (\lambda \mathscr{V}) \Phi(\mathscr{V}) = (\lambda a) \Phi(a) \,, \quad \vdash \bigcup \Delta = (\lambda a) \exists_{\mathscr{U}} (a \in \mathscr{U} \in \Delta)$$

where $\Phi(a)$ is $\begin{pmatrix} \mathscr{V} \\ a \end{pmatrix} \Phi(\mathscr{V})$, $\Phi(\mathscr{V})$ is $\begin{pmatrix} a \\ \mathscr{V} \end{pmatrix} \Phi(a)$, and the (distinct) variables a and $\mathscr{U}$ do not occur free in Δ. Note that

$$(3.10) \quad \vdash \Delta \cap (In \cup \{\Lambda\}) \neq \Lambda \supset \bigcap \Delta = \Lambda \,.$$

By $(3.9)_2$ we could use only unrestricted variables in D3.10, and by (3.10) the analogue does not hold for D3.11.

([5]) We write e.g. « $x \in X \in \Delta$ » for « $x \in X \wedge X \in \Delta$ », and « $x \in X \subseteq \Delta$ » for « $x \in X \wedge X \subseteq \Delta$ », as is customary.

CONVENTION 3.5. *If after using $\Phi(V_i)$ as a matrix, we write $\Phi(\Delta)$ where Δ is a term, then we understand that Δ is free for V_i in $\Phi(V_i)$ and that $\Phi(\Delta)$ is $\binom{V_i}{\Delta}\Phi(V_i)$.*

4. Axioms in EC^{∞} for classes and sets; some basic consequences.

First we define the successor $\mathfrak{S}(X)$ of the class X:

D4.1 $$\mathfrak{S}(X) \equiv_D X \cup \{X\}\,.$$

The axioms for classes and sets of EC^{∞} are AA4.1-10 below; AA.4-10 are very similar to the corresponding axioms 1.12 to 1.36 considered in [IST, p. 180]. If the (extensional) domain D_E of individuals is assumed to be empty ($In = \Lambda = 0$) as is in [IST], then A4.1 can be included in A4.8.

A4.1 $$\exists_u \forall_{\mathscr{A}} (\mathscr{A} \in In \supset \mathscr{A} \in u)\,,$$

A4.2 (*Identity*) $$\mathscr{U} = \mathscr{V} \supset \forall_{\mathscr{A}} (\mathscr{U} \in \mathscr{A} \equiv \mathscr{V} \in \mathscr{A})\,,$$

A4.3 (*Class-building*) $$\exists_X \forall_{\mathscr{V}} (\mathscr{V} \in X \equiv \mathscr{V} \in El \wedge p)$$ *where X has no free occurrences in p.*

A4.4 (*Power set*) $$\forall_u \exists_v \forall_x (x \subseteq u \supset x \in v)\,.$$

A4.5 (*Pairing*) $$\forall_{a,b} \exists_u (a, b \in u)\,.$$

A4.6 (*Union*) $$\forall_u \exists_v \forall_x (x \in u \supset x \subseteq v)\,.$$

A4.7 (*Regularity*) $$X \neq \Lambda \supset \exists_{\mathscr{A}} (\mathscr{A} \in X \wedge \mathscr{A} \cap X = \Lambda)\,.$$

A4.8 (*Infinity*) $$\exists_u [0 \in u \wedge \forall_x (x \in u \supset \mathfrak{S}x \in u)]\,.$$

As is well known, by A4.4 $\vdash X \subseteq x \supset X \in St$. Hence we can replace AA4.1,8 by the single axiom

A4.8′ $$\exists_u [\forall_{\mathscr{V}} (\forall_{\mathscr{A}} \mathscr{A} \notin \mathscr{V} \supset \mathscr{V} \in u) \wedge \forall_x (x \in u \supset \mathfrak{S}x \in u)]\,.$$

It asserts the existence of a set u that contains the elementless entities and is closed with respect to the successor operation. In case no individual exists, A4.8′ is equivalent to A4.8.

THEOR. 4.1. *We have* (2.1), i.e. $\vdash (N)\mathscr{V} \notin \Lambda$.

PROOF. We follow in part Sec. 2.2 of Suppes' book [7] which, unlike EC^{∞}, exludes proper classes according to Fraenkel, and assumes $=$ as a primitive notion.

From A4.3 with p replaced by $\mathscr{V} \neq \mathscr{V}$, Convention 3.3, and rule C with $\mathscr{W}$ we get (a) $\mathscr{W} \in Cl$ and $(b) \forall_{\mathscr{V}} (\mathscr{V} \in \mathscr{W} \equiv \mathscr{V} \in Cl \wedge \mathscr{V} \neq \mathscr{V})$, hence (c) $\forall_{\mathscr{V}} \mathscr{V} \notin \mathscr{W}$. This, (a), and D2.6 yield (d) $\mathscr{W} = \Lambda$. By D2.2 and (d), $\exists_{\mathscr{A}} \mathscr{A} \in \Lambda \supset \exists_{\mathscr{A}} \mathscr{A} \in \mathscr{W}$. So (c) yields $\forall_{\mathscr{A}} \notin \Lambda$ where $\mathscr{W}$ does not occur. Hence (2.1) holds. q.e.d.

By D2.6 and by D2.4 and A4.8 we respectively have

$$\vdash \forall_{\mathscr{A}} \mathscr{A} \notin X \supset X = \Lambda\,, \qquad \vdash \Lambda \in El\,. \tag{4.1}$$

From DD2.1,2 and A4.2 we respectively deduce

$$\begin{aligned} &\vdash \mathscr{A} = \mathscr{B} \supset (\exists_{\mathscr{V}} \mathscr{V} \in \mathscr{A} \equiv \exists_{\mathscr{V}} \mathscr{V} \in \mathscr{B})\,, \\ &\vdash \mathscr{A} = \mathscr{B} \supset (\mathscr{A} \in El \equiv \mathscr{B} \in El)\,. \end{aligned} \tag{4.2}$$

Now, to prove that

$$\vdash \forall_{\mathscr{A}} (\mathscr{A} \in X \equiv \mathscr{A} \in Y) \equiv X = Y\,, \tag{4.3}$$

we first assume (a) $\forall_{\mathscr{A}} (\mathscr{A} \in X \equiv \mathscr{A} \in Y)$ and (b) $\exists_{\mathscr{A}} (\mathscr{A} \in X \vee \mathscr{A} \in Y)$.

Then (c) $X = Y$ follows by D2.2. Now assume (a) and $\sim (b)$. Then $(4.1)_1$ yields $Y = \Lambda$ and $X = \Lambda$, so that by the symmetry and transitivity properties of $=$ [n. 2] we have (d) $X = Y$. So $\vdash (a) \supset \supset X = Y$.

To prove the converse implication, remark that (b) and $X = Y$ yield (a) by D2.2; furthermore $\sim (b)$ yields (a) trivially, so that $\vdash X = Y \supset (a)$. We conclude that (4.3) holds.

Definition D2.2 of identity may push us to weaken A4.2 into

$$\text{A4.2}' \qquad \mathscr{U} = \mathscr{V} \wedge \exists_{\mathscr{A}} (\mathscr{A} \in \mathscr{U} \vee \mathscr{A} \in \mathscr{V}) \supset \forall_{\mathscr{B}} (\mathscr{U} \in \mathscr{B} \equiv \mathscr{V} \in \mathscr{B})\,.$$

THEOR. 4.2. *For any matrix* $\Phi(\mathscr{A}, \mathscr{B}, \mathscr{C})$

$$\begin{aligned} &\vdash \mathscr{A} = \mathscr{B} \supset [\Phi(\mathscr{A}, \mathscr{B}, \mathscr{A}) \equiv \Phi(\mathscr{A}, \mathscr{B}, \mathscr{B})]\,, \\ &\vdash \mathscr{A} = \mathscr{B} \supset (\imath \mathscr{V})\Phi(\mathscr{A}, \mathscr{B}, \mathscr{A}) = (\imath \mathscr{V})\Phi(\mathscr{A}, \mathscr{B}, \mathscr{B})\,. \end{aligned} \tag{4.4}$$

PROOF. It is easy to prove (4.4) by induction on the length of $\Phi(\mathscr{A}, \mathscr{B}, \mathscr{C})$. The starting steps are those where $\Phi(\mathscr{A}, \mathscr{B}, \mathscr{C})$ is $\mathscr{V} \in \mathscr{C}$ or $\mathscr{C} \in \mathscr{V}$.

Let the first alternative hold and start with (a) $\mathscr{A} = \mathscr{B}$ and $\mathscr{V} \in \mathscr{A}$. Then $\exists_{\mathscr{W}} \mathscr{W} \in \mathscr{B}$ follows by $(4.2)_1$. Hence $\mathscr{A}, \mathscr{B} \in Cl$. Then by (4.3), (a) and $\mathscr{V} \in \mathscr{A}$ yield $\mathscr{V} \in \mathscr{B}$. Hence (a) yields $\mathscr{V} \in \mathscr{A} \supset \mathscr{V} \in \mathscr{B}$. Likewise it yields the converse implication. Hence $(4.4)_1$ holds in the present case.

Now let $\Phi(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be $\mathscr{C} \in \mathscr{V}$ and start with (a). Then by A4.2 $\forall_{\mathscr{W}}(\mathscr{A} \in \mathscr{W} \equiv \mathscr{B} \in \mathscr{W})$, hence $\mathscr{A} \in \mathscr{V} \equiv \mathscr{B} \in \mathscr{V}$. We conclude that $\vdash (a) \supset (\mathscr{A} \in \mathscr{V} \equiv \mathscr{B} \in \mathscr{V})$, which is $(4.4)_1$ in the present case. Thus the starting steps have been dealt with.

Now let $(4.4)_1$ $[(4.4)_2]$ hold in case the lenght of $\Phi(\mathscr{A}, \mathscr{B}, \mathscr{C})$ is $< n$ $[< n + 4]$. Then we can easily prove the validity of $(4.4)_1$ in the cases where $\Phi(\mathscr{A}, \mathscr{B}, \mathscr{C})$ has any of the forms $\mathscr{C} \in \Delta_1$, $\Delta_1 \in \mathscr{C}$, $\sim p$, $p \wedge q$, and $\forall_{\mathscr{W}} p$. Furthermore we can derive $(4.4)_2$ from the inductive hythesis $(4.4)_1$ by (2.2). q.e.d.

* * *

Theorem (4.3) also holds in [IST] (where $=$ is considered as a primitive notion). However the part of EC^{∞} where all variables are restricted to classes obviously differs from [IST] in that for $D \neq \Lambda$ we can set $X = \{a\}$ and $Y = \{b\}$ where a and b are individuals; then both $\forall_Z (Z \in X \equiv Z \in Y \equiv Z \in \Lambda)$ and $Y \neq \Lambda \neq X$ are true, and $a \neq b$ yields $X \neq Y$.

The ordered couple (a, b) is defined to be $\{\{a\}, \{a, b\}\}$. Furthermore we identify the n-tuple $(a_1, \ldots, a_n)$ with a_1 for $n = 1$, and with $((a_1, \ldots, a_{n-1}), a_n)$ for $n = 2, 3, \ldots$. The notions of n-ary relations and functions (Fn) are understood in the ususal way—cf. [IST, Secs. 3, 4]. The same holds for the *domain* Dmn R, *range* Rng R, and *field* Fld R of the largest binary relation belonging to the set R.

We use « $\times$ » for cartesian product and Y^2 for $Y \times Y$, so that $R \subseteq V^2$ means that R is a binary relation.

Following substantially [3], we introduce the notations $\mathscr{F}'a$ and $\mathscr{F}''X$ for the value of the (function) $\mathscr{F}$ at a and the $\mathscr{F}$-transform of the (class) X respectively.

D4.2,3 (ML^{∞})
$$\mathscr{F}'a =_D (\imath b)(a, b) \in \mathscr{F} ,$$
$$\mathscr{R}''X =_D (\lambda b) \exists_a [(a, b) \in \mathscr{R} \wedge a \in X] .$$

Now we can state the last two axions of EC^∞:

A4.9 (*Substitution*) $(\ )F \in Fn \wedge \mathrm{Dmn}\, F \in St \supset \mathrm{Rng}\, F \in St$.

A4.10 (*Relational axiom of choice*)

$$(\ ) R \subseteq V^2 \supset \exists_F [F \in Fn \wedge F \subseteq R \wedge \mathrm{Dmn}\, F = \mathrm{Dmn}\, R] .$$

It is useful to introduce the symbol $\{\Delta|p\}$:

D4.4 $$\{\Delta|p\} =_D (\lambda V_j)(\exists V_1, \ldots, V_n)(V_j = \Delta \wedge p)$$

where Δ is a term, $V_1, \ldots, V_n$ are the (distinct) variables having free occurrences in it, and V_j is the first variable different from V_1 to V_n without free occurrences in p.

Of course $\vdash \{\mathscr{V}|p\} = (\lambda\mathscr{V})p$. We can now introduce lambda expressions for n-ary relations and functions

D4.5 $$(\lambda\mathscr{A}_1, \ldots, \mathscr{A}_n)p =_D \{(\mathscr{A}_1, \ldots, \mathscr{A}_n)|p\} ,$$

D4.6 $$(\lambda\mathscr{A}_1, \ldots, \mathscr{A}_n)\Delta =_D (\lambda\mathscr{A}_1, \ldots, \mathscr{A}_n, \mathscr{B})(\mathscr{B} = \Delta) ,$$

where $\mathscr{A}_1$ to $\mathscr{A}_n$ are distinct variables and $\mathscr{B}$ is the first variable distinct from them that has no free occurrences in Δ. The theorems

(4.5) $$\vdash \mathscr{V} \in (\lambda\mathscr{V})p \equiv p \wedge \mathscr{V} \in El ,$$

(4.6) $$\vdash (V_1, \ldots, V_n) \in (\lambda V_1, \ldots, V_n)p \equiv p \wedge \bigvee_{i=1}^{n} V_i \in El$$

can be proved in substantially the usual way—cf. in particular (4.1) and A2.2. For $n = 1$ (4.6) is (4.5).

Let us explicitly note that—cf. (3.4)—we can prove [AA.4.1,8]

(4.5) $$\vdash \mathrm{In} = In , \quad \vdash \mathrm{In} \in St , \quad \mathrm{El} \notin El .$$

5. A straightforwrad way of constructing the analogue for EC^∞ of the theory of sets and transfinite numbers used in pure mathematics.

It is straightforward to express in EC^∞ the elementary set theory presented in [IST, Chap. 1] (Boolean algebra of classes, algebra of relations, functions, infinite Boolean operations, equivalence relations, and ordering). In order to give some hints how to do this and for later purposes, we now define the identity relation $\boldsymbol{I}$, the *$\boldsymbol{R}$-equivalence class $a/\boldsymbol{R}$* of the element ($a/\boldsymbol{R}$ may be a proper class), and the class $A//\boldsymbol{R}$ of the $\boldsymbol{R}$-equivalence classes having representatives in A—cf. [IST, Def. 7.3]:

DD5.1,2 (ML^{∞}) $\boldsymbol{I} =_D \{(a, b)|a = b\}$, $a/\boldsymbol{R} = \{b|(a, b) \in \boldsymbol{R}\}$,

D5.3 $A//\boldsymbol{R} =_D \{x|\exists_a(a \in A \wedge x = a/\boldsymbol{R})\}$.

We say—cf. [IST, Def. 8.12]—that (1) $\boldsymbol{R}$ is a *partial order* ($\boldsymbol{R} \in \boldsymbol{POrd}$) if $\boldsymbol{R}$ is a relation antisymmetric, reflexive, and transitive—cf. n. 30 below.

(2) is *well founded* ($\boldsymbol{R} \in \boldsymbol{WFnd}$) if $\boldsymbol{R}$ is an asymmetric relation every non-empty subset A of whose field has an $\boldsymbol{R}$-minimal element x ($A \cap \{y|(y, x) \in \boldsymbol{R}\} = \Lambda$, hence $\boldsymbol{R} \cap \boldsymbol{I} = \Lambda$).

(3) $\boldsymbol{R}$ is a *well-ordering* if $\boldsymbol{R} - \boldsymbol{I}$ is well-founded and $\boldsymbol{R}$ is a simple ordering (i.e. $\boldsymbol{R} \in \boldsymbol{POrd}$ and $\boldsymbol{R} \cup \boldsymbol{R}^{-1} = \mathrm{Fld}\, \boldsymbol{R} \times \mathrm{Fld}\, \boldsymbol{R}$).

It is also very simple to construct the analogue for EC^{∞} of the theory of ordinals and cardinals presented in [IST, Chaps. 2-4]. To give some hints we define *ε-transitive class* (*ε-Trns*), *ordinal class* (*Ord*) and *ordinal number* (Ord):

D5.4 (ML^{∞}) $\Delta \in \varepsilon\text{-}\boldsymbol{Trns} \equiv_D \Delta \in Cl \wedge \forall_x(x \in \Delta \supset x \subseteq \Delta)$,

D5.5 $\Delta \in \boldsymbol{Ord} \equiv_D \Delta \in \varepsilon\text{-}\boldsymbol{Trns} \wedge \Delta \subseteq \varepsilon\text{-}\boldsymbol{Trns}$.

In accordance with Convention 3.1

(5.1) $\vdash \mathrm{Ord} = (\lambda \mathscr{W})\mathscr{V} \in \boldsymbol{Ord}$, hence $\vdash \mathrm{Ord} = \boldsymbol{Ord} \cap St$.

The analogues for [IST] of the three definitions above constitute Def. 9.1 in [IST]. The basic (and only) difference between the former and the latter definitions consists of the inclusions of the condition $\Delta \in Cl$ in D5.4. The analogues for EC^{∞} of the theorems and proofs in [IST, Chap. 2] can be safely written in practically the same way, provided one be careful about the use of restricted variables and in particular one remember that $x \in y \in z \in \mathrm{Ord}$ implies that x, y, and z

are classes (sets)—see (5.2) below. The fact that under the assumption $In = \Lambda$ made in [IST], axiom A4.1 is embodied in the consequence $0 \in St$ of A4.8 will be relevant very seldom—see (5.9) below. In particular we can prove in EC^∞—cf. [IST, Theors. 9.2-9]—that

(5.2) $\qquad \vdash 0 \in \text{Ord}\,, \qquad \vdash x \in \text{Ord} \supset \mathfrak{S}x \in \text{Ord} \qquad [\text{D4.1}]\,,$

(5.3) $\qquad \vdash A \subseteq \text{Ord} \supset \bigcup A \in Ord\,, \qquad \vdash A \in Ord \supset \bigcup A \in \text{Ord}\,,$

(5.4) $\qquad \vdash \text{Ord} \in Ord\,, \qquad \vdash \text{Ord} \notin St\,,$

(5.5) $\qquad \vdash a, b \in \text{Ord} \supset a = b \vee a \in b \vee b \in a\,.$

We write « xRy » for « $(x, y) \in R$ »—cf. [IST]—and $A \upharpoonright F$ for the restriction of the function F to the class A. Then the general recursion principle [IST, Theor. 13.1] reads in EC^∞ as follows:

THEOR. 5.1. *Let R be a well founded relation such that, for every b in the Field of R, $\{a | (a, b) \in R\}$ is a set; and let F be a function of domain* $(\text{Fld}\,R) \times V$. *Then there is a unique function G such that* $\text{Dmn}\, G = \text{Fld}\, R$ *and for all* $a \in \text{Fld}\, R$

(5.6) $$G'a = F(a, \{b | bRa\} \upharpoonright G)\,.$$

We have only changed Monk's set variables x and y into our element variables. Practically so simple is the conversion of the proof. The analogue holds for the recursion theorems 13.2-8 in [IST], i.e. the general and usual recursion principles for ordinals, the same principles considered in connection with a parameter, the iteration principle, and primitive recursion.

All theorems of ordinal arithmetic, those stating equivalents for the axioms of choice in [IST, Chap. 3], and the theorems on cardinal numbers in [IST, Chap. 4] are carried over into EC^∞ together with their proofs by similar (extremely slight) changes, and by taking A4.1 into account as far as [IST, Theor. 15.18]—i.e. (5.8) below—is concerned. Let us mention in particular Theor. 13.10 in [IST] which is the assertion that *every well ordering whose field is a set, is similar with an ordinal number*; furthermore the following—cf. [IST, Def. 15.16 to Theor. 15.20]—where we start using *Greek lower case letters as variables restricted to ordinals*, unless otherwise noted:

THEOR. 5.2. *There is a unique function* ϱ *with domain* V, *such that for every element* a *we have* [6]

(5.7) $$\varrho a = \bigcap \{\alpha | \forall_b (b \in a \supset \varrho b < \alpha)\} \qquad (\forall_\alpha p \equiv_D \forall_x (\alpha \in \mathrm{Ord} \supset p)) .$$

We call ϱa the *rank* of a ($a \in El$). The rank of Λ or any individual is 0. The theorem

(5.8) $$\boldsymbol{M}_\alpha \in St \quad \textit{where} \quad \boldsymbol{M}_\alpha =_D \{a | \varrho a < \alpha\}$$

can be proved in EC^{∞} (by induction) as its analogue, Theor. 15.18, in [IST], except that we have to add at the outset of the proof that

(5.9) $$\vdash \boldsymbol{M}_0 = 0 , \qquad \vdash \boldsymbol{M}_1 = In \cup \{0\}$$

and that $\vdash \boldsymbol{M}_1 \in St$ by A4.1. Then we can follow the proof of Theor. 15.18 in [IST, p. 113] (on the basis of A4.4).

The analogues for EC^{∞} of [IST, Theor. 15.20] is rather straightforward:

(5.10) $$\vdash \alpha > 0 \supset \boldsymbol{M}_\alpha = \mathrm{In} \cup \bigcup_{\beta < \alpha} S\boldsymbol{M}_\alpha \qquad [\mathrm{D3.9}] ,$$

(5.11) $$\vdash \varrho(\boldsymbol{M}_\alpha) = \alpha , \qquad \vdash x \in \boldsymbol{M}_\alpha \supset x \subset \boldsymbol{M}_\alpha ,$$

(5.12) $$\vdash \alpha < \beta \supset \boldsymbol{M}_\alpha \subset \boldsymbol{M}_\beta \wedge \boldsymbol{M}_\alpha \in \boldsymbol{M}_\beta , \qquad \vdash \boldsymbol{M}_{\mathfrak{S}\alpha} = \mathrm{In} \cup S\boldsymbol{M}_\alpha ,$$

(5.13) $$\vdash \forall_\beta \exists_\gamma (\beta < \alpha \supset \beta < \gamma < \alpha) \supset \boldsymbol{M}_\alpha = \bigcup_{\beta < \alpha} \boldsymbol{M}_\beta , \qquad \vdash V = \bigcup_{\alpha \in Ord} \boldsymbol{M}_\alpha .$$

---

(6) If α and β are ordinals, then $\alpha < \beta \equiv_D \alpha \in \beta$ and $\alpha \leq \beta \equiv_D \alpha \in \beta \wedge \alpha = \beta$. Now, to prove Theor. 5.2, let the function F in (5.6) be defined by

(*) $$F(a, \mathscr{V}) =_D \begin{cases} \bigcap \{\alpha | \forall_b (b \in Dmn\mathscr{V} \supset \mathscr{V}'b < \alpha)\} & \text{if this is a set ,} \\ 0 & \text{otherwise .} \end{cases}$$

Now let $a \in El$. Then either $a \in In \cup \{0\}$, hence $\mathscr{I}_a$ with $\mathscr{I}_a =_D \{b | b \in a\}$ is the empty set (and by (*) $F(a, \mathscr{V}) = 0$); or $a \in St$ so that $\mathscr{I}_a = a$. We conclude that $\vdash \forall_a \mathscr{I}_a \in St$. Then (*) and Theor. 5.1 for $G = \varrho$ and $R = (\lambda a, b)\, a \in b$ [so that $\mathscr{V}'b < \alpha$ is equivalent to $(\mathscr{V}'b) R \alpha$] easily yield Theor. 5.2.

As far as the equivalents of the relational axiom of choice A4.10 are concerned, let us only mention the following:

Counting principle: For every set x there is an $\alpha \in \mathrm{Ord}$ and a one to one function F that maps x onto α.

Practically the theory of cardinals developed in [IST, Chap. 4] becomes its analogue for EC^{∞} by simply including the condition « A and B are classes » in the definition [IST, Def. 18.1] of equipotence: We say that A is *equipotent* with B if A and B are classes and there is a one-to-one mapping of A into B. We also say that A is a *cardinal* (*number*) if $A \in \mathrm{Ord}$ and A is not equipotent with any $\alpha \in A$.

By the counting principle, *for any set A there is a unique cardinal m ($m =_D |A|$) such that A and m are equipotent*—cf. [IST, Theor. 18.3].

6. On universes.

The extension of the « pure number theory » in [IST] to EC^{∞} is straightforward, as was shown in n. 5; this holds in particular for weakly and strongly inaccessible cardinals—cf. Def. 23.10 and Theor. 23.11 in [IST]. Somewhat less stranghtforward is the analogous extension of the theory of universes, which is closely related to those cardinals.

We understand that A is defined to be a *universe*, in EL^{∞} as well in the metalanguage, in case we have

$$(6.1) \quad \begin{cases} A \in St\,, \quad \omega \in A\,, \quad \mathrm{In} \in A\,, \quad (\ )a \in x \in A \supset a \in A\,, \\ (\ )x \in A \supset Sx \in A\,, \quad (\ )x \subseteq A \wedge |x| \neq |A| \supset x \in A\,, \end{cases}$$

where ω ist the first limit ordinal (>0).

This definition can be obtained from [IST, Def. 23.12] by restricting variables suitably, in connection with $(6.1)_4$, and by adding condition $(6.1)_3$. This addition is quite natural in that in [IST] $\mathrm{In} = \Lambda$, so that $(6.1)_3$ follows from $(6.1)_{2,4}$; furthermore it induces very slight changes in the theorems on universes in [IST]. More precisely Theor. 23.13 and its proof keep holding in MC^{∞} without any changes, except that some variables have to be suitably restricted in one of the aforementioned ways. Thus the following wffs are (syntactical) consequences (in EC^∞) of the hypothesis that A is a universe:

(6.2) $\quad (\ )x \in A \supset |x| < |A|\,, \qquad \omega < |A|\,, \qquad (\ )x \subseteq y \in A \supset x \in A\,,$

(6.3) $\quad (\ )a, b \in A \supset \{a, b\}\,, \qquad (a, b) \in A\,, \qquad (\ )x, y \in A \supset x \times y \in A\,,$

(6.4) $\quad (\ )x \in A \wedge f \in x \to A \supset (f, \text{Rng} f \in A)\,,$

(6.5) $\quad |A|$ *is strongly inaccessible*, $\quad (\ )x \in A \supset \bigcup x \in A\,,$

(6.6) $\quad (\ )I \in A \wedge x \in Fn \wedge \forall_i (i \in I \supset x_i \in A) \supset \bigcup_{i \in I} x_i \in A \wedge P_{i \in I} x_i \in A\,,$

where «$\times$» [«$P_{i\in I}$»] denotes cartesian [direct] product—cf. [IST, p. 55].

The analogue of [IST, Theor. 23.14] is the following:

THEOR. 6.1. *A is a universe iff $A = \boldsymbol{M}_\vartheta$ for some strongly inaccessible cardinal ϑ larger than* $|\text{In}|$.

This theorem differs from its analogue by the addition of the (obvious) condition $\vartheta > |\text{In}|$. The proof of $\Rightarrow$ reads as in [IST, p. 161]; the one of $\Leftarrow$ also, except that the condition $\vartheta > |\text{In}|$ is essential to deduce the inequality $\boldsymbol{M}_0 < \vartheta$ (which is the initial step of an induction) from the assumption $A = \boldsymbol{M}_\vartheta$ where ϑ is a strongly inaccessible cardinal.

Incidentally, from the proof of Theor. 6.1, which is a very slight generalization of the one of [IST, Theor. 23.14], it results that *if A is a universe, than $A = \boldsymbol{M}_\vartheta$ where $\vartheta = |A|$.*

It will cause no confusion to understand that e.g. $\boldsymbol{M}_\vartheta$ is defined within both EL^∞ and the metalanguage. The same holds for the *object system $\Psi_\alpha(x)$ of rank α based on the set x*:

(6.7) $\quad \Psi_1(x) =_D x \cup \{\Lambda\}\,, \qquad \Psi_\alpha(x) =_D x \cup \bigcup_{\beta<\alpha} S\Psi_\beta(x) \qquad (1 < \alpha \in \text{Ord})\,.$

Obviously $\Psi_\alpha(x)$ is the set of the objects of «rank $< \alpha$ relative to x», furthermore $\boldsymbol{M}_\alpha = \Psi_\alpha(\text{In})$—cf. $(5.9)_2$, (5.10).

Let us say that A is a *partial universe*, briefly $A \in PUniv$, if conditions $(6.1)_{1,2,4,5,6}$ hold, i.e. A is a universe according to [IST, Def. 23.12]. A universe—defined by conditions (6.1) in A—is a partial universe; in addition *if $A \in PUniv$, then* (6.2-6) *hold* and—cf. the proof of [IST, Theor. 23.14] and $(6.5)_1$—$\Psi_{\vartheta_1}(x) \subseteq A$ where *ϑ_1 is the first strongly inaccessible cardinal larger than the rank ϱx of x.*

Furthermore, by the ε-transitivity of the partial universe A —cf. $(6.1)_4$—it is easy to see that the following analogue of Theor. 6.1 or [IST, Theor. 23.14] holds.

THEOR. 6.2. *(a) If A is a partial universe, then*

$$(6.8) \qquad A = \Psi_\vartheta(\mathscr{I}) \quad \textit{where} \quad \mathscr{I} = A \cap \mathrm{In}\,, \ \vartheta = |A|\,.$$

(b) If $(6.8)_1$ holds where $\mathscr{I} \subseteq \mathrm{In}$ and ϑ is a strongly inaccessible cardinal larger than $|\mathscr{I}|$, then A is a partial universe and $(6.8)_{2,3}$ hold.

PROOF. To prove (a) let $(6.8)_{2,3}$ hold and let A be a partial universe, so that ϑ is strongly inaccessible—cf. $(6.5)_1$. Then we can deduce the first of the assertions

$$(6.9) \qquad \Psi_\vartheta(\mathscr{I}) \subseteq A\,, \qquad A \subseteq \boldsymbol{M}_\vartheta\,, \qquad A \subseteq \Psi_\vartheta(\mathscr{I})$$

by the reasoning obtained from the deduction of $\boldsymbol{M}_\vartheta \subseteq A$ within the proof (of the part $\Rightarrow$) of Theor. 23.14 in [IST, p. 161], by means of the replacement $\boldsymbol{M}_\alpha \to \Psi_\alpha(\mathscr{I})$. We can deduce $(6.9)_2$ as in the same proof.

Now let $(6.9)_3$ not hold, as an hypothesis for reductio ad absurdum. Then there is an element b of $A - \Psi_\vartheta(\mathscr{I})$ with the least rank.

Since $b \in A$, by $(6.9)_2$ $\varrho b < \vartheta$. We cannot have $\varrho b = 0$, for this yields $b \in \mathscr{I} \cup \{0\}$ and $\vdash \mathscr{I} \cup \{0\} \subseteq \Psi_\vartheta(\mathscr{I})$. Then b is a non-empty set.

By $(6.1)_4$ $b \subseteq A$. Then the case $b \not\subseteq \Psi_\vartheta(\mathscr{I})$ obviously contrasts with the above minimum property of ϱb. Hence $b \subseteq \Psi(\mathscr{I})$. Since $\varrho b < \vartheta$, this yields $b \in \Psi_{\varrho b \dot{+} 1}(\mathscr{I})$ by (6.7), so that (since $\varrho b + 1 < \vartheta$) $b \in \Psi_\vartheta(\mathscr{I})$, which contrasts with an assertion above [7]. Then $(6.9)_3$ must hold. This and $(6.9)_1$ yield $(6.8)_1$. Thus part (a) holds.

Now let the assumptions in part (b) hold. Then the validity of conditions $(6.1)_{1,2,4,5}$ is easily checked. To check $(6.1)_6$ suppose that (a) $x \subseteq \Psi_\vartheta(\mathscr{I})$ and (b) $|x| < |\Psi_\vartheta(\mathscr{I})|$ hold. Since ϑ is strongly inaccessible, by induction one easily deduces

$$(6.10) \qquad (\ )\,\alpha < \vartheta \supset |\Psi_\alpha(\mathscr{I})| < \vartheta \qquad (\vartheta \leqslant |\Psi_\vartheta(\mathscr{I})|)\,.$$

Hence (c) $|\Psi_\vartheta(\mathscr{I})| = \vartheta$, so that by the assumption $(6.8)_1$ we have thesis $(6.8)_3$. From (b) and (c) we deduce $|x| < \vartheta$. Now we can re-

(7) By « $\dot{+}$ » [« $+$ »] we denote the sum of ordinals [cardinals]—cf. [IST].

mark, as in [IST, p. 162], that, since ϑ is regular,

$$\bigcup_{y\in x} \varrho y < \vartheta\,, \qquad \text{so that} \qquad \varrho x \leqslant \Big(\bigcup_{y\in x} \varrho y\Big) + 1 < \vartheta\,.$$

Hence $x \in \Psi_{\vartheta}(\mathscr{I})$, which by assumption $(6.8)_1$ yields $x \in A$. So $(6.1)_6$ holds.

We can prove $\forall_{\alpha}\ [\alpha < \vartheta \supset \mathscr{I} = \Psi_{\alpha}(\mathscr{I}) \cap \mathrm{In}]$ by induction. This and $(6.8)_1$ yield $(6.8)_2$. q.e.d.

7. A semantical system, $\Sigma_{\vartheta}(EL^{\infty})$, for EL^{∞}.

We call *semantical system* for EL^{∞} (of cardinality ϑ) any triple

(7.1) $$\Sigma_{\vartheta}(EL^{\infty}) = (O^{\infty}, \vartheta, a_E) \qquad [O^{\infty} \in (\vartheta + 1 \to V)]$$

where ϑ is a strongly inaccessible cardinal and where for some (extensional) domain of individuals D_E we have $[O^{\infty}_{\alpha} = O^{\infty}(\alpha)]$

(7.2) $$O^{\infty}_{\alpha} = \Psi_{\alpha\dot{+}1}(D_E)\,, \qquad a_E \in O_0 (= D_E)\,.$$

Incidentally from (6.7) we see that for any limit ordinal $\omega\tau < \vartheta$ (i.e. with $\tau < \vartheta$)

(7.3) $$O^{\infty}_{\beta} \subset \bigcup_{\gamma\in\omega\tau} O^{\infty}_{\gamma} = \Psi_{\omega\tau}(D_E \cup \{\Lambda\}) \subset O^{\infty}_{\omega\tau} \qquad \text{for } \beta < \omega\tau\,.$$

In particular O^{∞}_{ω} contains a set, A, having an element of rank m for every $m < \omega$, so that $A \neq O^{\infty}_{\gamma}$ for all $\gamma < \omega$. The replacement of $\alpha + 1$ with α in (7.2) would cause e.g. O^{∞}_{ω} to contain only members of the sets O^{∞}_{γ} with $\gamma < \omega$. By (7.2) O^{∞}_{α} is the class of extensions of rank $\leqslant \alpha$. The *sets in* $\Sigma_{\vartheta}(EL^{\infty})$ constitute $\Psi_{\vartheta}(D_E)$ while $O^{\infty}_{\vartheta} - \Psi_{\vartheta}(D_E)$ is the class of the *proper classes in* $\Sigma_{\vartheta}(EL^{\infty})$.

In accordance with [GIMC] we say that $V[M]$ is a value-assignment [model] for EL^{∞} if it is a function from the variables [constants] of EL^{∞} into O^{∞}_{ϑ} [for which $M(c_1) = \Lambda$].

We define the *extensional designatum* $\bar{\Delta} = \mathrm{des}_{MV}(\Delta)$ of the expression Δ in EL^{∞} in the semantical system (7.1), at (the model) M and (the value-assignment) V, recursively by the conditions (1) to (4) below, where 1 and 0 stand for true and false respectively and where we understand that $\bar{\Delta}_i = \text{des}_{MV}(\Delta_i)$ $(i = 1, 2)$:

(1) *If* Δ *is* $\boldsymbol{V}_i$ $[c_i]$, *then* $\bar{\Delta}$ *is* $V(\boldsymbol{V}_i)[\boldsymbol{M}(c_i)]$.

(2) *If* Δ *is* $\Delta_1 \in \Delta_2$, *then* $\bar{\Delta}$ *is* 1 *or* 0 *according to whether or not* $\bar{\Delta}_1 \in \bar{\Delta}_2$:

(3) *If* Δ *is* $\sim\Delta_1$, $\Delta_1 \wedge \Delta_2$, *or* $(\forall \boldsymbol{V}_i)\Delta_1$ *where* Δ_1 *and* Δ_2 *are matrices, then* $\bar{\Delta}$ *is in order* $1 - \bar{\Delta}_1$, $\bar{\Delta}_1 \cdot \bar{\Delta}_2$, *or the product of the numbers* $\text{des}_{MV'}(\Delta_1)$ *extended to all value-assignments* V' *with* $V'(\boldsymbol{V}_j) = V(\boldsymbol{V}_j)$ *for* $j \neq i$.

(4) *If* Δ *is* $(\imath\, \boldsymbol{V}_i)(\Delta_1)$ *where* Δ_1 *is a matrix, then either* (a') *there is exactly one* $b \in O_V^\infty$ *for which* $\text{des}_{MV'}(\Delta_1) = 1$ *where* $V'(\boldsymbol{V}_i) = b$ *and* $V'(\boldsymbol{V}_j) = V(\boldsymbol{V}_j)$ *for* $j \neq i$, *and* (a'') $\bar{\Delta} = b$; *or* (b') *the exact uniqueness condition* (a') *on* b *above fails to hold and we have* (b'') $\bar{\Delta} = a_E$.

CHAPTER 2

THE INTERPRETED TYPE-FREE MODAL LANGUAGE ML^∞

8. The type free modal language ML^∞. Quasi intensions.

The primitive symbols of ML^∞ are those of EL^∞ [n. 2] with the addition of «$=$» (contingent identity) and «N» (necessity). We define the expressions or wffs of ML^∞ (i.e. its terms and matrices) recursively by conditions (1)-(4) in n. 2 (for EL^∞) and the following two additional conditions

(5) *If* Δ *and* Δ_1 *are terms, then* $(\Delta = \Delta_1)$ *is a matrix.*

(6) *If* p *is matrix, then such is* $N(p)$.

To construct a semantical system for ML^∞, let D and Γ be two disjoint sets of individuals with $|D| \geqslant 1$ and $|\Gamma| > 1$. We consider Γ as the class of elementary possible cases, briefly Γ-cases. Furthermore we assume that the class Ord of ordinal numbers (for the metalanguage) contains some strongly inaccessible cardinal ϑ—cfr. [IST, pp. 159]—larger than $|D \cup \Gamma|$, so that the set $\Psi_\vartheta(D \cup \Gamma)$—cf. (7.2)—is a universe (n. 6). Then we can consider $\Psi_\vartheta(D \cup \Gamma)$ to be the universe related to the semantical system $\Sigma_\vartheta(EL^\infty)$ for EL^∞ ($\text{des}_{MV}(\text{In}) = D \cup \Gamma$). In $\Sigma_\vartheta(EL^\infty)$ $\Psi_\vartheta(D \cup \Gamma)$ is the class of elements, i.e. sets and individuals (for EL^∞), whereas the subsets of the same class that are not elements of its—i.e. the elements of $\Psi_{\vartheta\dot{+}1}(D \cup \Gamma) - \Psi_\vartheta(D \cup \Gamma)$—are the proper classes (for EL^∞).

We want to define the analogue $\Sigma_\vartheta(ML^\infty)$ of $\Sigma_\vartheta(EL^\infty)$ for ML^∞. To this end let $\mathscr{D}$ be a mapping of Γ onto $\boldsymbol{S}(D)$ and assume that [8]

(8.1) $$D = \bigcup_{\gamma\in\Gamma} \mathscr{D}(\gamma)\,, \qquad (D \cap \Gamma = \Lambda,\ \mathscr{D} \in (\boldsymbol{S}D)^\Gamma)$$

and

(8.2) $$\mathscr{D}(\gamma) \cap (\gamma') = \{a_E\} \qquad \text{for } \gamma, \gamma' \in \Gamma \text{ and } \gamma \neq \gamma',$$

where a_E ($\in D$) will be used in connection with descriptions (whithout making any Γ-case privileged). Incidentally the substitution of Λ for $\{a_E\}$ in (8.2) and the addition of the condition $a_E \in D$ would make a Γ-case γ privileged, via the condition $a_E \in \mathscr{D}(\gamma)$.

We say that f is a *quasi intension of modal rank* $\leqslant 0$, and we write $f \in QI_0$, if f is a mapping of a subset Γ_1 of Γ into D, such that $f(\gamma) \in \mathscr{D}(\gamma)$ for all $\gamma \in \Gamma_1$ (and we shall consider these *QIs* as binary relations for the sake of homogeneousness). Using the direct product $P_{\gamma\in\Gamma} A_\gamma$ of any family A_γ—cf. [IST, p. 55]—we can write

(8.3) $$QI_0 =_D \{\xi \mid \exists_\eta [\xi \subseteq \eta \in P_{\gamma\in\Gamma}\mathscr{D}(\gamma)]\} \quad \text{hence} \quad \Lambda \in QI_0 \subset \boldsymbol{S}(\Gamma \times D)\,.$$

In the Γ-cases out of the domain of $f(\in QI_0) f$ represents (the extension of) Λ; if $f(\gamma) = a_E (\gamma \in \text{Dmn} f)$, then in γ f represents the nonexisting object, i.e. nothing. To identify a_E with Λ may cause confusion —cf. [GIMC, fn. 27, p. 106].

Looking forward to stating the rule (δ_3) of quasi intensional designation [n. 9], we now define the class QI_α of *quasi intensions of modal rank* $\leqslant \alpha$ for $0 < \alpha \in \text{Ord}$ by transfinite induction using formal nota-

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(8) By «B^A» we denote the class of the mappings of A into B.

tions in the metalanguage (as we did in (8.3)):

$$(8.4)\quad \begin{cases} QI_\alpha =_D \{\xi | \xi \in \Gamma \times V \quad \text{and,} \quad \text{for all } \gamma \in \Gamma, \\ \text{either} \quad \exists_\eta [\eta \in QI_0 \wedge (\{\gamma\} \times V) \cap \xi = (\{\gamma\} \cap V) \cap \eta] \\ \text{or} \quad \forall_\zeta [(\gamma, \zeta) \in \xi \supset \zeta \in \bigcup_{\beta<\alpha} QI_\beta]\} \quad (0 < \alpha < \text{Ord}) . \end{cases}$$

Thus, if $\xi \in QI_\alpha$ and $\gamma \in \Gamma$, then either $\gamma \notin \text{Fld}\, \xi$ and ξ represents Λ in γ, or $\gamma \in \text{Fld}\, \xi$ and there is a unique $b \in \mathscr{D}(\gamma)$ such that $(\gamma, b) \in \xi$ and ξ represents b in γ, or else there are some $\zeta \in \bigcup_{\beta<\alpha} QI_\beta$ such that $(\gamma, \zeta) \in \xi$ and ξ represents, in γ, the class of these ζ's (or the property holding only for them).

If α is the least ordinal for which $\xi \in QI_\alpha$ and $\xi \neq \Lambda$, then we call α the *modal rank* ($\text{mr}\, \xi$) of ξ. It is useful to set $\text{mr}\, \Lambda = -1$.

THEOR. 8.1. *The (ordinary) rank of any nonempty set $\xi \in QI_\alpha$, where α is the modal rank* $\text{mr}\, \xi$ *of ξ, is a function* $\text{rm}\, \alpha$ *of α (the rank of the QIs of modal rank α), defined for $\alpha < \vartheta$:*

$$(8.5)\qquad \text{rm}\, \alpha = 3 \dotplus \omega\tau \dotplus 3n \qquad (\alpha = \omega\tau \dotplus n,\ n \in \omega) .$$

PROOF. First suppose $\alpha = 0$. Then by $(8.3)_2$ ξ is a non-empty set of couples of elements of Γ and D. Hence $\text{rm}\, \alpha = 3$.

Now assume $\alpha > 0$. Then (since $\xi \in QI_\alpha - QI_\beta$ for all $\beta < \alpha$) for every $\delta < \alpha$ there is an ordinal β such that $\delta \leqslant \beta < \alpha$ and $\xi \cap (\Gamma \times QI_\beta)$ has at least one element of the form (γ, ξ_γ) with $\gamma \in \Gamma$ and $\xi_\gamma \in QI_\beta$, whose rank is $\text{rm}\, \beta \dotplus 2$. Now we easily see that by (8.4) we have

$$(8.6)\qquad \text{rm}\, \alpha = \bigcup_{\beta<\alpha} [\text{rm}\, \beta \dotplus 2] \dotplus 1 .$$

Hence $\text{rm}\, \alpha = \alpha$ if α is a limit number, and $\text{rm}\, (\alpha \dotplus 1) = \text{rm}\, \alpha \dotplus 3$. This yields (8.5). q.e.d.

Let ϑ be a strongly inaccessible cardinal larger than $|D|$ and $|\Gamma|$. Then $\Psi_{\vartheta \dotplus 1}(D \cup \Gamma)$ [n. 6] can be considered as a universe $\mathfrak{U}$ (for $\text{In} = D \cup \Gamma$). By (8.5), (7.2), and (6.7)

$$(8.7)\qquad QI_{\omega\tau \dotplus n} \subset O^{\infty}_{3 \dotplus \omega\tau \dotplus 3n} \qquad (D_E = \Gamma \cup D,\ n \in \omega) .$$

Since $\vartheta = \omega\vartheta$, this and (7.2) yield

$$(8.8) \qquad \mathrm{rm}\,\vartheta = \vartheta\,, \qquad QI_\vartheta \subset O_\vartheta^\infty = \Psi_{\vartheta \dot{+} 1}(D_E) \qquad \text{for } D_E = D \cup \Gamma\,.$$

We say that the triple (determined by D, Γ, a_E, and ϑ)

$$(8.9) \qquad \Sigma_\vartheta(ML^\infty) = \big((\lambda\alpha)QI_\alpha, \vartheta, a_M\big) \qquad \text{with } a_M =_D \{(\gamma, a_E) | \gamma \in \Gamma\} \in QI_0$$

is a *semantical system* for ML^∞, and that any $\xi \in QI_\vartheta$ with $\mathrm{mr}\,\xi < \vartheta$ [$\mathrm{mr}\,\xi = \vartheta$] is a QI in $\Sigma_\vartheta(ML^\infty)$ for elements [non-elements].

Incidentally, since QI_α has elements of modal rank α,

$$(8.10) \qquad QI_\alpha \cap (O^\infty_{\mathrm{rm}\alpha} - O^\infty_\beta) \neq \Lambda \qquad \text{for } \beta < \mathrm{rm}\,\alpha\,.$$

Let us remark that the classes O^∞_α ($\alpha \in \mathrm{Ord}$) are cumulative analogues of the classes of extensions O^ν_t considered in [GIMC, n. 7, p. 25] for EL^ν. Furthermore in [GIMC, n. 7] it was observed that the QIs of the type (1), i.e. those for properties of individuals of type 1, could be identified with functions from Γ-cases (i.e. elements of Γ) to classes of QIs of type 1 (for individuals); however in [GIMC] another alternative choice for these QIs was preferred to keep their levels as low as possible.

The choice (8.3) of the QIs for ML^∞ is in accordance with the latter alternative. These QIs have low levels by (8.7) and $(8.8)_1$. This fact is important for MC^∞, because $(8.8)_1$ yields $\mathrm{rm}\,[\mathrm{rm}\,\vartheta] = \mathrm{rm}\,\vartheta$ and in connection with this it will allow us to define suitable (direct) analogues of the QIs in $\Sigma_\vartheta(ML^\infty)$ both for sets and classes, within MC^∞ itself; furthermore this definition is basilar for carrying over to MC^∞ a large part of the theory for MC^ν developed in [GIMC, Chap. IX].

An analogue for ML^∞ for the alternative possible choice of the QIs for ML^ν, discarded in [GIMC], is afforded by the function QI'_α of α that is defined for $\alpha \leqslant \vartheta$ as follows:

$$(8.11) \qquad \begin{cases} QI'_0 = P_{\gamma\in\Gamma}[\mathscr{D}(\gamma) \cup \{\Lambda\}]\,, \\ QI'_\alpha =_D P_{\gamma\in\Gamma}[\mathscr{D}(\gamma) \cup S\mathscr{U}_\alpha] \qquad \text{for } \mathscr{U}_\alpha = \bigcup_{\beta<\alpha} QI'_\beta\,. \end{cases}$$

These QIs are simpler and give rise to a perhaps simpler definition of equivalent QIs [n. 9]. However the analogues of the QIs (8.11) can be defined within MC^∞ itself only for elements (and not for e.g. proper classes). This is related to the fact that the analogue rm′ of the function rm [Theor. 8.1] fulfills, in contrast to the analogue of (8.5), the condition

$$\text{(8.12)} \qquad \mathrm{rm}'(\omega\tau \dotplus n) = \omega\tau \dotplus 3 \dotplus 3n \ (n \in \omega); \qquad \text{hence } \mathrm{rm}'\vartheta = \vartheta \dotplus 3 .$$

Let us explicitly prove equality $(8.12)_2$ that has the main interest. To this end we first note that $\varrho\mathscr{U}_\alpha < \varrho\mathscr{U}_\beta < \vartheta$ (ϑ is strongly inaccessible) for $\alpha < \beta < \vartheta$. Hence $\varrho\mathscr{U}_\vartheta = \vartheta$.

Now assume $\xi \in QI'_\vartheta - \mathscr{U}_\vartheta$ (so that ϑ is the modal rank of ξ). Then ξ is a set of couples (γ, a_γ) where $\gamma \in \Gamma$ and either $a_\gamma \subseteq \mathscr{U}_\gamma$ or $a_\gamma \in D$. If ϱa_γ were $< \vartheta$ for all $\gamma \in \Gamma$, then (since $|\Gamma| < \vartheta$ and ϑ is regular) $\beta = \bigcup_{\gamma \in \Gamma} \varrho a_\gamma < \vartheta$ would hold. Then $a_\gamma \in \mathscr{U}_\beta$ for all $\gamma \in \Gamma$ (obviously there is an α with $\alpha \leqslant \beta \leqslant \mathrm{rm}'\alpha$; hence $\mathrm{mr}'a_\gamma \leqslant \beta$). Then $\xi \in QI_{\beta\dotplus 1}$, hence $\xi \in \mathscr{U}_\vartheta$, in contrast to an assumption above. We conclude that $\varrho a_\gamma = \vartheta$ for some $\gamma \in \Gamma$, so that $\varrho\xi = \vartheta \dotplus 3$. Thus (8.11) has been proved.

9. Equivalent *QIs*, *L*-determinate *QIs*, and extensions. Rules of intensional designation for ML^∞.

DEF. 9.1. *We say that ξ and η are equivalent QIs in the Γ-case γ ($\xi =_\gamma \eta$) iff $\{\zeta | (\gamma, \zeta) \in \xi\} = \{\zeta | (\gamma, \zeta) \in \eta\}$:*

$$\text{(9.1)} \quad \xi =_\gamma \eta \ \textit{iff} \ \xi, \eta \in QI_\vartheta, \quad \gamma \in \Gamma, \quad \textit{and} \quad (\forall \zeta)[(\gamma, \zeta) \in \xi \ \textit{iff} \ (\gamma, \zeta) \in \eta] .$$

THEOR. 9.1. *For $\gamma \in \Gamma$, $=_\gamma$ is an equivalent relation.*

THEOR. 9.2. *$\xi =_\gamma \eta$ for all $\gamma \in \Gamma$ iff $\xi = \eta$ and $\xi, \eta \in QI_\vartheta$.*

These theorems are the analogues for ML^∞ of Theors. 10.1,2 in [GIMC] for ML^ν. The theorems above can be proved very easily, unlike Theor. 10.2 for ML^ν, in accordance with the fact that the *QIs* for ML^∞ have one type unlike those for ML^ν.

Let $QI_\alpha - QI_0 (0 < \alpha \leqslant \vartheta)$, so that by (8.3,4) ξ is a (binary) relation with first members in Γ; let us set

$$\text{(9.2)} \qquad \xi^\dagger(\gamma) =_D \{\zeta | (\gamma, \zeta) \in \xi\} \quad (\text{hence } \xi = \{(\gamma, \zeta) | \zeta \in \xi^\dagger(\gamma)\}) \quad \text{for } \gamma \in \Gamma.$$

Then $\xi^{\dagger}$ is a function of domain Γ. In case this function is constant, we call ξ an *L-determinate QI(L-QI) of modal rank* $\leqslant \alpha$. This is rather in harmony with Carnap [2, § 22]. Furthermore we call *L-determinate QI of modal rank* $\leqslant 0$ any $\xi \in QI_0$ [hence $\xi \subseteq \eta$ for some $\eta \in D^{\Gamma}$] such that $\xi(\gamma) \neq a_M(\gamma)$ holds for at most one $\gamma \in \Gamma$—cf. $(8.9)_2$. Incidentally (9.1) yields

$$(9.3) \qquad \xi =_{\gamma} \eta \text{ iff } \xi^{\dagger}(\gamma) = \eta^{\dagger}(\gamma) \qquad \text{with } \gamma \in \Gamma \text{ and } \xi, \eta \in QI_{\vartheta}\,.$$

If $\xi =_{\gamma} \eta$, $\eta \in L\text{-}QI$, and either not $\xi =_{\gamma} a_M$ or $\eta = a_M$, then we say that η is the *extension* of ξ in γ and that the modal rank of η is the *extensional rank* of ξ in γ.

Our identification of extensions with L-determinate QIs is rather in harmony with [2, § 23]. Another natural way of defining the extension of ξ in γ would be to identify it with the class $E_{\xi,\gamma}$ of QIs that are equivalent to ξ in γ. However $E_{\xi,\gamma} \subseteq QI_{\vartheta}$ and $E_{\xi,\gamma} \notin QI_{\vartheta}$. Therefore we preferred the first definition.

THEOR. 9.3. *Let $\xi \in QI_{\vartheta}$. Then (a) for every $\gamma \in \Gamma$ there is exactly one extension, η_{γ}, of ξ in γ. Furthermore (b) the modal rank* mr ξ *of ξ is $< \vartheta$ (ϑ is a strongly inaccessible cardinal larger than $|\Gamma|$) iff* rm $\eta_{\gamma} < \vartheta$ *for all $\gamma \in \Gamma$.*

PROOF. Thesis (*a*) holds obviously, so that $\xi =_{\gamma} \eta_{\gamma}$ for all $\gamma \in \Gamma$. If $\operatorname{mr} \xi < \vartheta$, the condition $\operatorname{mr} \eta_{\gamma} < \vartheta$ for all $\gamma \in \Gamma$ obviously holds. Now assume this condition. Then by (8.3,4) and (9.1) $\varrho\xi \leqslant \bigcup_{\gamma \in \Gamma} \varrho\eta_{\gamma}$. Since $|\Gamma| < \vartheta$ and for all $\gamma \in \Gamma$ we have $\varrho\eta_{\gamma} < \vartheta$, this yields $\varrho\xi < \vartheta$. Hence $\operatorname{mr} \xi \leqslant \varrho\xi < \vartheta$. q.e.d.

THEOR. 9.4. *Assume $\xi_{\gamma} \in QI_{\vartheta}$ for all $\gamma \in \Gamma$. Then (a) there is an η such that $\eta =_{\gamma} \xi_{\gamma}$ for all $\gamma \in \Gamma$; and (b) if the modal rank of ξ_{γ} is $< \vartheta$ for all $\gamma \in \Gamma$, then that of η also is $< \vartheta$.*

PROOF. By $(9.2)_2$ we have $\eta = \{(\gamma, \zeta) | \zeta \in \xi_{\gamma}^{\dagger}\}$, hence thesis (*a*) holds. Thesis (*b*) follows by Theor. 9.3. q.e.d.

* * *

Let $\mathscr{M} = (\mathscr{M}_1, \mathscr{M}_2, \ldots)$ and $\mathscr{V} = (\mathscr{V}_1, \mathscr{V}_2, \ldots)$ be two denumerable successions of elements of QI_{ϑ} $\left(\mathscr{M}, \mathscr{V} \in (QI_{\vartheta})^{\omega}\right)$ with $\mathscr{M}_1 = \Lambda$, to be used as a model and value-assignment (i.e. to assign values to constants and variables) respectively. Then *the intensional designatum* $\tilde{\Delta} = \widetilde{\operatorname{des}}_{\mathscr{M}\mathscr{V}}(\Delta)$ *of the wff* Δ *in* ML^∞ *at* (*the model*) $\mathscr{M}$ *and* (*the value assignment*) $\mathscr{V}$ is defined recursively by the following table where Δ, Δ_1, and Δ_2 are wffs and where the definitions

$$(9.4) \qquad \tilde{\Delta} = \widetilde{\operatorname{des}}_{\mathscr{M}\mathscr{V}}(\Delta)\,, \qquad \tilde{\Delta}_i = \widetilde{\operatorname{des}}_{\mathscr{M}\mathscr{V}}(\Delta_i) \qquad (i = 1, 2)$$

are understood:

Rule	If Δ is	Then $\tilde{\Delta}$ is
(δ_1)	V_i or c_i	V_i or $\mathscr{M}_i$ respectively ($\mathscr{M}_1 = \Lambda$)
(δ_2)	$\Delta_1 = \Delta_2$	The class of the Γ-cases γ for which $\tilde{\Delta}_1 =_\gamma \tilde{\Delta}_2$.
(δ_3)	$\Delta_1 \in \Delta_2$	The class of the Γ-cases γ for which $(\gamma, \tilde{\Delta}_1) \in \tilde{\Delta}_2$
(δ_4)	$\sim\Delta_1$ or $(\Delta_1 \wedge \Delta_2)$	$\Gamma - \tilde{\Delta}_1$ or $\tilde{\Delta}_1 \cap \tilde{\Delta}_2$ respectively.
(δ_5)	$(\forall V_i)\Delta_1$	The class of the Γ-cases γ such that $\gamma \in \in \widetilde{\operatorname{des}}_{\mathscr{M}\mathscr{V}'}(\Delta_1)$ for every value-assignment $\mathscr{V}'$ with $\mathscr{V}'_j = \mathscr{V}_j$ for $j \neq i$.
(δ_6)	$N(\Delta_1)$	Γ if $\tilde{\Delta}_1 = \Gamma$; otherwise Λ.
(δ_7)	$(\imath V_i)(\Delta_1)$	The QI ξ such that for every $\gamma \in \Gamma$, either conditions (1) to (3) below hold: (1) there is a $\mathscr{V}'$ (in $(QI_\vartheta)^\omega$) such that $\mathscr{V}'_j = \mathscr{V}_j$ for $i \neq j$ and $\gamma \in \widetilde{\operatorname{des}}_{\mathscr{M}\mathscr{V}'}(\Delta_1)$, (2) if $\gamma \in \widetilde{\operatorname{des}}_{\mathscr{M}\mathscr{V}''}(\Delta_1)$ and $\mathscr{V}''_j = \mathscr{V}_j$ for $j \neq i$, then $\mathscr{V}''_i =_\gamma \mathscr{V}'_i$, and (3) $\xi =_\gamma \mathscr{V}'_i$ for every $\mathscr{V}'_i$ as in (1); or at least one of conditions (1) and (2) fails to hold and $\xi =_\gamma a_M$—cf. $(8.9)_2$.

It is easy to realize the effective existence of the function $\widetilde{\operatorname{des}}_{\mathscr{M}\mathscr{W}}(\Delta)$ of $\mathscr{M}$, $\mathscr{V}$, and Δ provided one use thesis (*a*) in Theor. 9.3 to check the existence of the QI $\widetilde{\operatorname{des}}_{\mathscr{M}\mathscr{V}}[(\imath V_i)\Delta_1]$.

On the basis of (δ_1) we shall use « Λ » for « c_1 » in wffs in ML^∞. Let $\gamma \in \Gamma$ and $\mathscr{M}, \mathscr{V} \in (QI_\vartheta)^\omega$ with $\mathscr{M}_1 = \Lambda$. Then we say that the matrix p (in ML^∞) *holds* or is *true* in γ at (the model) $\mathscr{M}$ and (the value-assignement) $\mathscr{V}$ in case $\gamma \in \widetilde{\operatorname{des}}_{\mathscr{M}\mathscr{V}}(p)$; in the opposite case we say that p is *false* in γ at $\mathscr{M}$ and $\mathscr{V}$. Furthermore we say that p is *logically valid* ($\Vdash p$) if it is true in every Γ-case and at all models and value assignments; we also say that q is a logical (semantical) consequence of p ($p \Vdash q$) in case at all models and value assignments, q holds in all Γ-cases where p does. This is equivalent to $\Vdash p \supset q$.

10. Some conventions. Introduction of $^{\cap}$, $^{\cup}$, $\exists^{(1)}$, $\exists^{(1)\cap}$, $\exists_1$, and $\exists_1^{\cap}$.

In connection with ML^{∞} we use the metalinguistic notations $\vee$, $\supset$, $\equiv$, $(\exists V_i)$, (V_i), $(\forall V_{r_1}, \ldots, V_{r_n})$ in the usual way, i.e. as in connection with EL^{∞} [n. 2]. Let us add that we use the notations $=^{\cap}$, $\supset^{\cap}$, and $\equiv^{\cap}$ for strict equality, strict implication, and strict equivalence respectively—cf. [GIMC, Defs. 6.3-5] and we use $\in^{\cap}$ in the corresponding way. We mean that e.g. the following definitions are understood

DD10-3 $\quad \Delta =^{\cap} \Delta_1 \equiv_D N\Delta = \Delta_1, \quad p \supset^{\cap} q \equiv_D N(p \supset q), \quad \Delta \in^{\cap} \Delta_1 \equiv_D N\Delta \in \Delta_1 .$

The sign $^{\cup}$ will be used, as in [GIMC], e.g. in the combinations $=^{\cup}$, $\supset^{\cup}$, and $\in^{\cup}$ according to the definitions

DD10.4,5 $\quad \Delta =^{\cup} \Delta_1 \equiv_D \Diamond \Delta = \Delta_1 , \quad p \supset^{\cup} q \equiv_D \Diamond (p \supset q) ,$

D10.6 $\quad \Delta \in \Delta_1 \equiv_D \Diamond \Delta \in \Delta_1 .$

The cohesive powers of $\supset^{\cap}$, $\equiv^{\cap}$, ... and the ones of $\supset^{\cup}$, $\equiv^{\cup}$, ... are equivalent to those of $\supset$, $\equiv$, ... respectively.

We understand that the non existing object a^* and the notions, or metalinguistic predicates, of element (El), individual (In), class (Cl), and set (St) are introduced within ML^{∞} as in EL^{∞}, i.e. by means of DD2.1,3,6-8. This is meant by the sign « (ML^{∞}) » written in these definitions.

Let us now introduce in ML^{∞} the operators $(\exists^{(1)}\mathscr{V})[(\exists^{(1)\cap}\mathscr{V})]$ (there is at most one [a strictly unique] $\mathscr{V}$ such that) and $(\exists_1\mathscr{V})[(\exists_1^{\cap}\mathscr{V})]$ (there is exactly one [a strictly unique] $\mathscr{V}$ such that):

D10.7 $\quad (\exists^{(1)}\mathscr{V})\Phi(\mathscr{V}) \equiv_D \forall_{\mathscr{U}\mathscr{V}} [\Phi(\mathscr{U})\Phi(\mathscr{V}) \supset \mathscr{U} = \mathscr{V}] ,$

D10.8 $\quad (\exists^{(1)\cap}\mathscr{V})\Phi(\mathscr{V}) \equiv_D \forall_{\mathscr{U}\mathscr{V}} [\Phi(\mathscr{U})\Phi(\mathscr{V}) \supset \mathscr{U} =^{\cap} \mathscr{V}] ,$

D10.9,10 $\quad (\exists_1\mathscr{V})p \equiv_D \exists_{\mathscr{V}} p(\exists^{(1)}\mathscr{V})p , \quad (\exists_1^{\cap}\mathscr{V})p \equiv_D \exists_{\mathscr{V}} p(\exists^{(1)\cap}\mathscr{V})p ,$

where $\mathscr{U}$ is the first variable without any occurrences in $\Phi(\mathscr{V})$.

* * *

We consider Convention 3.2 on the combination of *El*, *Cl*, ... with $\cup, \cap$, ... as holding for ML^{∞} also—hence the matrices (3.1) are valid (in ML^{∞}). We keep (in ML^{∞}) the definitions DD3.1-11 for $(\lambda\mathcal{V})p$ and the symbols for class operations, we keep Convention 3.1 on corresponding italicized and non-italicized symbols (such as *El* and El) —cf. fn. 3; furthermore we also accept Convention 3.3 on restricted variables in connection with ML^{∞}, except that the restrictions are meant as holding necessarily, so that (3.5-7) are turned into

$$\text{(10.1)} \quad \begin{aligned} &\forall_X \Phi(X) =_D (\forall V_i)[V_i \in^{\cap} Cl \supset \Phi(V_i)]\,, \\ &(\imath X)\Phi(X) =_D (\imath V_i)\,[V_i \in^{\cap} Cl \wedge \Phi(V_i)]\,, \end{aligned}$$

$$\text{(10.2)} \quad \begin{aligned} &\forall_a \Phi(a) =_D (\forall V_i)[V_i \in^{\cap} El \supset \Phi(V_i)]\,, \\ &(\imath a)\Phi(a) =_D (\imath V_i)\,[V_i \in^{\cap} El \wedge \Phi(V_i)]\,, \end{aligned}$$

$$\text{(10.3)} \quad \begin{aligned} &\forall_x \Phi(x) \equiv_D (\forall V_i)[V_i \in^{\cap} St \supset \Phi(V_i)]\,, \\ &(\imath x)\Phi(x) =_D (\imath V_i)\,[V_i \in^{\cap} St \wedge \Phi(V_i)]\,. \end{aligned}$$

CONVENTION 10.1. *As in* [GIMC] *we call* « *N* » *the modal quantifier. By* « (*N*) » [« ( ) »] *we denote any string of quantifiers that may* [*cannot*] *include* « *N* ».

The extensionalization $\Delta^{(e)}$ of a class, Δ, is defined to be the class of the elements that equal (i.e. are equiextensional with) the elements of Δ; we also define the notion *Ext* of being extensional and the modal striction $\Delta^{\cap}$ [modal sum $\Delta^{\cup}$] of Δ—cf. fn. (5):

$$\text{DD.10.11-14} \quad \begin{cases} \Delta^{(e)} =_D (\lambda\mathcal{U})\,\exists_{\mathcal{W}}(\mathcal{U} = \mathcal{W} \in \Delta)\,, \quad \Delta \in Ext \equiv_D \Delta = \Delta^{(e)}, \\ \Delta^{\cap} =_D (\lambda\mathcal{U})\, N\mathcal{U} \in \Delta\,, \quad \Delta^{\cup} =_D (\lambda\mathcal{U}) \diamond \mathcal{U} \in \Delta\,, \end{cases}$$

where the distinct variables $\mathcal{U}$ and $\mathcal{W}$ are the first variables without free occurrences in Δ.

CONVENTION 10.2. *We shall use the analogues for italicized symbols of notations such as* « $F^{(e)}$ », « $F^{\cap}$ », *and* «$F^{\cup}$ » *as shorts for the corresponding analogues of the R.H.S.s* (*right hand sides*) *of* DD10.11,13,14.

E.g. we understand, under a suitable choice of $\mathscr{W}$, the validity of

$$\text{(10.4)}\quad \begin{aligned} &\Delta \in Cl^{\frown} \cap \mathrm{El}^{\smile} \equiv N\Delta \in Cl \diamond \Delta \in \mathrm{El}\,, \\ &\Delta \in El^{(e)} \equiv_D \exists_{\mathscr{W}} (\mathscr{W} \in El \wedge \mathscr{W} = \Delta)\,. \end{aligned}$$

By $(10.4)_2$ « $El^{(e)}$ » expresses the important notion of equiextensionality with an element. Theor. 9.4 (*b*) yields the first of the assertions

$$\text{(10.5)}\qquad \Vdash F \in^{\frown} El^{(e)} \supset F \in El\,, \qquad \Vdash F \in^{\smile} El \supset F \in^{\frown} El\,,$$

which constitute the main result of this section. The second follows easily from D2.1, D10.3, and rule (δ_3) in n. 9.

11. On collections, relations, $\{\Delta|p\}$, and relational and functional lambda expressions.

We define *collections*, $\{\}$, and *intensional collections* $\{\}^{(i)}$ by

$$\text{D11.1}\qquad \{\Delta_1, \ldots, \Delta_n\} =_D (\lambda\mathscr{A}) \bigvee_{i=1}^{n} \mathscr{A} = \Delta_i\,,$$

$$\text{D11.2}\qquad \{\Delta_1, \ldots, \Delta_n\}^{(i)} =_D (\lambda\mathscr{A}) \bigvee_{i=1}^{n} \mathscr{A} = {}^{\frown}\Delta_i\,,$$

where $\mathscr{A}$ is a suitable variable—cf. Def. 18.13 in [GIMC]. A satisfactory modal analogue of Kuratowski's definition of (ordered) n-tuple, $(\Delta_1, \ldots, \Delta_n)$, can be based on $\{\}^{(i)}$:

$$\text{D11.3}\qquad (\Delta_1, \Delta_2) =_D \{\{\Delta_1\}^{(i)}, \{\Delta_1, \Delta_2\}^{(i)}\}^{(i)},$$

$$\text{D11.4}\qquad (\Delta_1, \ldots, \Delta_{n+1}) =_D \begin{cases} \Delta_1 & (n = 0)\,, \\ ((\Delta_1, \ldots, \Delta_n), \Delta_{n+1}) & (n > 0)\,. \end{cases}$$

In Chap. 3 we shall prove some basic syntactical theorems on the notions being introduced now. The corresponding semantical theorems are immediate consequences of the former. This induces us to avoid presenting many proofs. For instance we may quickly say that $(\Delta_1, \ldots, \Delta_n) = (\Delta_1', \ldots, \Delta_n')$ yields $\Delta_i = {}^{\frown}\Delta_1'$ $(i = 1, \ldots, n)$ and that, since $|\Gamma|\geqslant 2$, $\{a_1, \ldots, a_n\}$ is a proper class while $\{\Delta_1, \ldots, \Delta_n\}^{\frown}$ and $\{\Delta_1, \ldots, \Delta_n\}^{(i)}$ are sets.

We consider the definitions D4.2,3 of $\mathscr{F}'a$ and $\mathscr{R}''\mathscr{C}$ as holding in ML^∞ also; D4.4-6 have to be turned into D11.5-7 below:

D11.5 $$\{\Delta|p\} = (\lambda\mathscr{V})\exists_{\mathscr{A}_1,\ldots,\mathscr{A}_n}(p\wedge\mathscr{V} = {}^{\frown}\Delta)$$

where the distinct variables $\mathscr{A}_1$ to $\mathscr{A}_n$ are the free variables in the term Δ, so that these variables are bound in $\{\Delta|p\}$.

Now we can define the n-ary relational and n-ary functional lambda expressions:

D11.6 $$(\lambda a_1, \ldots, a_n)\, p =_D \{(a_1, \ldots, a_n)|p\}\,,$$

D11.7 $$(\lambda a_1, \ldots, a_n)\Delta =_D (\lambda a_1, \ldots, a_n, b)\, b = {}^{\frown}\Delta\,,$$

where $a_1, \ldots, a_n$ are distinct variables and b is the first variable that does not occur free in the term Δ and is distinct from a_1 to a_n.

The lambda expressions above have the usual basic substitution properties—see (18.3) in Part II.

We can now define the cartesian product $X\times Y$ of the classes X and Y, and the n-th cartesian power X^n of X:

D11.8 $$\Delta\times\Delta_1 = (\lambda a, b)(a\in\Delta\wedge b\in\Delta_1)\,,$$

D11.9 $$\Delta^1 =_D \Delta\,, \qquad \Delta^{n+1} =_D \Delta^n\times\Delta \qquad (n = 1, 2, \ldots)$$

where the variables a and b are suitably chosen. Of course «$\Delta\subseteq V^n$» *means* (*in* ML^∞) *that Δ is an n-ary relation.* We now define the *converse* $\breve{\Delta}$ (or Δ^{-1}) of the relation Δ, *domain* (Dmn), *range* (Rng) and *field* (Fld):

D11.10 $$\Delta^{-1} =_D \breve{\Delta} =_D \{(a, b)|(b, a)\in\Delta\}\,,$$

D11.11,13 $$\begin{cases} \mathrm{Dmn}\, R =_D (\lambda a)\,\exists_b (a, b)\in R\,, \qquad \mathrm{Rng}\, R =_D \mathrm{Dmn}\, \breve{R}\,, \\ \mathrm{Fld}\, R =_D \mathrm{Dmn}\, R\cup\mathrm{Rng}\, R\,. \end{cases}$$

Now we define the restriction $\mathscr{A}\upharpoonleft\mathscr{R}$ [$\mathscr{R}\upharpoonright\mathscr{A}$] of (the relational part of) $\mathscr{R}$ to the class $\mathscr{A}$, in its first [second] members:

D11.14,15 $$\mathscr{A}\upharpoonleft\mathscr{R} =_D (\mathscr{A}\times V)\cap\mathscr{R}\,, \qquad \mathscr{R}\upharpoonright\mathscr{A} = (V\times\mathscr{A})\cap\mathscr{R}\,.$$

12. Inner identity of n-tuples and modally constant, modally separated, and absolute n-ary attributes in ML^{∞}.

We define Δ and Δ_1 are *innerly identical n-tuples*,

D12.1 $$\Delta =_n \Delta_1 \equiv_D \exists_{a_1,\ldots,a_n\, b_1,\ldots,b_n}[\Delta = (a_1, \ldots, a_n) \wedge \wedge \Delta_1 = (b_1, \ldots, b_n) \bigvee_{i=1}^{n} a_i = b_i]$$

where the a's and b's are suitable distinct variables, because intuitively the condition $\Delta =_n \Delta_1$ is useful, it follows from $\Delta = \Delta_1$ but by no means does it imply $\Delta = \Delta_1$ unless $n = 1$.

The notion *Ext* of extensionality [D10.12], which is much used, is completely useless in connection with relations by what we said about n-tuples ($n > 1$). The same holds for $\Delta^{(e)}$ in case $\Delta \subseteq V^n$ ($n > 1$). Therefore we now define *n-ary extensionalization* ($F^{(ne)}$) and *n-ary extensionality* (Ext_n) on the basis of D12.1—cf. DD10.11,12:

DD12.2,3 $$\Delta^{(ne)} =_D (\lambda a)\exists_b (a =_n b \wedge b \in \Delta)\,, \quad \Delta \in Ext_n \equiv_D \Delta = \Delta^{(ne)}$$

where b is the first variable that does not occur free in Δ.

The notions $\Delta^{(ne)}$, $\Delta^{\frown}$, and $\Delta^{\smile}$ [DD10.13,14] are satisfactory and are in accordance with the corresponding definitions in [GIMC] also in case Δ is an n-ary relation ($n > 0$). The preliminary definition D12.1 also allows us to introduce the notions of a *modally constant attribute* (*MConst*) and a *modally separated* [*absolute*] *n-ary attribute* ($MSep_n$ [Abs_n]) in substantial accordance with [GIMC, Defs. 13.1 and 18.8,9]:

D12.4 $$\Delta \in MSep_n \equiv_D \Delta \subseteq V^n \forall_{a,b}(a, b \in \Delta \diamond a =_n b \supset a =^{\frown} b)\,,$$

D12.5,6 $$\Delta \in MConst \equiv_D \Delta \in Cl^{\frown} \wedge \Delta^{\smile} = \Delta^{\frown}, \quad Abs_n =_D MConst \cap MSep_n\,.$$

In a similar way D12.1 allows us to define in ML^{∞} the notions of quasi *modally constant attributes* (*QMConst*) and *quasi modally separated* [*absolute*] *n-ary attributes* ($QMSep_n$ [$QAbs_n$]) in accordance with [GIMC, Defs. 24.1-3]. Obvious semantical analogues of the syntactical theorems on *MConst*, *MSep*, and *Abs* proved in [GIMC, n. 41] hold in ML^{∞}.

CONVENTION 12.1. (*a*) *We shall use* « *Ext* », »*MSep* », *and* « *Abs* » *for* « Ext_1 », « $MSep_1$ », *and* « Abs_1 » *respectively.*

(*b*) « $Pred \in Ext$ » [« $Pred \in MConst$ »], *where* « *Pred* » *stands for any italicized predicate such as El or Cl, is an abbreviation of* « $Pred = Pred^{(e)}$ », [« $Pred^{\frown} = Pred^{\smile}$ »]—cf. Conv. 3.2. *The analogue holds for* e.g. *MSep and Abs.*

Now, for instance the obvious semantical analogues for ML^{∞} of [*GIMC*, Th. 41.1, (IV), (VIII), (IX)] can be expressed as follows:

$$(12.1) \qquad \Vdash MConst \in Abs\,, \qquad \Vdash Abs_n \in Abs\,, \qquad \Vdash MSep_n \in Ext\,.$$

At this point let us mention the analogues of [GIMC, The. 41.1, (V) (VI)]:

$$(12.2) \qquad \begin{array}{l} \Vdash F \subseteq G \wedge F \in MConst \wedge G \in Abs \supset F \in Abs\,; \\ \Vdash \Phi \subseteq MConst \wedge \Phi \in MConst \supset \Phi \in Abs\,. \end{array}$$

Remark that $(12.2)_{1\cdot 2}$ are meaningful and true even if F, G, and Φ are replaced by italicized predicates.

13. Functions.

We want to define (a *general or intensional*) *n-ary function* (Fn_n) (9) and the more particular notions of an *extensionally univocal* (or *extensionally invariant*) *n-ary function* (Fnc_n), and an *absolute n-ary function* (AFn_n). To this end we first define (*intensional*) *univocality* (Un) and n-ary *extensional univocality*—or *invariance*—(EUn_n)—cf. D12.1:

$$\text{DD13.1,2} \qquad \Delta \in \left\{ \begin{array}{c} Un \\ EUn_n \end{array} \right. \equiv_D \forall_{abcd}[(a, b), (c, d) \in \Delta \wedge \left\{ \begin{array}{l} a =^{\frown} c \supset b =^{\frown} d] \\ a =_n c \supset b = d]\,, \end{array} \right.$$

$$\text{DD13.3-5} \qquad \left\{ \begin{array}{l} \Delta \in Fn_n \equiv_D \Delta \subseteq V^{n+1} \wedge \Delta \in Un \cap MConst\,, \\ Fnc_n =_D Fn_n \cap EUn_n\,, \qquad AFn_n =_D Fn_n \cap MSep_{n+1} \end{array} \right.$$

(9) D13.3 is in accordance with the modal definition of functions in terms of relations proposed in [GIMC, n. 14, p. 54].

and we shall use the notations EUn, Fn, Fnc, and AFn for EUn_1 to AFn_1 respectively.

By D13.3 and D11.11-13

$$(13.1)\qquad \begin{array}{l} \Vdash \mathscr{F} \in Fn_n \supset (\text{Dmn}\,\mathscr{F},\ \text{Rng}\,\mathscr{F},\ \text{Fld}\,\mathscr{F} \in MConst) \wedge \text{Dmn}\,\mathscr{F} \subseteq V^n, \\ \Vdash Fn_n \in MConst\,. \end{array}$$

By DD13.3,5 and D12.6 we deduce the first of the theorems

$$(13.2)\qquad \begin{cases} \Vdash AFn_n \subset Abs_{n+1}\,, & \Vdash Abs_{n+1} \cap Un \subset EUn_{n+1}^{\frown}\,, \\ \Vdash AFn_n \subset Fnc_n\,, & \Vdash Fnc_n \notin MConst\,. \end{cases}$$

The second follows easily from DD12.4-6 and DD13.1,2 while $(13.2)_{1,2}$ yield the third. To prove $(13.2)_4$ we remark that

$$\Vdash x \neq y \wedge x =^{\smile} y \supset F \in (Fnc - Fnc^{\frown})$$

where

$$F =_D \{(x, 0),\ (y, 1)\}^{(i)} \qquad \text{and} \qquad 1 =_D \{0\}^{(i)}\,.$$

The notion AFn_n holds for all mathematical n-ary functions, and other functions too. Ordinary non-mathematical n-ary functions—such as the i-th co-ordinate $x_i(M, \tau)$ of the particle M at the instant τ, in the Galileian frame $\mathscr{G}$—are extensionally invariant and by this are included in Fnc_n. Of course by D12.3 and DD13.1-4 the condition $\mathscr{F} \in Fn \cap Ext_{n+1}$ does not at all L-imply $\mathscr{F} \in Fnc_n$. In fact

$$(13.3)\qquad \Vdash \mathscr{F} \in MConst \cap Ext_n \supset \mathscr{F} =^{\frown} \Lambda \vee \mathscr{F} =^{\frown} V^n\,.$$

The notion Fn_n holds for the most general n-ary (intensional) functions such as $\mathscr{F}$,—cf. (13.6,7) below. It is a cumulative analogue of the intensional notion of an ary function used in [GIMC].

To show some examples, we first introduce the identity relation $\boldsymbol{I}$ (among elements) and the successor function $\mathfrak{S}$—cf. D4.1:

$$(13.4)\qquad \boldsymbol{I} =_D (\lambda a, b)a = b\,, \quad \mathfrak{S} =_D (\lambda X)(X \cup \{X\}^{(i)}) \quad [\text{D11.7}]\,.$$

It is easy to see that

$$(13.5)\qquad \Vdash \boldsymbol{I} \in EUn \cap \overline{Un} \cap \overline{Fn}\,, \quad \Vdash \boldsymbol{I}^{\frown} \in Fnc\,, \quad \Vdash \mathfrak{S} \in Fn \cap \overline{Fnc}\,.$$

Now we state that the constant $c_{9(n+1)}$ denotes (in ML^{∞}) the natural number n, but we simply use « n » for it. We understand D2.4 and the definition $n+1 =_D \mathfrak{S}n$ which can be written as follows

$$(13.6)\qquad 0 =_D \Lambda\,, \quad 1 =_D \{\Lambda\}^{(i)}\,, \quad n+1 =_D n \cup \{n\}^{(i)}\,.$$

As another example we consider the (intensional) function

$$(13.7)\qquad \Vdash \mathscr{F}_1 =_D (\lambda a, b)(a =^{\frown} 0 \wedge b =^{\frown} 1 \vee \sim a =^{\frown} 0 \wedge b =^{\frown} 0)\,,$$

by which $(13.8)_1$ below holds.

$$(13.8)\qquad \Vdash \mathscr{F}_1 \in Fn \cap \overline{Fnc} \cap \overline{EUn}\,, \qquad \Vdash EUn \not\subset Un\,.$$

We deduce $(13.8)_2$ from $(13.5)_1$.

14. Equipotence. Hints at intrinsic extensionality and ordinals.

With a view to defining equipotence let us observe, first, that by requiring that $\mathscr{F}$ and $\breve{\mathscr{F}}$ should have the property Fn, Fnc, or AFn we consider three notions of a $1-1$ *mapping*. Second, we take care of classes X and Y that are *iso-extensionalizable*: $X^{(e)} = Y^{(e)}$. Now let us define (*intensional*) *equipotence*, $\approx$, *total equipotence* $\approx^{(t)}$, and *extensional equipotence*, $\approx^{(e)}$:

$$\text{DD14.1,2}\quad \left.\begin{matrix} X \approx Y \\ X \approx^{(t)} Y \end{matrix}\right\} \equiv_D \exists_{\mathscr{F}} \left(\mathscr{F}, \breve{\mathscr{F}} \in \left\{ \begin{matrix} Fn \\ Fnc \end{matrix} \right. \wedge X = \mathrm{Dmn}\,\mathscr{F} \wedge Y = \mathrm{Rng}\,\mathscr{F} \right)$$

$$\text{D14.3}\quad X \approx^{(e)} Y \equiv_D \exists_{\mathscr{F}} [\mathscr{F}, \breve{\mathscr{F}} \in Fnc \wedge X^{(e)} = (Dmn\,\mathscr{F})^{(e)} \wedge Y^{(e)} = (Rng\,\mathscr{F})^{(e)}].$$

The replacement of Fnc with AFn in D14.3 would not alter the notion $\approx^{(e)}$ which is perhaps the most natural among the three extensional equivalence relations that were proposed above as modal explicata of $1-1$ mappings. However any non-empty set x is extensionally equipotent with a proper class, $x^{(e)}$, via $x \times x$. Instead neither $\approx^{(t)}$ nor $\approx$ hold between a set and a proper class. Let us add that $\approx$ to $\approx^{(e)}$ are related by the theorems

$$(14.1)\qquad \Vdash \approx^{(t)} \subset \approx^{(e)}\,, \quad \Vdash \approx^{(t)} \subset \approx\,, \quad \Vdash \sim (\approx \subseteq \approx^{(e)})\,.$$

Furthermore, after defining the intensionalization $\mathscr{A}^{(I)}$ of $\mathscr{A}$ by

D14.4 $$\mathscr{A}^{(I)} =_D \{\{a\}^{(i)} | a \in \mathscr{A}\},$$

we can assert that

(14.2) $$\Vdash X_1 \approx X_2 \equiv X_1^{(I)} \approx X_2^{(I)} \equiv X_1^{(I)} \approx^{(t)} X_2^{(I)},$$

where the first occurrence of $\approx$ cannot be replaced by $\approx^{(t)}$ or $\approx^{(e)}$.
The fact that $\Vdash (\ )x \neq \Lambda \supset x^{(e)} \notin El$ and more generally

(14.3) $$\Vdash V^n \cap x \neq \Lambda \supset x^{(ne)} \notin El \quad [\mathrm{D}12.2]$$

appears troublesome especially within the theory of cardinals. Therefore it is natural to try and schematize into $\mathscr{R}^{(ne)\frown}$ the intuitive (self-explanatory) notion of the *n-ary intrinsic extensionalization* of $\mathscr{R}$; however the second of the theorems

(14.4) $$\Vdash (\ )x^{(ne)\frown} \in St, \quad \Vdash \Lambda =^{\smile} V^n \cap X \supset \Lambda = X^{(ne)\frown}$$

is not very satisfactory. Therefore we shall also consider the n-ary attributes $\mathscr{R}^{\smile(ne)\frown}$ and $\mathscr{R}^{\smile(ne)\frown} \cap \mathscr{R}^{(ne)}$ [n. 26]. The latter seems to me satisfactory to the above end.

* * *

We spoke of equipotence with a view to the applications of the theory of (transfinite) cardinal numbers, to be developed syntactically later. Our definition of cardinal shall not involve equipotence in accordance with [IST]. In order to hint briefly at the theory of ordinal and cardinal numbers based on ML^{∞}(or MC^{∞}) to be presented in Part 3, we accept definition D5.4 of ε-transitivity (ε-transitivity (ε-*Trns*) and we say that $\mathscr{A}$ is an *ordinal* (*class*) (*Ord*) iff $\mathscr{A}$ and its arbitrary element (if any) are classes and are both ε-transitive and modally constant:

D14.5 $$\mathscr{A} \in Ord \equiv_D \mathscr{A} \in MConst \cap \varepsilon\text{-}Trns \wedge \mathscr{A} \subseteq MConst \cap \varepsilon\text{-}Trns .$$

By Conv. 3.1 Ord means *ordinal number.* One can prove

(14.5) $$\Vdash 0, n, \mathrm{Ord} \in Ord, \quad \Vdash \mathrm{Ord} \subset Ord, \quad \Vdash \mathrm{Ord}, Ord \in Abs .$$

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