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PRIME DIVISORS OF q -BINOMIAL COEFFICIENTS

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Introduction.

1. The q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{j=1}^k \frac{q^{n-j+1} - 1}{q^j - 1}$$

for q an indeterminate and n a non-negative integer. It is known that the q -binomial coefficient is a polynomial in q and that for $q=1$ it reduces to the ordinary binomial coefficient. For additional properties and references see [2].

In this paper we generalize some recently proved results for ordinary binomial coefficients to q -binomial coefficients. In section 2 we consider the problem of determining if there are q -binomial coefficients divisible by a specified factor, and we generalize a theorem of Simmons [5], who considered this problem for ordinary binomial coefficients. In section 3 we find formulas for the number of q -binomial coefficients divisible by a fixed power of a prime, thus generalizing results of Carlitz [1] and the author [3], [4].

In section 2 we assume the following, which we call conditions (1.1).

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- (1.1) Let p_1, \dots, p_k be prime numbers and let q be a rational number such that when q is reduced to its lowest terms, p_i does not divide the numerator or denominator for $i=1, \dots, k$. Let $e(i)$ be the smallest positive integer such that $q^{e(i)} \equiv 1 \pmod{p_i}$ and let $p_i^{h(i)}$ be the highest power of p_i dividing $q^{e(i)} - 1$. If $p_i^{h(i)} = 2$, let $p_i^{t(i)}$ be the highest power of p_i dividing $q + 1$.

In section 3 we assume:

- (1.2) Assume (1.1) holds, with $k=1$. We use the notation $p=p_1$, $e=e(1)$, $h=h(1)$, $t=t(1)$.

Throughout this paper we shall use the following rule, which is due to Fray [2], for determining the highest power of a prime p dividing a q -binomial coefficient. Suppose conditions (1.2) hold. Then any positive integer n can be written uniquely as

$$(1.3) \quad n = a_0 + e(a_1 + a_2p + \dots + a_k p^{k-1})$$

where

$$0 \leq a_0 < e, \quad 0 \leq a_i < p \quad (i=1, \dots, k).$$

Similarly

$$(1.4) \quad \begin{aligned} r &= b_0 + e(b_1 + b_2p + \dots + b_k p^{k-1}) \\ n+r &= c_0 + e(c_1 + c_2p + \dots + c_k p^{k-1}). \end{aligned}$$

We can write

$$\begin{aligned} a_0 + b_0 &= \varepsilon_0 e + c_0 \\ \varepsilon_0 + a_1 + b_1 &= \varepsilon_1 p + c_1 \\ &\dots \\ \varepsilon_{k-2} + a_{k-1} + b_{k-1} &= \varepsilon_{k-1} p + c_{k-1} \\ \varepsilon_{k-1} + a_k + b_k &= c_k \end{aligned}$$

where each ε_i is either zero or one. If $p^h > 2$ then the highest power of p dividing $\begin{bmatrix} n+r \\ r \end{bmatrix}$ is p^s where

$$(1.5) \quad s = \varepsilon_0 h + \varepsilon_1 + \dots + \varepsilon_{k-1}.$$

If $p^h=2$ then the highest power of p dividing $\begin{bmatrix} n+r \\ r \end{bmatrix}$ is p^s where

$$(1.6) \quad s = \varepsilon_1 t + \varepsilon_2 + \dots + \varepsilon_{k-1}.$$

2. Specified divisors of q -binomial coefficients. Simmons [5] has shown that if r and N are any positive integers then there are infinitely many $m \geq r$ such that

$$\left(\begin{bmatrix} m \\ r \end{bmatrix}, N \right) = 1.$$

This result can easily be generalized to q -binomial coefficients.

THEOREM 2.1. Let N and r be positive integers and let p_1, \dots, p_k be the prime divisors of N . Assume conditions (1.1) hold. Then there are infinitely many $m \geq r$ such that

$$\left(\begin{bmatrix} m \\ r \end{bmatrix}, N \right) = 1.$$

PROOF. For each p_i we write

$$(2.1) \quad r = b_0 + e_i(b_1 + b_2 p_i + \dots + b_{f(i)} p_i^{f(i)-1})$$

where

$$0 \leq b_0 < e_i, \quad 0 \leq b_j < p_i \quad (j=1, \dots, f(i)).$$

Let d be any positive integer and let

$$m = r + d \prod_{i=1}^k e_i p_i^{f(i)}.$$

By (1.5) and (1.6) it is clear that

$$\left(\begin{bmatrix} m \\ r \end{bmatrix}, N \right) = 1$$

Theorem 2.1 says that for arbitrary primes p_1, \dots, p_k there are an infinite number of positive integers m such that p_i^0 is the highest power of p_i dividing $\begin{bmatrix} m \\ r \end{bmatrix}$ for $i=1, \dots, k$, provided (1.1) holds. It seems natural to ask the following question: If p_1, \dots, p_k are arbitrary primes and $g(1), \dots, g(k)$ are arbitrary non-negative integers, are there an infinite number of positive integers m such that $r_i^{g(i)}$ is the highest power of p_i dividing $\begin{bmatrix} m \\ r \end{bmatrix}$? We shall prove that the answer is always yes for ordinary binomial coefficients. It is clear that the answer is not always yes for q -binomial coefficients, however. For example, if $p_1=3, q=8, g(1)=1$ and $r=1$, then $e(1)=2$ and since

$$\begin{bmatrix} m \\ 1 \end{bmatrix} = (8^m - 1)/7$$

it is clear that $3^{e(1)}$ is not the highest power of 3 dividing $\begin{bmatrix} m \\ 1 \end{bmatrix}$ for any m . In fact, by (1.5), if $p_1^{h(1)} > 2, r < e(1)$ and $0 < g(1) < h(1)$, then $p_1^{g(1)}$ will not be the highest power of p_1 dividing $\begin{bmatrix} m \\ r \end{bmatrix}$ for any m . By (1.6), if $p_1^{h(1)} = 2, r = 1$, and $0 < g(1) < t(1)$ then $p_1^{t(1)}$ will not be the highest power of p_1 dividing $\begin{bmatrix} m \\ r \end{bmatrix}$ for any m .

THEOREM 2.2. Let r be a positive integer, p_1, \dots, p_k prime numbers and $g(1), \dots, g(k)$ non-negative integers. Assume conditions (1.1) hold. If $p_i^{h(i)} > 2$, assume $r \geq e(i)$ and/or $g(i) \geq h(i)$. If $p_i^{h(i)} = 2$, assume $r > 1$ and/or $g(i) \geq t(i)$. Then if $(e(i), e(j)) = 1$ for $i \neq j$ there are infinitely many positive integers m such that the highest power of p_i dividing $\begin{bmatrix} m \\ r \end{bmatrix}$ is $p_i^{g(i)}$ ($i=1, \dots, k$).

PROOF. We again use expansions (2.1), assuming $b_{f(i)} \neq 0$. If $r = b_0$ we say that $f(i) = 0$. Let

$$\begin{aligned} S_i &= r \text{ if } g(i) = 0 \\ &= b_0 + e_i(b_1 + \dots + b_{f(i)-1}p_i^{f(i)-2} + p_i^{f(i)+g(i)-1}) \\ &\text{if } f(i) \geq 2, g(i) > 0, \end{aligned}$$

$$\begin{aligned} &= b_0 + e_i p_i^{g(i)} \text{ if } f(i) = 1, g(i) > 0, p_i^{h(i)} > 2, \\ &= e_i p_i^{g(i) - h(i)} \text{ if } f(i) = 0, g(i) \geq h(i), p_i^{h(i)} > 2, \\ &= p_i^{g(i) - t(i) + 1} \text{ if } f(i) = 1, g(i) \geq t(i), p_i^{h(i)} = 2. \end{aligned}$$

By the Chinese Remainder Theorem, the system of congruences

$$(2.2) \quad x \equiv S_i \pmod{e_i p_i^{(i) + g(i) + 1}}$$

has an infinite number of positive simultaneous solutions. If m is such a solution, it is clear by (1.5) and (1.6) that $p_i^{g(i)}$ is the highest power of p_i dividing $\begin{bmatrix} m \\ r \end{bmatrix}$.

Theorem 2.2 could be stated more generally by replacing the condition that $(e(i), e(j)) = 1$ for $i \neq j$ by the condition that congruences (2.2) have a simultaneous solution.

COROLLARY. Let $N = p_1^{g(1)} \dots p_k^{g(k)}$ be any positive integer and let r be a positive integer. If the hypotheses of Theorem 2.2 are satisfied, then there are an infinite number of positive integers m such that

$$\begin{bmatrix} m \\ r \end{bmatrix} = NM, M \not\equiv 0 \pmod{p_i} \quad (i = 1, \dots, k).$$

We note that the conclusions of Theorem 2.2 and its corollary always hold for ordinary binomial coefficients.

3. The number of q -binomial coefficients divisible by a fixed power of a prime. L. Carlitz [1] has defined $\theta_j(n)$ as the number of binomial coefficients

$$\binom{n}{s} \quad (s = 0, 1, \dots, n)$$

divisible by exactly p^j , where p is a prime number, and he has found formulas for $\theta_j(n)$ for certain values of j and n . The writer [3], [4] has also considered this problem. In particular, if we write

$$n = c_0 + c_1p + \dots + c_r p^r (0 \leq c_i < p)$$

then we have the formulas

$$(3.1) \quad \theta_0(n) = (c_0 + 1)(c_1 + 1) \dots (c_r + 1)$$

$$(3.2) \quad \theta_1(n) = \sum_{i=0}^{r-1} (c_0 + 1) \dots (c_{i-1} + 1)(p - c_i - 1)c_{i+1}(c_{i+2} + 1) \dots (c_r + 1).$$

Assume that we have conditions (1.2) and let $\alpha_j(n)$ denote the number of q -binomial coefficients

$$\begin{bmatrix} n \\ s \end{bmatrix} \quad (s = 0, 1, \dots, n)$$

divisible by exactly p^j . Fray [2] has proved that if n has expansion (1.3) then

$$\alpha_0(n) = (a_0 + 1)(a_1 + 1) \dots (a_k + 1)$$

which is a special case of our next theorem.

We note that if $p > 2$ and $j > h + k + 1$, or if $p = 2$ and $j > y + k - 2$, where y is t if $p^h = 2$ and $y = 1$ if $p = 2, h > 1$, then

$$\alpha_j(n) = 0.$$

In the next theorem we assume $j \leq h + k - 1$ if $p > 2$ and $j \leq y + k - 2$ if $p = 2$.

THEOREM 3.1. Assume (1.2) holds and n is a positive integer having expansion (1.3). For $m = 1, \dots, k$ define

$$n_m = a_m + a_{m+1}p + \dots + a_k p^{k-m}.$$

If $p > 2$ then

$$\begin{aligned} \alpha_j(n) = & (a_0 + 1)\theta_j(n_1) + (e - a_0 - 1)a_1\theta_{j-h}(n_2) + \\ & + (e - a_0 - 1) \sum_{m=1}^{j-h} (p - a_1) \dots (p - a_m)a_{m+1}\theta_{j-h-m}(n_{m+2}). \end{aligned}$$

If $p = 2, h > 1$, then

$$\alpha_j(n) = \theta_j(n).$$

If $p^h=2$, then

$$\alpha_j(n) = (a_1 + 1)\theta_j(n_2) + (1 - a_1)a_2\theta_{j-t}(n_3) + \sum_{m=2}^{j-t+1} (1 - a_1)(2 - a_2) \dots (2 - a_m)a_{m+1}\theta_{j-t-m+1}(n_{m+2}).$$

PROOF. If $p > 2$ we use (1.5). Let r have expansion (1.4). If $\begin{bmatrix} n \\ r \end{bmatrix}$ is to be divisible by exactly p^j then we consider the possibilities for ϵ_i . If $\epsilon_0=0$ there are a_0+1 choices for b_0 , namely

$$b_0 = 0, 1, \dots, a_0,$$

and clearly, by (1.5), there are

$$(a_0 + 1)\theta_j(n_1)$$

ways of writing r . If $\epsilon_0 = \dots = \epsilon_m = 1$, $\epsilon_{m+1} \neq 1$, then there are $e - a_0 - 1$ choices for b_0 ,

$$b_0 = a_0 + 1, a_0 + 2, \dots, e - 1,$$

there are $p - a_i$ choices for b_i , $i = 1, \dots, m$,

$$b_i = a_i, a_i + 1, \dots, p - 1,$$

and there are a_{m+1} choices for b_{m+1} ,

$$b_{m+1} = 0, \dots, a_{m+1} - 1.$$

By (1.5) it is clear that the number of choices for r is

$$(e - a_0 - 1)(p - a_1) \dots (p - a_m)a_{m+1}\theta_{j-h-m}(n_{m+2}).$$

Note that we let $\theta_{j-h-k+1}(n_{k+1}) = 1$ if $j - h - k + 1 \geq 0$. The proof is similar for the case $p = 2$. Note that for this case $e = 1$, $n = n_1$.

For example, if $p > 2$ and either $h > j$ or $a_1 = \dots = a_{j-h+1} = 0$ then

$$\alpha_j(n) = (a_0 + 1)\theta_j(n_1).$$

Thus if $p > 2$ and $s > j$

$$\alpha_j(ep^s) = \theta_j(p^s) = p^{j-1}(p-1),$$

$$\alpha_j(eap^s) = \theta_j(ap^s) = ap^{j-1}(p-1) \quad (0 \leq a < p).$$

Also

$$\begin{aligned} \alpha_1(n) &= (a_0 + 1)\theta_1(n_1) \text{ if } h > 1, p > 2, \\ &= (a_0 + 1)\theta_1(n_1) + (e - a_0 - 1)a_1\theta_0(n_2) \text{ if } h = 1, p > 2, \\ &= (a_1 + 1)\theta_1(n_2) \text{ if } p^h = 2, \\ &= \theta_1(n) \text{ if } p = 2, h > 1. \end{aligned}$$

Thus by (3.1) and (3.2) we have for $0 \leq a_0 < e$, $0 \leq a_i < p$ ($i = 1, 2$),

$$\begin{aligned} \alpha_1(a_0 + e(a_1 + a_2p)) & \\ &= (a_0 + 1)(p - a_1 - 1)a_2 \text{ if } p > 2, h > 1, \\ &= (a_0 + 1)(p - a_1 - 1)a_2 + (e - a_0 - 1)a_1(a_2 + 1) \text{ if } p > 2, h = 1, \\ &= 0 \text{ if } p^h = 2, \\ &= (1 - a_1)a_2 \text{ if } p = 2, h > 1. \end{aligned}$$

By using Theorem 3.1 and the formulas for $\theta_j(n)$ found in [1], [3] and [4], we could write out many more formulas for $\alpha_j(n)$.

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