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MAXIMUM PRINCIPLES FOR SOME QUASILINEAR SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We present proofs and extensions of a maximum principle announced by Horáček and Výborný [1] for a quasilinear, non-hyperbolic, second order partial differential operator of the form

$$\sum a_{ij}(x,\ u,\ \text{grad}\, u)D_{ij}u - a(x,\ u,\ \text{grad}\, u).$$

The assumptions on the coefficients are less stringent than previously required. From this basic theorem, we derive an interior maximum principle, a boundary maximum principle, and a uniqueness theorem for the elliptic case.

1. Introduction.

Horáček and Výborny [1] announced a maximum principle for a quasilinear, non-hyperbolic second order partial differential operator of the form

$$\sum a_{ij}(x,\ u,\ \text{grad}\, u)D_{ij}u - a(x,\ u,\ \text{grad}\, u).$$

This theorem generalized results of Redheffer [2] and Výborný [3] for such equations. Redheffer required that the differences $|a_{ij}(x, u, 0) - a_{ij}(x, u, \text{grad}\, u)|$ and $|a(x, u, 0) - a(x, u, \text{grad}\, u)|$ be bounded by a function g of $|\text{grad}\, u|$ that was positive, increasing, and satisfied the condition

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$$\int_0 \frac{1}{g} = \infty .$$

Výborný connected these differences with Redheffer's potentials $c(x)$, by assuming the existence of a smooth positive function τ on $\overline{G}$ that was zero on the boundary. He proved a maximum principle for a boundary point that required the above differences to be bounded by a product of the above function g and a function B of τ that was positive and satisfied

$$\int_0 B(t)dt < \infty .$$

In [1], this was carried further: the differences were bounded by a continuous function f of τ and $|\operatorname{grad} u|$, satisfying, among other things, the condition that the initial value problem $\varphi' = cf(t, \varphi)$, $\varphi(0) = 0$ had unique solution zero on some interval $[0, A]$, where c was a certain constant. In the present paper, we improve Theorem 1 of [1], and also prove an interior maximum principle, a boundary maximum principle, and a uniqueness theorem. The uniqueness theorem corrects that announced in [1].

2. Notation, definitions, and conditions.

We list the following for later reference.

2.1. Let a, A, and b be real numbers and let F be a real-valued function defined on $(a, \infty) \times [0, \infty)$. A function φ will be considered a solution of the initial value problem $\varphi'(t) = F(t, \varphi(t))$, $\varphi(a) = b$ on the interval $[a, A]$ if φ is continuous on $[a, A]$, differentiable on $(a, A]$, $\varphi(a) = b$, and $\varphi'(t) = F(t, \varphi(t))$ for all $t \in (a, A]$.

As usual, derivatives at endpoints of intervals are interpreted as one-sided derivatives.

2.2. Throughout this paper, we shall let f denote a continuous non-negative function on $(0, \infty) \times [0, \infty)$ satisfying

(i) $f(t, 0) = 0$ for all $t \in (0, \infty)$,

(ii) there exists $\delta>0$ such that for each $t\in(0, \delta)$, we have

$$0\leqq \liminf_{\substack{\varphi_2\to 0_+ \\ 0\leqq\varphi_1\leqq\varphi_2}} \frac{f(t, \varphi_2)-f(t, \varphi_1)}{\varphi_2}\leqq\infty,$$

(iii) there exist constants $A>0$ and $c>0$ such that for each $\varepsilon>0$, there is a solution φ_ε to the problem $\varphi'=cf(t, \varphi)$ on $[0, A]$ with $0<\varphi_\varepsilon(t)\leqq\varepsilon$ for $t\in[0, A]$.

Notice that condition (ii) holds if

(ii*) f is non-decreasing in its second variable.

Condition (iii) holds if the following condition holds.

(iii*) f is continuous on $(0, \infty)\times[0, \infty)$, and there exist $A>0$ and $c>0$ such that the initial value problem

$$(*)\qquad \varphi'=cf(t, \varphi),\ \varphi(0)=0$$

has only the zero solution on $[0, A]$.

We prove that (iii*) implies (iii). Let $\varepsilon>0$ and consider the initial value problem

$$(**)\qquad \varphi'=cf(t, \varphi),\ \varphi(A)=\varepsilon.$$

By Peano's existence theorem, there is a solution φ_ε to (**) on some interval $[a, A]$, where $a>0$. With respect to the open set $Q=(0, A)\times(0, 2\varepsilon)$, this function can be extended to the left as a solution over a maximal interval $(\alpha, A]$. Since φ_ε is non-decreasing to the right, $(t, \varphi_\varepsilon(t))$ tends to the point $(\alpha, \varphi_\varepsilon(\alpha))\in\partial Q$, where

$$\varphi_\varepsilon(\alpha)\equiv\lim_{t\to\alpha}\varphi_\varepsilon(t);$$

also, $\varphi_\varepsilon(t)\leqq\varepsilon$ and φ_ε is continuous on $[\alpha, A]$. Clearly, $\varphi_\varepsilon(t)>0$ for $t\in[\alpha, A]$; otherwise, we could define a non-trivial solution to (*). Therefore, $\alpha=0$ and φ_ε is a solution to $\varphi'=cf(t, \varphi)$ on $[0, A]$ with $0<\varphi_\varepsilon\leqq\varepsilon$.

2.3. We shall let G be an open, connected domain in R_n and shall denote by E a differential operator of the form

$$Eu(x)=\sum_{i,j=1}^{n} a_{ij}(x,\ u(x),\ \operatorname{grad} u(x))D_{ij}u(x)-a(x,\ u(x),\ \operatorname{grad} u(x)),$$

where u is any twice differentiable function. For simplicity, we suppose that a and a_{ij} $(i,\ j=1,\ \dots,\ n)$ are functions defined on $G\times R_1\times R_n$.

We shall refer to the following conditions on u:

(i) $u\in C(\overline{G})\cap C^2(G)$,

(ii) $Eu\geqq 0$ in G,

(iii) $a(x,\ u(x),\ 0)\geqq 0$ in G,

(iv) $\Sigma a_{ij}(x,\ u(x),\ 0)\lambda_i\lambda_j\geqq 0$ for all $\lambda\in R_n$, and x in G,

(v) $|D_{ij}u(x)|\leqq K$ for $x\in G$ and $i,\ j=1,\ \dots,\ n$, where K is a positive constant.

2.4. Let B be a continuous, positive function on $(0,\ \infty)$ with

$$\int_0^a B(s)ds<\infty$$

for all $a<0$. Without loss of generality, we assume that B is bounded away from zero by a positive constant B_0.

2.5. Let τ be a function on $\overline{G}$ satisfying the conditions

(i) $\tau=0$ on ∂G, $\tau>0$ on G;

(ii) $\tau\in C^1(\overline{G})\cap C^2(G)$;

(iii) $|\operatorname{grad}\tau|\leqq M$ on $\overline{G}$ and $|\operatorname{grad}\tau|\geqq m>0$ on ∂G;

(iv) τ can be extended to a continuously differentiable function on an open set containing $\overline{G}$.

Condition (iv) is satisfied if ∂G is piecewise continuously differentiable. Partial derivatives at boundary points are understood in (iii) as limits of corresponding partial derivatives from the interior.

3. Basic theorem.

THEOREM 3.1. *Let E, G, and u satisfy the conditions of 2.3. Let* $y \in \partial G$ *and* $u(x) < u(y)$ *for all* $x \in \overline{G} - \{y\}$. *Suppose there exist functions f, B, and* τ *satisfying the conditions of* 2.2, 2.4 *and* 2.5 *except possibly* 2.5 (iv). *Further, suppose*

(i) $\liminf_{\substack{x \to y \\ x \in G}} \sum_{i,j=1}^{n} a_{ij}(x, u(x), 0) D_i \tau(x) D_j \tau(x) = \beta_1 > 0,$

(ii) $| a_{ij}(x, u(x), 0) - a_{ij}(x, u(x), \operatorname{grad} u(x)) | \leqq$

$\leqq f(\tau(x), | \operatorname{grad} u(x) |)$, for $i, j = 1, \ldots, n$,

$a(x, u(x), 0) - a(x, u(x), \operatorname{grad} u(x)) \leqq f(\tau(x), | \operatorname{grad} u(x) |)$,

(iii) $\sum_{i,j=1}^{n} a_{ij}(x, u(x), 0) D_{ij} \tau(x) \geqq -B(\tau(x))$

for all $x \in G$,
where constant c of 2.2 *satisfies*

$$c > \frac{M}{\beta_1}(n^2K + 1).$$

Then

$$\limsup_{\substack{x \to y \\ x \in l}} \frac{u(x) - u(y)}{| x - y |} < 0,$$

where l is any half ray emanating from y at an angle less than $\frac{\pi}{2}$ *with the inner normal n at y.*

PROOF. Choose β so that $0 < \beta < \beta_1$ and $\frac{M}{\beta}(n^2K + 1) < c$. There is an open ball N centered at y such that $\sum_{i,j=1}^{n} a_{ij}(x, u(x), 0) D_i \tau(x) D_j \tau(x) > \beta$ on $\overline{N} \cap G$. Choose ν such that $\frac{M}{\beta}(1 + \nu)(n^2K + 1) < c$. There is A_1, $0 < A_1 \leqq A$, for which $\exp \left(\frac{1}{\beta} \int_0^{A_1} B(s) ds \right) < 1 + \nu$. We can take N small enough that $\tau(x)<\min\{A_1, \delta\}$ on $\overline{N}\cap G$. (Recall that δ is the constant from 2.2 (ii)). Let $\varepsilon>0$, to be chosen later. Let φ be the corresponding solution of the problem $\varphi'=cf(t, \varphi)$ on $[0, A]$ guaranteed by 2.2 (iii). We define the auxiliary function w on $\overline{N}\cap\overline{G}$ by $w(x)=u(x)+z(\tau(x))$ with

$$z(\tau)=\frac{1}{c_1}\int_0^\tau \varphi(t)\exp\left(\frac{1}{\beta}\int_0^t B(s)ds\right)dt$$

and

$$c_1=M\exp\left(\frac{1}{\beta}\int_0^{A_1} B(s)ds\right).$$

Now

$$z'(\tau)=\frac{1}{c_1}\varphi(\tau)\exp\left(\frac{1}{\beta}\int_0^\tau B(s)ds\right)>0$$

on $[0, A_1]$, and

$$z''(\tau)=\frac{1}{c_1}\left[\varphi'(\tau)+\varphi(\tau)\frac{B(\tau)}{\beta}\right]\exp\left(\frac{1}{\beta}\int_0^\tau B(s)ds\right)>0$$

on $(0, A]$ because $\varphi>0$ on $[0, A_1]$ and $B(t)>0$ on $(0, A]$.

We shall show by contradiction that w cannot attain its maximum over $\overline{N\cap G}$ at an interior point of that set. Suppose, on the contrary, there is a maximum point x_0 in $N\cap G$. Let E_0 be the linear operator associated with E and u and acting on w, defined by $E_0w(x)=\Sigma a_{ij}(x, u(x), 0)D_{ij}w(x)$. Since x_0 is an interior maximum, $E_0w(x_0)\leqq 0$ (see, for example, Miranda [4], p. 4). We shall now show that

$$E_0w(x_0)>0.$$

Let $\beta_2=\Sigma a_{ij}(x_0, u(x_0), 0)D_i\tau(x_0)D_j\tau(x_0)$. Then $\beta_2>\beta$ and there exists $\mu>0$ such that

$$\frac{f(t, \varphi_2)-f(t, \varphi_1)}{\varphi_2}>-\frac{B_0(\beta_2-\beta)}{(n^2K+1)c_1\beta}$$

for all φ_1 and φ_2 satisfying $0\leqq\varphi_1\leqq\varphi_2<\mu$.

Let us restrict ε so that $\varphi(\tau(x_0))<\mu$.

At x_0, we have $0=\operatorname{grad} w=\operatorname{grad} u+z' \operatorname{grad} \tau$; so that

$$|\operatorname{grad} u(x_0)|=z'(\tau(x_0))\,|\operatorname{grad} \tau(x_0)|\leqq Mz'(\tau(x_0))=$$

$$=\frac{M}{c_1}\varphi(\tau(x_0))\exp\left(\frac{1}{\beta}\int_0^{\tau(x_0)}B(s)ds\right)<\varphi(\tau(x_0)).$$

Therefore,

$$\frac{f(\tau(x_0),\ \varphi(\tau(x_0)))-f(\tau(x_0),\ |\operatorname{grad} u(x_0)|)}{\varphi(\tau(x_0))}>-\frac{B(\tau(x_0))(\beta_2-\beta)}{(n^2K+1)c_1\beta},$$

so that

$$(n^2K+1)[f(\tau(x_0),\ \varphi(\tau(x_0)))-f(\tau(x_0),\ |\operatorname{grad} u(x_0)|)]$$

$$>-\varphi(\tau(x_0))\cdot\frac{1}{c_1}\cdot\frac{B(\tau(x_0))}{\beta}\cdot(\beta_2-\beta)>z''(\tau(x_0))(\beta-\beta_2),$$

giving

$$-(n^2K+1)f(\tau(x_0),\ |\operatorname{grad} u(x_0)|)+z''(\tau(x_0))\cdot\beta_2$$

$$>-(n^2K+1)f(\tau(x_0),\ \varphi(\tau(x_0)))+z''(\tau(x_0))\cdot\beta.$$

This implies that

$$E_0w(x_0)\geqq E_0w(x_0)-Eu(x_0)=$$

$$=\Sigma a_{ij}(x_0,\ u(x_0),\ 0)D_{ij}w(x_0)-\Sigma a_{ij}(x_0,\ u(x_0),\ \operatorname{grad} u(x_0))D_{ij}u(x_0)+$$

$$+a(x_0,\ u(x_0),\ \operatorname{grad} u(x_0))\geqq$$

$$\geqq\Sigma[a_{ij}(x_0,\ u(x_0),\ 0)-a_{ij}(x_0,\ u(x_0),\ \operatorname{grad} u(x_0))]D_{ij}u(x_0)+$$

$$+[a(x_0,\ u(x_0),\ \operatorname{grad} u(x_0))-a(x_0,\ u(x_0),\ 0)]+$$

$$+z''(\tau(x_0))\beta_2+z'(\tau(x_0))\Sigma a_{ij}(x_0,\ u(x_0),\ 0)D_{ij}\tau(x_0)\geqq$$

$$\geqq-(n^2K+1)f(\tau(x_0),\ |\operatorname{grad} u(x_0)|)+\beta_2z''(\tau(x_0))-z'(\tau(x_0))B(\tau(x_0))>$$

$$>-(n^2K+1)f(\tau(x_0),\ \varphi(\tau(x_0))+\beta z''(\tau(x_0))-z'(\tau(x_0))B(\tau(x_0)).$$

Now

$$\beta z''(\tau)-B(\tau)z'(\tau)=\frac{\beta}{c_1}\exp\left(\frac{1}{\beta}\int_0^\tau B(s)ds\right)\varphi'(\tau)\geqq\frac{\beta}{c_1}\varphi'(\tau)$$

and also

$$\frac{c_1}{\beta}(n^2K+1)=\frac{M}{\beta}(n^2K+1)\exp\left(\frac{1}{\beta}\int_0^{A_1} B(s)ds\right)<\frac{M}{\beta}(n^2K+1)(1+\nu)<c.$$

Thus,

$$E_0w(x_0)>-(n^2K+1)f(\tau(x_0),\ \varphi(\tau(x_0)))+\frac{\beta}{c_1}\varphi'(\tau(x_0))=$$

$$=\frac{\beta}{c_1}\left[\varphi'(\tau(x_0))-\frac{c_1}{\beta}(n^2K+1)f(\tau(x_0),\ \varphi(\tau(x_0)))\right]\geqq$$

$$\geqq\frac{\beta}{c_1}\left[\varphi'(\tau(x_0))-cf(\tau(x_0),\ \varphi(\tau(x_0)))\right]=0.$$

From this contradiction, we conclude that w can attain its maximum only on $\partial(N\cap G)$. We now show that by taking ε small enough, this maximum can only be attained on $N\cap\partial G$. There exists $\eta>0$ such that $u(x)<u(y)-\eta$ on $\overline{G}\cap\partial N$. Restricting ε further, we choose $\varepsilon<\frac{M\eta}{A_1}$; so that

$$z(\tau)=\frac{1}{c_1}\int_0^\tau\varphi(t)\exp\left(\frac{1}{\beta}\int_0^t B(s)ds\right)dt\leqq\frac{\tau}{c_1}\varphi(\tau)\exp\left(\frac{1}{\beta}\int_0^{A_1} B(s)ds\right)\leqq$$

$$\leqq\frac{A_1\varepsilon}{M}<\eta$$

on $[0,\ A_1]$. Then

$$w(x)=u(x)+z(\tau(x))<u(y)$$

on $\overline{G}\cap\partial N$. Therefore, the maximum of w is attained only on $N\cap\partial G$. Since $w=u$ there, $w(x)\leqq u(y)$ on $\overline{N\cap G}$. In particular, for $x\in l\cap N$, we have

$$\frac{u(x)-u(y)}{|x-y|} \leqq \frac{z(\tau(y))-z(\tau(x))}{|x-y|}.$$

Therefore,

$$\limsup_{\substack{x \to y \\ x \in l}} \frac{u(x)-u(y)}{|x-y|} \leqq -z'(0)\,|\operatorname{grad} \tau(y)| \cos(ln) \leqq$$

$$\leqq -mz'(0)\cos(ln) < 0.$$

This proves the theorem.

REMARK 3.1. All the conditions on u listed in 2.3 and the conditons of (ii) in the statement of the theorem need be assumed only in some neighbourhood of y. Also, the conditions on τ listed in 2.5 can be replaced by the following:

There exists a neighbourhood N of y and a function τ defined on $N \cap \overline{G}$ satisfying

(i) $\tau = 0$ on $\partial G \cap N$ and $\tau > 0$ in $G \cap N$;

(ii) $\tau \in C^1(\overline{G} \cap N) \cap C^2(G \cap N)$;

(iii) $|\operatorname{grad} \tau(x)| \leqq M$ in $G \cap N$ and
$|\operatorname{grad} \tau(x)| \geqq m > 0$ on $\partial G \cap N$.

In view of the above, we may weaken the assumption « $a(x, u(x), 0) \geqq 0$ » to « $a(x, u(x), 0) \geqq 0$ if $u(x) > 0$ » if we assume that $u(y) > 0$.

REMARK 3.2. If we modify the hypothesis of Theorem 3.1 so that

$$Eu \leqq 0, \quad a(x, u(x), 0) \leqq 0, \quad u(x) > u(y)$$

for all $x \in \overline{G}$ with $x \neq y$, and

$$a(x, u(x), 0) - a(x, u(x), \operatorname{grad} u(x)) \geqq -f(\tau(x), |\operatorname{grad} u(x)|),$$

while leaving the other conditions as they are, then

$$\liminf_{\substack{x \to y \\ x \in l}} \frac{u(x)-u(y)}{|x-y|} > 0.$$

REMARK 3.3. If $a_{ij}(x, u, \text{grad}\, u) = a_{ij}(x, u)$, we can drop the assumption that $D_{ij}u(x)$ is bounded and the theorem remains valid. In the proof, the difference

$$a_{ij}(x, u(x), 0) - a_{ij}(x, u(x), \text{grad}\, u(x))$$

is zero, so that we require only $c > \frac{M}{\beta_1}$.

REMARK 3.4. The existence condition on f (see 2.2 (iii)) is essential. Consider the operator $Eu = u'' - \alpha a(x, u')$ on $(0, 1)$, where

$$a(x, y) = \begin{cases} 0 \text{ if } x \leqq 0 \text{ or } y \leqq 0, \\ 2y/x \text{ if } 0 \leqq y \leqq x^2, \\ 2x \text{ if } x^2 \leqq y. \end{cases}$$

Let $\tau(x) = x$, $B(t) \equiv 1$, and $f(t, \varphi) = \alpha a(t, \varphi)$. Using these functions, one can show that for $0 < \alpha < 1$ the hypothesis of the minimum principle (Remark 3.2) holds at $x = 0$, but that for $\alpha \geqq 1$ the only condition that does not hold is (iii) of 2.2. In the latter case, the function $u = \frac{1}{3}\alpha x^3$ satisfies $Eu = 0$, but $u'(0) = 0$.

REMARK 3.5. Theorem 3.1 is a generalization of Theorem 2 of Výborný [3]. If the hypothesis of Výborný's theorem holds, then so does the hypothesis of Theorem 3.1: let $f(t, \varphi) = B(t)g(\varphi)$ for $0 < t < \infty$ and $0 < \varphi < \infty$, and $f(t, 0) = 0$.

4. An extension of Theorem 3.1.

As it stands, Theorem 3.1 does not contain as a special case the linear operator treated by Pucci in [5]. In Pucci's theorem, the domain is a sphere S,

$$Eu(x) = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij}u(x) + \Sigma b_i(x) D_i u(x) + c(x)u(x),$$

and $\tau(x) = r_0 - |x - \eta|$, where r_0 and η are the radius and center of S.

The conditions on the coefficients are as follows:

(A) $\liminf_{\substack{x\to y\\ x\in S}} \Sigma a_{ij}(x)D_i\tau(x)D_j\tau(x)>0$;

(B) there exists a continuous, positive, decreasing function $B(\tau)$ defined for $0<\tau<r_0$, such that $\int_0 B(t)dt<\infty$ and

$$\liminf_{\substack{x\to y\\ x\in S}} \frac{b_i(x)D_i\tau(x)}{B(\tau(x))}>-1;$$

(C) $c(x)\leqq 0$ and

$$\liminf_{\substack{x\to y\\ x\in S}} \frac{c(x)\tau(x)}{B(\tau(x))}>-1,$$

where B is the function of condition (B).

He concludes that u cannot attain a non-negative maximum at $y\in\partial S$ unless either u is constant or

$$\liminf_{\substack{x\to y\\ x\in l}} \frac{u(x)-u(y)}{|x-y|}<0,$$

where l is as in Theorem 3.1.

We remark, in passing, that if $u(y)>0$, then the second part of condition (C) may be dropped.

If $b_i\equiv 0$ for $i=1, \ldots, n$, $c(x)\leqq 0$, and $u(y)>0$, then Pucci's hypothesis implies ours, since there will be a neighbourhood of y where $a(x, u(x), 0)=-c(x)u(x)\geqq 0$. However, if $u(y)=0$, the inequality $a(x, u(x), 0)=-c(x)u(x)\geqq 0$ may not be satisfied in any neighbourhood of y.

If $n=1$ and $c\equiv 0$, the hypothesis of Theorem 3.1 follows from Pucci's hypothesis if we let

$$f(\tau(x), |u'(x)|)=B(\tau(x))\cdot|u'(x)|.$$

However, if $n>1$, Pucci's condition (B) does not necessarily imply that

$$a(x, u(x), 0)-a(x, u(x), \operatorname{grad} u(x))=\Sigma b_iu_i\leqq f(\tau(x), |\operatorname{grad} u(x)|)$$

for some functions τ and f.

In order to include Pucci's theorem, we modify Theorem 3.1 by adding extra terms to E.

THEOREM 4.1. *Suppose the hypothesis of Theorem* 3.1 *holds except that we replace E by E^+ where*

$$E^+u(x)=\Sigma a_{ij}(x,\ u(x),\ \operatorname{grad} u(x))D_{ij}u(x)-a(x,\ u(x),\ \operatorname{grad} u(x))+$$

$$\Sigma b_i(x,\ u(x),\ \operatorname{grad} u(x))D_i u(x)+c(x,\ u(x),\ \operatorname{grad} u(x))\cdot u(x).$$

The functions b_i and c are defined on $G\times R_1\times R_n$, $c(x)\leqq 0$ in some neighbourhood of y,

$$\liminf_{\substack{x\to y\\ x\in G}} \frac{1}{B(\tau(x))}\Sigma b_i(x,\ u(x),\ \operatorname{grad} u(x))D_i\tau(x)>-\infty,$$

and

$$\liminf_{\substack{x\to y\\ x\in G}} \frac{c(x,\ u(x),\ \operatorname{grad} u(x))\cdot\tau(x)}{B(\tau(x))}>-\infty,$$

where B and τ are the functions of Theorem 3.1. *Moreover, we assume $u(y)\geqq 0$.*

Then the conclusion of Theorem 3.1 *holds.*

NOTE. If $c\equiv 0$, we can remove the condition $u(y)\geqq 0$. Also, trivially, we may use different functions B_1 and B_2 for the last inequalities, so long as they satisfy the conditions of 2.4.

PROOF. The proof follows that of Theorem 3.1 except that we use the auxiliary function

$$z(\tau)=\frac{1}{c_1}\int_0^\tau \varphi(t)\exp\left[\left(\frac{2+n}{\beta}\right)\int_0^t B(s)ds\right]dt,$$

where

$$c_1=M\exp\left[\left(\frac{2+n}{\beta}\right)\int_0^{A_1} B(t)dt\right],$$

and use the auxiliary operator defined by

$$E_0^+ w(x) = \Sigma a_{ij}(x, u(x), 0) D_{ij} w(x) + \Sigma b_i(x, u(x), \operatorname{grad} u(x)) D_i w(x) + + c(x, u(x), \operatorname{grad} u(x)) \cdot w(x).$$

REMARK 4.1. The counterexamples provided by Pucci [5] show that the bounds on the growth of the coefficients c and b_i, $i = 1, \ldots, n$, are essential.

REMARK 4.2. Similar extensions can be made to the theorems of the following sections. However, for simplicity, we consider only the original operator E.

5. The interior maximum principle.

THEOREM 5.1. *Let G and E be as in 2.3. Let u be a function satisfying conditions* (ii)-(iv) *of* 2.3 *and* (i′) $u \in C^2(G)$. *Suppose that G, u, and the coefficients of E satisfy the following interior condition.*

(*IC*) *To each sphere S with $\bar{S} \subset G$, there correspond*

(a) *a constant γ_s satisfying*

$$\sum_{i,j=1}^{n} a_{ij}(x, u(x), 0) \lambda_i \lambda_j \geqq \gamma_s |\lambda|^2 > 0$$

for all $x \in S$ and all $\lambda \in R_n$; and

(b) *functions f_s, B_s, and τ_s satisfying the conditions of* 2.2, 2.4 *and* 2.5 (*except possibly for* iv) *with constants M_s, m_s, c_s, and so on, such that*

$$|a_{ij}(x, u(x), 0) - a_{ij}(x, u(x), \operatorname{grad} u(x)| \leqq f_s(\tau_s(x), |\operatorname{grad} u(x)|),$$

$$a(x, u(x), 0) - a(x, u(x), \operatorname{grad} u(x)| \leqq f_s(\tau_s(x), |\operatorname{grad} u(x)|),$$

and

$$\sum_{i,j=1}^{n} a_{ij}(x, u(x), 0) D_{ij} \tau_s(x) \geqq -B_s(\tau_s(x))$$

for all $x \in S$ (or just for all x within a distance η_s of ∂S, where η_s is some positive constant depending on S). Let the constants involved satisfy the inequality

$$c_s > \frac{M_s}{m_s^2 \gamma_s}(n^2 K_s + 1),$$

where $K_s = \sup\{|D_{ij}u(x)| : x \in S\}$.

We conclude that u cannot attain its maximum in the interior of G unless u is constant.

PROOF. Suppose u is not constant on G but $u(x_0) = \max\{u(x) : x \in G\}$ for some $x_0 \in G$. Then, there are x_1 and x_2 in G such that $u(x_1) < u(x_2) = u(x_0)$ and $|x_1 - x_2| < \operatorname{dist}(x_1, \partial G)$. There is an open sphere S_1 about x_1 in which $u(x) < u(x_0)$. Expand S_1 if necessary, until its surface touches a point x_3 where $u(x_3) = u(x_0)$ but $u(x) < u(x_0)$ for $x \in S_1$. Note that we have ensured $x_3 \in G$. Let S be a subsphere of S_1 with $\partial S \cap \partial S_1 = \{x_3\}$. We may apply Theorem 3.1 to the sphere S at x_3 because

$$\beta_S \equiv \liminf_{x \to x_3} \sum_{i,j=1}^{n} a_{ij}(x, u(x), 0) D_i \tau_s(x) D_j \tau_s(x) \geqq \gamma_s |\operatorname{grad} \tau_s(x_3)|^2 \geqq m_s^2 \gamma > 0.$$

Thus, $D_\nu u(x_3) < 0$ where ν is the inner normal to ∂S at x_3, contrary to the fact that x_3 is an interior maximum. This proves the theorem.

REMARK 5.1. We can weaken the uniform ellipticity condition IC (a) to

$$\sum_{i,j=1}^{n} a_{ij}(x, u(x), 0) \lambda_i \lambda_j > 0$$

for all $x \in G$ and $\lambda \in R_n$, provided that the coefficients $a_{ij}(x, u(x), 0)$ are continuous in x on G and provided that the inequality involving the constants is replaced by the stronger condition that for each S there is a sequence $c_{sk} \to \infty$ such that f_s satisfies 2.2 (iii) for each c_{sk}, in this case, there will be a positive constant $\gamma(x_3)$ and a neighbourhood of x_3 in which

$$\Sigma a_{ij}(x, u(x), 0) D_i \tau_s(x) D_j \tau_s(x) > \gamma(x_3) > 0.$$

In applying Theorem 3.1, we confine ourselves to this neighbourhood and take k_0 large enough that

$$c_{sk_0} > \frac{M_s}{\gamma(x_3)}(n^2K_s+1).$$

REMARK 5.2. We may weaken the interior condition (IC) to the following. To each sphere S with $\bar{S}\subset G$ and each point $y\in\partial S$, there correspond

(a) a neighbourhood N_{sy} of y;

(b) a constant γ_{sy} such that

$$\Sigma a_{ij}(x, u(x), 0)\lambda_i\lambda_j \geqq \gamma_{sy} |\lambda|^2 > 0$$

for all $x\in S\cap N_{sy}$ and $\lambda\in R_n$; and

(c) functions f_{sy} , B_{sy} , and τ_{sy} satisfying the conditions of 2.2, 2.4, and 2.5 except (iv), with constants M_{sy} , m_{sy} , c_{sy} , and so on, such that

$$|a_{ij}(x, u(x), 0)-a_{ij}(x, u(x), \operatorname{grad} u(x))| \leqq f_{sy}(\tau_{sy}(x), |\operatorname{grad} u(x)|),$$

$$a(x, u(x), 0)-a(x, u(x), \operatorname{grad} u(x)) \leqq f_{sy}(\tau_{sy}(x), |\operatorname{grad} u(x)|),$$

and

$$\Sigma a_{ij}(x, u(x), 0)D_{ij}\tau_{sy}(x) \geqq -B_{sy}(\tau_{sy}(x))$$

for all $x\in S\cap N_{sy}$. Let the constants involved satisfy the inequality

$$c_{sy} > \frac{M_{sy}}{m_{sy}^2\gamma_{sy}}(n^2K_{sy}+1),$$

where $K_{sy}=\sup\{|D_{ij}u(x)| : x\in S\cap N_{sy}\}$.

REMARK 5.3. There is a corresponding interior minimum principle. If we modify the hypothesis of Theorem 5.1 so that $Eu\leqq 0$, $a(x, u(x), 0)\leq 0$, and

$$a(x, u(x), 0)-a(x, u(x), \operatorname{grad} u(x)) \geqq -f_s(\tau_s, |\operatorname{grad} \tau(x)|),$$

while leaving the other conditions as they are, then u cannot attain its minimum in the interior of G unless u is constant.

REMARK 5.4. Theorem 5.1 is a generalization of Theorem 4 of Redheffer [2]. If Redheffer's hypothesis holds then so does the hypothesis of Theorem 5.1. For a sphere S with $\overline{S}\subset G$, let $\gamma_s=\frac{1}{L^2}$; $\tau_s(x)=r^2-|x-\overline{x}|^2$, where r and $\overline{x}$ are the radius and center of S;

$$B_s(t)=(2n)\max\{|a_{ii}(x, u(x), 0)| : 0\leqq i\leqq n, x\in\overline{S}\};$$

and $f(t, \varphi)=g(\varphi)$ for $0<t<\infty$ and $0<\varphi<\infty$, and $f(t, 0)=0$.

6. The boundary maximum principle.

Before stating the main result of the section, Theorem 6.2, we modify Theorem 3.1, so that the hypothesis no longer requires $u(x)<u(y)$ for points $x\neq y$ on the boundary ∂G.

THEOREM 6.1. *Suppose the hypothesis of Theorem* 3.1 *holds on* G *except that* $u(x)<u(y)$ *on* G *instead of on* $\overline{G}-\{y\}$, *and condition* (iii) *is replaced by the two conditions*

(i) $D_{ij}\tau$ *is bounded on* G *for each* i, $j=1, \ldots, n$ (*at least in some neighbourhood of* y) *by* $B(\tau)$,

(ii) B *is non-increasing and* $a_{ij}(x, u(x), 0)$ *is continuous at* y *for all* i, $j=1, \ldots, n$.

Suppose also

(iii) f *is a non-increasing function of its first variable* t (*at least in some neighbourhood of* $t=0$) *and condition* (iv) *of* 2.5 *holds for* τ.

Then the conclusion of Theorem 3.1 *holds.*

PROOF. The proof consists in deforming G in a neighbourhood of y in such a way that $u(x)<u(y)$ on the boundary of the deformed domain and Theorem 3.1 can be applied.

Since $|\operatorname{grad}\tau(x)|>0$ on ∂G, we have $D_i\tau(y)\neq 0$ for some i, say n. Without loss of generality, let $D_n\tau(y)>0$. Let N be a sphere centred at y in which τ is continuously differentiable, $D_n\tau(x)>0$, and conditions (i)-(iii) of the hypothesis hold; for condition (iii) this means that $f(t, \varphi)$ is a non-increasing function of t for all t with

$$0 \leqq t \leqq \sup\{\tau(x) : x \in N \cap G\}.$$

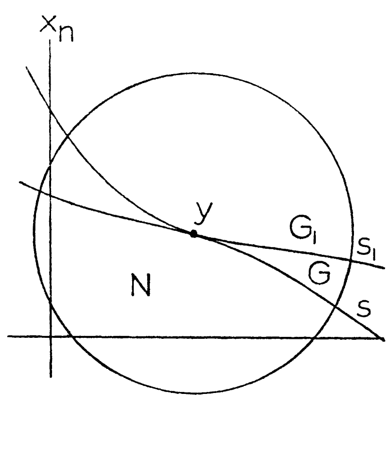
Define the transformation $g : R_n \rightarrow R_n$ by

$$g(x) \equiv (g_1(x), \ldots, g_n(x)) = (x_1, \ldots, x_{n-1}, x_n + \sum_{i=1}^{n-1} (y_i - x_i)^2).$$

Let h be the inverse of g, and let G_1 be the image of G under g. The implicit functions theorem guarantees the existence of a sphere S in R_{n-1} with center $(y_1, \ldots, y_{n-1})$ and a unique continuous function $s(x_1, \ldots, x_{n-1})$ defined on S such that $y_n = s(y_1, \ldots, y_{n-1})$ and

$$\tau(x_1, \ldots, x_{n-1}, s(x_1, \ldots, x_{n-1})) = 0 \text{ for } (x_1, \ldots, x_{n-1}) \in S;$$

if one takes N small enough, the equation $x_n = s(x_1, \ldots, x_{n-1})$ represents ∂G in N, and no other points of ∂G lie in N. Since $D_n\tau(x)>0$ in $G \cap N$, $G \cap N$ lies in the positive x_n direction from the graph of s. The image s_1 of s under g is the boundary of G_1; the point y remains fixed and for any other point in S satisfying $(x_1, \ldots, x_{n-1}) \neq (y_1, \ldots, y_{n-1})$, we have $s_1(x_1, \ldots, x_{n-1}) > s(x_1, \ldots, x_{n-1})$. Define the function τ_1 on $\overline{G}_1 \cap N$ by $\tau_1(x) = \tau(h(x))$.



We verify that if N is small enough, the hypothesis of Theorem 3.1 holds in $G_1 \cap N$. Obviously, τ_1 satisfies conditions (i) and (ii) of 2.5. We check (iii). Choose $M_1 > M$ such that $\frac{M_1}{\beta_1}(n^2K+1) < c$, and let $\eta < \min\left\{\frac{3m^2}{4}, M_1{}^2 - M^2\right\}$. Now,

$$\big|\,|\operatorname{grad}\tau_1(x)|^2 - |\operatorname{grad}\tau(x)|^2\,\big| \leqq \big|\,|\operatorname{grad}\tau(h(x))|^2 - |\operatorname{grad}\tau(x)|^2\,\big| +$$

$$+\left|\, 4D_n\tau(h(x))\sum_{i=1}^{n-1} D_i\tau(h(x))(y_i - x_i) + 4[D_n\tau(h(x))]^2 \sum_{i=1}^{n-1}(y_i - x_i)^2\,\right| \leqq$$

$$\leqq \big|\,|\operatorname{grad}\tau(h(x))|^2 - |\operatorname{grad}\tau(x)|^2\,\big| + 4M^2(n-1)(\operatorname{diam} N) + 4M^2(\operatorname{diam} N)^2.$$

We may take N small enough that this expression is less than η for $x \in N \cap \overline{G}$. Then $|\operatorname{grad}\tau_1(x)| < M_1$ for $x \in N \cap \overline{G}_1$ and $|\operatorname{grad}\tau_1(x)| > \frac{m}{2}$ for $x \in N \cap \partial G_1$. Now, we show that conditions (i)-(iii) of the hypothesis of Theorem 3.1 hold.

(i) $\liminf_{\substack{x \to y \\ x \in G_1}} \Sigma a_{ij}(x, u(x), 0) D_i\tau_1(x) D_j\tau_1(x) =$

$$\Sigma a_{ij}(y, u(y), 0) D_i\tau(y) D_j\tau(y) = \beta_1 .$$

(ii) The monotonicity of f and the fact that $\tau_1(x) \leqq \tau(x)$ in $N \cap G_1$ imply that the inequalities in (ii) hold.

(iii) Calculation of the second derivatives of τ_1 and application of conditions (i) and (ii) of the hypothesis give us that

$$\Sigma a_{ij}(x, u(x), 0) D_{ij}\tau_1(x) \geqq -B(\tau_1(x)) \cdot T$$

on $N \cap G_1$, where T is a constant. Thus $B_1(t) = B(t) \cdot T$ defines a function satisfying 2.4 and condition (iii) of Theorem 3.1.

We conclude that the hypothesis of Theorem 3.1 holds on $N \cap G_1$. Since G_1 contains an interval of a half ray l with endpoint y if the same is true of G, Theorem 6.1 is proved.

THEOREM 6.2 (Boundary maximum principle). *Let G, E and u satisfy* 2.3 *and suppose that*

$$\sum_{i,j=1}^{n} a_{ij}(x,\ u(x),\ 0)\lambda_i\lambda_j \geqq \gamma\,|\,\lambda\,|^2 > 0$$

for all $x\in G$ *in some neighbourhood of* ∂G *and all* $\lambda\in R_n$. *Suppose that part* (b) *of the interior condition* (*IC*) *of Theorem* 5.1 *holds with* γ *for* γ_s *and, in addition, the following boundary condition* (*BC*) *holds.*

(*BC*) *There are functions* f, B *and* τ *satisfying* 2.2, 2.4 *and* 2.5 *such that*

(a) $|\,a_{ij}(x,\ u(x),\ 0)-a_{ij}(x,\ u(x),\ \mathrm{grad}\,u(x)\,|\leqq f(\tau(x),\ |\,\mathrm{grad}\,u(x)\,|)$

and

$$a(x,\ u(x),\ 0)-a(x,\ u(x),\ \mathrm{grad}\,u(x))\leqq f(\tau(x),\ |\,\mathrm{grad}\,u(x)\,|)$$

in G *in some neighbourhood of* ∂G;

(b) f *is non-increasing in the first variable* t, *at least in some neighbourhood of* $t=0$;

(c) *the constant* c *associated with* f *satisfies*

$$c>\frac{M}{m^2\gamma}(n^2K+1);$$

(d) $D_{ij}\tau$ *is bounded in some neighbourhood of* ∂G *for each* $i,\ j=1,\ \ldots,\ n$ by $B(\tau)$;

(e) B *is non-increasing and* $a_{ij}(x,\ u(x),\ 0)$ *is continuous at* y *for all* $i,\ j=1,\ \ldots,\ n$.

Then u *does not attain its maximum at any point* y *of* ∂G *unless either* u *is constant in* G *or*

$$\limsup_{\substack{x\to y\\ x\in l}} \frac{u(x)-u(y)}{|\,x-y\,|}<0,$$

where l *is any half ray of the type described in Theorem* 3.1.

PROOF. This is a simple consequence of Theorems 5.1 and 6.1.

REMARK 6.1. Uniform ellipticity can be weakened to

$$\Sigma a_{ij}(x,\ u(x),\ 0)\lambda_i\lambda_j>0$$

for all $x\in\overline{G}$ and $\lambda\in R_n$ provided that the coefficients $a_{ij}(x,\ u(x),\ 0)$ are continuous in x on $\overline{G}$ and provided that the constants c and c_s corresponding to f in (BC) and each f_s in (IC) can be chosen arbitrarily large, as described in Remark 5.1.

There is also a boundary minimum principle.

7. Application to a boundary value problem.

In the usual way, Theorems 5.1 and 6.2 give us the following uniqueness theorem.

THEOREM 7.1. *Let* $u\in C^1(\overline{G})\cap C^2(G)$ *with* $|D_{ij}u(x)|\leqq K$ *for all* $x\in G$. *Let* u *be a solution of the boundary value problem*

$$Eu=0 \quad on \quad G,$$

$$b(x)=\alpha(x,\ u(x),\ \text{grad}\, u(x))D_l u(x)+\beta(x,\ u(x),\ \text{grad}\, u(x))\cdot u(x)=0$$

$$on \quad \partial G,$$

where

$$\alpha(x,\ u(x),\ \text{grad}\, u(x))\geqq 0,\ \beta(x,\ u(x),\ \text{grad}\, u(x))\leqq 0,$$

and

$$|\alpha(x,\ u(x),\ \text{grad}\, u(x))|+|\beta(x,\ u(x),\ \text{grad}\, u(x))|>0;$$

l denotes a vector forming an acute angle with the inner normal to ∂G at x (l may vary with x). Suppose, also, that

$$\Sigma a_{ij}(x,\ u(x),\ 0)\lambda_i\lambda_j\geqq\gamma\,|\lambda|^2>0 \text{ for all } x\in G,$$

$u(x)\cdot a(x,\ u(x),\ 0)$ on G, and both the interior and boundary conditions, (IC) and (BC), hold with absolute value signs around

$$a(x,\ u(x),\ 0)-a(x,\ u(x),\ \text{grad}\, u).$$

Then u is constant in G.

PROOF. It follows from Theorems 5.1 and 6.2 and their corresponding minimum principles (Remarks 5.3 and 6.1) that u cannot attain a positive maximum or a negative minimum on $\overline{G}$ unless u is constant.

REMARK 7.1. If $\alpha=0$ at any point in ∂G, then $u\equiv 0$ in G.

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