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MAXIMUM PRINCIPLES FOR SOME QUASILINEAR SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

M. A. Dow and R. Výborný

ABSTRACT. We present proofs and extensions of a maximum principle announced by Horáček and Výborný [1] for a quasilinear, non-hyperbolic, second order partial differential operator of the form

$$\sum a_{ij}(x, u, \text{grad } u)\partial_{ij}u - a(x, u, \text{grad } u).$$

The assumptions on the coefficients are less stringent than previously required. From this basic theorem, we derive an interior maximum principle, a boundary maximum principle, and a uniqueness theorem for the elliptic case.

1. Introduction.

Horáček and Výborný [1] announced a maximum principle for a quasilinear, non-hyperbolic second order partial differential operator of the form

$$\sum a_{ij}(x, u, \text{grad } u)\partial_{ij}u - a(x, u, \text{grad } u).$$

This theorem generalized results of Redheffer [2] and Výborný [3] for such equations. Redheffer required that the differences $|a_{ii}(x, u, 0)| - a_{ij}(x, u, \text{grad } u)|$ and $|a(x, u, 0)| - a(x, u, \text{grad } u)|$ be bounded by a function $g$ of $|\text{grad } u|$ that was positive, increasing, and satisfied the condition

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Výborný connected these differences with Redheffer's potentials $c(x)$, by assuming the existence of a smooth positive function $\tau$ on $\mathbb{C}$ that was zero on the boundary. He proved a maximum principle for a boundary point that required the above differences to be bounded by a product of the above function $g$ and a function $B$ of $\tau$ that was positive and satisfied

$$\int_0^1 \frac{1}{g} \, dt = \infty.$$  

In [1], this was carried further: the differences were bounded by a continuous function $f$ of $\tau$ and $|\operatorname{grad} u|$, satisfying, among other things, the condition that the initial value problem $\varphi' = cf(t, \varphi)$, $\varphi(0) = 0$ had unique solution zero on some interval $[0, A]$, where $c$ was a certain constant. In the present paper, we improve Theorem 1 of [1], and also prove an interior maximum principle, a boundary maximum principle, and a uniqueness theorem. The uniqueness theorem corrects that announced in [1].

2. **Notation, definitions, and conditions.**

We list the following for later reference.

2.1. Let $a$, $A$, and $b$ be real numbers and let $F$ be a real-valued function defined on $(a, \infty) \times [0, \infty)$. A function $\varphi$ will be considered a solution of the initial value problem $\varphi' = F(t, \varphi)$, $\varphi(a) = b$ on the interval $[a, A]$ if $\varphi$ is continuous on $[a, A]$, differentiable on $(a, A]$, $\varphi(a) = b$, and $\varphi'(t) = F(t, \varphi(t))$ for all $t \in (a, A]$. As usual, derivatives at endpoints of intervals are interpreted as one-sided derivatives.

2.2. Throughout this paper, we shall let $f$ denote a continuous non-negative function on $(0, \infty) \times [0, \infty)$ satisfying

(i) $f(t, 0) = 0$ for all $t \in (0, \infty)$,
(ii) there exists $\delta > 0$ such that for each $t \in (0, \delta)$, we have

$$0 \leq \liminf_{\varphi_2 \to 0^+} \frac{f(t, \varphi_2) - f(t, \varphi_1)}{\varphi_2} \leq \infty,$$

(iii) there exist constants $A > 0$ and $c > 0$ such that for each $t \in (0, A)$, there is a solution $\varphi(t)$ to the problem $\varphi' = cf(t, \varphi)$ on $[0, A]$ with $0 < \varphi(t) \leq \varepsilon$ for $t \in [0, A]$.

Notice that condition (ii) holds if

(ii*) $f$ is non-decreasing in its second variable.

Condition (iii) holds if the following condition holds.

(iii*) $f$ is continuous on $(0, \infty) \times [0, \infty)$, and there exist $A > 0$ and $c > 0$ such that the initial value problem

(*)

$$\varphi' = cf(t, \varphi), \quad \varphi(0) = 0$$

has only the zero solution on $[0, A]$.

We prove that (iii*) implies (iii). Let $\varepsilon > 0$ and consider the initial value problem

(**)

$$\varphi' = cf(t, \varphi), \quad \varphi(A) = \varepsilon.$$

By Peano's existence theorem, there is a solution $\varphi(t)$ to (***) on some interval $[a, A]$, where $a > 0$. With respect to the open set $Q = (0, A) \times (0, 2\varepsilon)$, this function can be extended to the left as a solution over a maximal interval $(a, A]$. Since $\varphi(t)$ is non-decreasing to the right, $(t, \varphi(t))$ tends to the point $(a, \varphi(a)) \in \partial Q$, where

$$\varphi(a) = \lim_{t \to a^-} \varphi(t);$$

also, $\varphi(t) \leq \varepsilon$ and $\varphi(t)$ is continuous on $[a, A]$. Clearly, $\varphi(t) > 0$ for $t \in [a, A]$; otherwise, we could define a non-trivial solution to (*). Therefore, $\alpha = 0$ and $\varphi(t)$ is a solution to $\varphi' = cf(t, \varphi)$ on $[0, A]$ with $0 < \varphi(t) \leq \varepsilon$.

2.3. We shall let $G$ be an open, connected domain in $\mathbb{R}^n$ and
shall denote by $E$ a differential operator of the form

$$Eu(x) = \sum_{i,j=1}^{n} a_{ij}(x, u(x), \text{grad } u(x))D_{ij}u(x) - a(x, u(x), \text{grad } u(x)),$$

where $u$ is any twice differentiable function. For simplicity, we suppose that $a$ and $a_{ij}$ ($i, j = 1, \ldots, n$) are functions defined on $G \times R_1 \times R_n$.

We shall refer to the following conditions on $u$:

(i) $u \in C(\overline{G}) \cap C^2(G)$,
(ii) $Eu \geq 0$ in $G$,
(iii) $a(x, u(x), 0) \geq 0$ in $G$,
(iv) $\sum a_{ij}(x, u(x), 0)\lambda_i \lambda_j \geq 0$ for all $\lambda \in R_n$, and $x$ in $G$,
(v) $|D_{ij}u(x)| \leq K$ for $x \in G$ and $i, j = 1, \ldots, n$, where $K$ is a positive constant.

2.4. Let $B$ be a continuous, positive function on $(0, \infty)$ with

$$\int_{0}^{a} B(s)ds < \infty$$

for all $a<0$. Without loss of generality, we assume that $B$ is bounded away from zero by a positive constant $B_0$.

2.5. Let $\tau$ be a function on $\overline{G}$ satisfying the conditions

(i) $\tau = 0$ on $\partial G$, $\tau > 0$ on $G$;
(ii) $\tau \in C(\overline{G}) \cap C^2(G)$;
(iii) $|\text{grad } \tau| \leq M$ on $\overline{G}$ and $|\text{grad } \tau| \geq m > 0$ on $\partial G$;
(iv) $\tau$ can be extended to a continuously differentiable function on an open set containing $\overline{G}$.

Condition (iv) is satisfied if $\partial G$ is piecewise continuously differentiable. Partial derivatives at boundary points are understood in (iii) as limits of corresponding partial derivatives from the interior.
3. Basic theorem.

**Theorem 3.1.** Let $E$, $G$, and $u$ satisfy the conditions of 2.3. Let $y \in \partial G$ and $u(x) < u(y)$ for all $x \in G - \{y\}$. Suppose there exist functions $f$, $B$, and $\tau$ satisfying the conditions of 2.2, 2.4 and 2.5 except possibly 2.5 (iv). Further, suppose

(i) $\liminf_{x \to y} \sum_{i,j=1}^{n} a_{ij}(x, u(x), 0)D\tau(x)D_{ij}\tau(x) = \beta_i > 0$,

(ii) $|a_{ij}(x, u(x), 0) - a_{ij}(x, u(x), \text{grad } u(x))| \leq f(\tau(x), |\text{grad } u(x)|)$, for $i, j = 1, \ldots, n$,

(iii) $\sum_{i,j=1}^{n} a_{ij}(x, u(x), 0)D\tau(x) \geq -B(\tau(x))$

for all $x \in G$,

where constant $c$ of 2.2 satisfies

$$c > \frac{M}{\beta_1} (n^2K + 1).$$

Then

$$\limsup_{x \to y} \sup_{x \in I} \frac{u(x) - u(y)}{|x - y|} < 0,$$

where $I$ is any half ray emanating from $y$ at an angle less than $\frac{\pi}{2}$ with the inner normal $n$ at $y$.

**Proof.** Choose $\beta$ so that $0 < \beta < \beta_1$ and $\frac{M}{\beta} (n^2K + 1) < c$. There is an open ball $N$ centered at $y$ such that $\sum_{i,j=1}^{n} a_{ij}(x, u(x), 0)D\tau(x)D_{ij}\tau(x) > \beta$ on $N \cap G$. Choose $\nu$ such that $\frac{M}{\beta} (1 + \nu)(n^2K + 1) < c$. There is $A_1$, $0 < A_1 \leq A$, for which $\exp \left( \frac{1}{\beta} \int_0^{A_1} B(s)ds \right) < 1 + \nu$. We can take $N$ small enough
that \( \tau(x) \leq \min \{ A_1, \delta \} \) on \( \overline{N \cap G} \). (Recall that \( \delta \) is the constant from 2.2 (ii)). Let \( \varepsilon > 0 \), to be chosen later. Let \( \varphi \) be the corresponding solution of the problem \( \varphi' = cf(t, \varphi) \) on \( [0, A] \) guaranteed by 2.2 (iii). We define the auxiliary function \( w \) on \( \overline{N \cap G} \) by \( w(x) = u(x) + z(\tau(x)) \) with

\[
z(\tau) = \frac{1}{c_1} \int_0^\tau \varphi(t) \exp \left( \frac{1}{\beta} \int_0^t B(s)ds \right) dt
\]

and

\[
c_1 = M \exp \left( \frac{1}{\beta} \int_0^{A_1} B(s)ds \right).
\]

Now

\[
z'(\tau) = \frac{1}{c_1} \varphi(\tau) \exp \left( \frac{1}{\beta} \int_0^\tau B(s)ds \right) > 0
\]

on \( [0, A_1] \), and

\[
z''(\tau) = \frac{1}{c_1} \left[ \varphi'(\tau) + \varphi(\tau) \frac{B(\tau)}{\beta} \right] \exp \left( \frac{1}{\beta} \int_0^\tau B(s)ds \right) > 0
\]

on \( (0, A] \) because \( \varphi > 0 \) on \( [0, A_1] \) and \( B(t) > 0 \) on \( (0, A] \).

We shall show by contradiction that \( w \) cannot attain its maximum over \( \overline{N \cap G} \) at an interior point of that set. Suppose, on the contrary, there is a maximum point \( x_0 \) in \( N \cap G \). Let \( E_0 \) be the linear operator associated with \( E \) and \( u \) and acting on \( w \), defined by \( E_0 w(x) = \Sigma a_i(x, u(x), 0)D_i w(x) \). Since \( x_0 \) is an interior maximum, \( E_0 w(x_0) \leq 0 \) (see, for example, Miranda [4], p. 4). We shall now show that

\[
E_0 w(x_0) > 0.
\]

Let \( \beta_2 = \Sigma a_i(x_0, u(x_0), 0)D_i \tau(x_0)D_i \tau(x_0) \). Then \( \beta_2 > \beta \) and there exists \( \mu > 0 \) such that

\[
\frac{f(t, \varphi_2) - f(t, \varphi_1)}{\varphi_2} > \frac{B_0(\beta_2 - \beta)}{(n^2K+1)c_1\beta}
\]
for all $\varphi_1$ and $\varphi_2$ satisfying $0 \leq \varphi_1 \leq \varphi_2 < \mu$.

Let us restrict $\varepsilon$ so that $\varphi(\tau(x_0)) < \mu$.

At $x_0$, we have $0 = \nabla w = \nabla u + z' \nabla \tau$; so that

$$\| \nabla u(x_0) \| = z'(\tau(x_0)) \| \nabla \tau(x_0) \| \leq Mz'(\tau(x_0)) =$$

$$= \frac{M}{c_1} \varphi(\tau(x_0)) \exp \left( \frac{1}{\beta} \int_0^{\tau(x_0)} B(s) ds \right) < \varphi(\tau(x_0)).$$

Therefore,

$$\frac{f(\tau(x_0), \varphi(\tau(x_0))) - f(\tau(x_0), \| \nabla u(x_0) \|)}{\varphi(\tau(x_0))} > \frac{B(\tau(x_0))(\beta_2 - \beta)}{(n^2K + 1)c_1\beta},$$

so that

$$(n^2K + 1)[f(\tau(x_0), \varphi(\tau(x_0))) - f(\tau(x_0), \| \nabla u(x_0) \|)]$$

$$> - \varphi(\tau(x_0)) \cdot \frac{1}{c_1} \cdot \frac{B(\tau(x_0))}{\beta} \cdot (\beta_2 - \beta) > z''(\tau(x_0))(\beta - \beta_2),$$

giving

$$-(n^2K + 1)f(\tau(x_0), \| \nabla u(x_0) \|) + z''(\tau(x_0)) \cdot \beta_2$$

$$> -(n^2K + 1)f(\tau(x_0), \varphi(\tau(x_0))) + z''(\tau(x_0)) \cdot \beta.$$

This implies that

$$E_0 \omega(x_0) \geq E_0 \omega(x_0) - Eu(x_0) =$$

$$= \Sigma a_{ij}(x_0, u(x_0), 0)D_{ij}w(x_0) - \Sigma a_{ij}(x_0, u(x_0), \nabla u(x_0))D_{ij}u(x_0) +$$

$$+ a(x_0, u(x_0), \nabla u(x_0)) \geq$$

$$\geq \Sigma[a_{ij}(x_0, u(x_0), 0) - a_{ij}(x_0, u(x_0), \nabla u(x_0))]D_{ij}u(x_0) +$$

$$+ [a(x_0, u(x_0), \nabla u(x_0)) - a(x_0, u(x_0), 0)] +$$

$$+ z''(\tau(x_0)) \beta_2 + z'(\tau(x_0))\Sigma a_{ij}(x_0, u(x_0), 0)D_{ij}\tau(x_0) \geq$$

$$\geq -(n^2K + 1)f(\tau(x_0), \| \nabla u(x_0) \|) + \beta_2 z''(\tau(x_0)) - z'(\tau(x_0))B(\tau(x_0)) >$$

$$> -(n^2K + 1)f(\tau(x_0), \varphi(\tau(x_0))) + \beta z''(\tau(x_0)) - z'(\tau(x_0))B(\tau(x_0)).$$
Now
\[ \beta z''(\tau) - B(\tau)z'(\tau) = \frac{\beta}{c_1} \exp \left( \frac{1}{\beta} \int_{0}^{\tau} B(s) \, ds \right) \phi'(\tau) = \frac{\beta}{c_1} \phi'(\tau) \]
and also
\[ \frac{c_1}{\beta} (n^2 K + 1) = \frac{M}{\beta} (n^2 K + 1) \exp \left( \frac{1}{\beta} \int_{0}^{A_1} B(s) \, ds \right) < \frac{M}{\beta} (n^2 K + 1)(1 + \nu) < c. \]
Thus,
\[ E_0 w(x_0) > -(n^2 K + 1)f(\tau(x_0), \varphi(\tau(x_0))) + \frac{\beta}{c_1} \phi'(\tau(x_0)) = \]
\[ = \frac{\beta}{c_1} \left[ \phi'(\tau(x_0)) - \frac{c_1}{\beta} (n^2 K + 1)f(\tau(x_0), \varphi(\tau(x_0))) \right] \geq \]
\[ \geq \frac{\beta}{c_1} \left[ \phi'(\tau(x_0)) - c(f(\tau(x_0), \varphi(\tau(x_0)))) \right] = 0. \]

From this contradiction, we conclude that \( w \) can attain its maximum only on \( \partial(N \cap G) \). We now show that by taking \( \epsilon \) small enough, this maximum can only be attained on \( N \cap \partial G \). There exists \( \eta > 0 \) such that \( u(x) < u(y) - \eta \) on \( \overline{G} \cap \partial N \). Restricting \( \epsilon \) further, we choose \( \epsilon < \frac{M \eta}{A_1} \), so that
\[ z(\tau) = \frac{1}{c_1} \int_{0}^{\tau} \varphi(t) \exp \left( \frac{1}{\beta} \int_{0}^{t} B(s) \, ds \right) \, dt \leq \frac{\tau}{c_1} \varphi(\tau) \exp \left( \frac{1}{\beta} \int_{0}^{A_1} B(s) \, ds \right) \leq \]
\[ \leq A_1 \epsilon < \eta \]
on \([0, A_1]\). Then
\[ w(x) = u(x) + z(\tau(x)) < u(y) \]
on \( \overline{G} \cap \partial N \). Therefore, the maximum of \( w \) is attained only on \( N \cap \partial G \). Since \( w = u \) there, \( w(x) \leq u(y) \) on \( \overline{N \cap G} \). In particular, for \( x \in I \cap N \),
we have
\[ \frac{u(x) - u(y)}{|x - y|} \leq \frac{z(\tau(y)) - z(\tau(x))}{|x - y|}. \]

Therefore,
\[ \lim_{x \to y} \sup_{x \in I} \frac{u(x) - u(y)}{|x - y|} \leq -z'(0) |\text{grad } \tau(y)| \cos (ln) \leq -mz'(0) \cos (ln) < 0. \]

This proves the theorem.

**Remark 3.1.** All the conditions on \( u \) listed in 2.3 and the conditions of (ii) in the statement of the theorem need be assumed only in some neighbourhood of \( y \). Also, the conditions on \( \tau \) listed in 2.5 can be replaced by the following:

There exists a neighbourhood \( N \) of \( y \) and a function \( \tau \) defined on \( N \cap \mathcal{G} \) satisfying

(i) \( \tau = 0 \) on \( \partial G \cap N \) and \( \tau > 0 \) in \( G \cap N \);

(ii) \( \tau \in C^1(\mathcal{G} \cap N) \cap C^2(G \cap N) \);

(iii) \( |\text{grad } \tau(x)| \leq M \) in \( G \cap N \) and

\( |\text{grad } \tau(x)| \geq m > 0 \) on \( \partial G \cap N \).

In view of the above, we may weaken the assumption « \( a(x, u(x), 0) \geq 0 \) » to « \( a(x, u(x), 0) \geq 0 \) if \( u(x) > 0 \) » if we assume that \( u(y) > 0 \).

**Remark 3.2.** If we modify the hypothesis of Theorem 3.1 so that
\[ Eu \leq 0, \quad a(x, u(x), 0) \leq 0, \quad u(x) > u(y) \]
for all \( x \in \mathcal{G} \) with \( x \neq y \), and
\[ a(x, u(x), 0) - a(x, u(x), \text{grad } u(x)) \geq -f(\tau(x), |\text{grad } u(x)|), \]
while leaving the other conditions as they are, then
\[ \lim_{x \to y} \inf_{x \in I} \frac{u(x) - u(y)}{|x - y|} > 0. \]
REMARK 3.3. If $a_{ij}(x, u, \text{grad } u) = a_{ij}(x, u)$, we can drop the assumption that $D_{ij}u(x)$ is bounded and the theorem remains valid. In the proof, the difference

$$a_{ij}(x, u(x), 0) - a_{ij}(x, u(x), \text{grad } u(x))$$

is zero, so that we require only $c > \frac{M}{\beta_1}$.

REMARK 3.4. The existence condition on $f$ (see 2.2 (iii)) is essential. Consider the operator $Eu = u'' - \alpha a(x, u')$ on $(0, 1)$, where

$$a(x, y) = \begin{cases} 
0 & \text{if } x \leq 0 \text{ or } y \leq 0, \\
2y/x & \text{if } 0 \leq y \leq x^2, \\
2x & \text{if } x^2 \leq y.
\end{cases}$$

Let $\tau(x) = x$, $B(t) = 1$, and $f(t, \varphi) = -\alpha a(t, \varphi)$. Using these functions, one can show that for $0 < \alpha < 1$ the hypothesis of the minimum principle (Remark 3.2) holds at $x = 0$, but that for $\alpha \geq 1$ the only condition that does not hold is (iii) of 2.2. In the latter case, the function $u = \frac{1}{3} \alpha x^3$ satisfies $Eu = 0$, but $u'(0) = 0$.

REMARK 3.5. Theorem 3.1 is a generalization of Theorem 2 of Výborný [3]. If the hypothesis of Výborný's theorem holds, then so does the hypothesis of Theorem 3.1: let $f(t, \varphi) = B(t)g(\varphi)$ for $0 < t < \infty$ and $0 < \varphi < \infty$, and $f(t, 0) = 0$.

4. An extension of Theorem 3.1.

As it stands, Theorem 3.1 does not contain as a special case the linear operator treated by Pucci in [5]. In Pucci's theorem, the domain is a sphere $S$,

$$Eu(x) = \sum_{i, j=1}^{n} a_{ij}(x) D_{ij}u(x) + \sum b_i(x) D_iu(x) + c(x)u(x),$$

and $\tau(x) = r_0 - |x - \eta|$, where $r_0$ and $\eta$ are the radius and center of $S$. 
The conditions on the coefficients are as follows:

(A) \( \lim \inf_{x \to y} \sum_{x \in S} a_{ij}(x)D_{ij}(x) > 0; \)

(B) there exists a continuous, positive, decreasing function \( B(\tau) \) defined for \( 0 < \tau < r_0 \), such that \( \int_0 B(t)dt < \infty \) and

\[
\lim \inf_{x \to y} \frac{b_i(x)D_i(x)}{B(\tau(x))} > -1;
\]

(C) \( c(x) \leq 0 \) and

\[
\lim \inf_{x \to y} \frac{c(x)\tau(x)}{B(\tau(x))} > -1,
\]

where \( B \) is the function of condition (B).

He concludes that \( u \) cannot attain a non-negative maximum at \( y \in \partial S \) unless either \( u \) is constant or

\[
\lim \inf_{x \to y} \frac{u(x) - u(y)}{|x - y|} < 0,
\]

where \( l \) is as in Theorem 3.1.

We remark, in passing, that if \( u(y) > 0 \), then the second part of condition (C) may be dropped.

If \( b_i \equiv 0 \) for \( i = 1, \ldots, n \), \( c(x) \leq 0 \), and \( u(y) > 0 \), then Pucci's hypothesis implies ours, since there will be a neighbourhood of \( y \) where \( a(x, u(x), 0) = -c(x)u(x) \geq 0 \). However, if \( u(y) = 0 \), the inequality \( a(x, u(x), 0) = -c(x)u(x) \geq 0 \) may not be satisfied in any neighbourhood of \( y \).

If \( n = 1 \) and \( c \equiv 0 \), the hypothesis of Theorem 3.1 follows from Pucci's hypothesis if we let

\[
f(\tau(x), |u'(x)|) = B(\tau(x)) \cdot |u'(x)|.
\]

However, if \( n > 1 \), Pucci's condition (B) does not necessarily imply that

\[
a(x, u(x), 0) - a(x, u(x), \text{grad } u(x)) = \sum b_i u_i \leq f(\tau(x), |\text{grad } u(x)|)
\]

for some functions \( \tau \) and \( f \).
In order to include Pucci’s theorem, we modify Theorem 3.1 by adding extra terms to $E$.

**THEOREM 4.1.** Suppose the hypothesis of Theorem 3.1 holds except that we replace $E$ by $E^+$ where

$$E^+ u(x) = \sum a_i(x, u(x), \nabla u(x))D_iu(x) - a(x, u(x), \nabla u(x)) + \sum b_i(x, u(x), \nabla u(x))D_iu(x) + c(x, u(x), \nabla u(x)) \cdot u(x).$$

The functions $b_i$ and $c$ are defined on $G \times R_1 \times R_n$, $c(x) \leq 0$ in some neighbourhood of $y$,

$$\lim_{x \to y} \inf_{x \in G} \frac{1}{B(y(x))} \sum b_i(x, u(x), \nabla u(x))D_i\tau(x) > -\infty,$$

and

$$\lim_{x \to y} \inf_{x \in G} \frac{c(x, u(x), \nabla u(x)) \cdot \tau(x)}{B(y(x))} > -\infty,$$

where $B$ and $\tau$ are the functions of Theorem 3.1. Moreover, we assume $u(y) \geq 0$.

Then the conclusion of Theorem 3.1 holds.

**NOTE.** If $c \equiv 0$, we can remove the condition $u(y) \geq 0$. Also, trivially, we may use different functions $B_1$ and $B_2$ for the last inequalities, so long as they satisfy the conditions of 2.4.

**PROOF.** The proof follows that of Theorem 3.1 except that we use the auxiliary function

$$z(\tau) = \frac{1}{c_1} \int_0^\tau \varphi(t) \exp \left[ \left( \frac{2+n}{\beta} \right) \int_0^t B(s) ds \right] dt,$$

where

$$c_1 = M \exp \left[ \left( \frac{2+n}{\beta} \right) \int_0^{A_1} B(t) dt \right],$$
and use the auxiliary operator defined by

\[ E_0^+ w(x) = \sum a_{ij}(x, u(x), 0)D_{ij}w(x) + \sum b_i(x, u(x), \text{grad } u(x))D_iw(x) + c(x, u(x), \text{grad } u(x)) \cdot w(x). \]

**Remark 4.1.** The counterexamples provided by Pucci [5] show that the bounds on the growth of the coefficients \( c \) and \( b_i, i = 1, \ldots, n, \) are essential.

**Remark 4.2.** Similar extensions can be made to the theorems of the following sections. However, for simplicity, we consider only the original operator \( E \).

5. **The interior maximum principle.**

**Theorem 5.1.** Let \( G \) and \( E \) be as in 2.3. Let \( u \) be a function satisfying conditions (ii)-(iv) of 2.3 and \( (i') u \in C^2(G) \). Suppose that \( G, u, \) and the coefficients of \( E \) satisfy the following interior condition.

\((\text{IC})\) To each sphere \( S \) with \( \overline{S} \subset G \), there correspond

(a) a constant \( \gamma \), satisfying

\[ \sum_{i,j=1}^n a_{ij}(x, u(x), 0)\lambda_i \lambda_j \geq \gamma |\lambda|^2 > 0 \]

for all \( x \in S \) and all \( \lambda \in \mathbb{R}^n \); and

(b) functions \( f_s, B_s, \) and \( \tau_s \) satisfying the conditions of 2.2, 2.4 and 2.5 (except possibly for iv) with constants \( M_s, m_s, c_s \), and so on, such that

\[ |a_{ij}(x, u(x), 0) - a_{ij}(x, u(x), \text{grad } u(x))| \leq f_s(\tau_s(x), |\text{grad } u(x)|), \]

\[ a(x, u(x), 0) - a(x, u(x), \text{grad } u(x)) | \leq f_s(\tau_s(x), |\text{grad } u(x)|), \]

and

\[ \sum_{i,j=1}^n a_{ij}(x, u(x), 0)D_{ij}\tau_s(x) \geq -B_s(\tau_s(x)) \]
for all \( x \in \mathcal{S} \) (or just for all \( x \) within a distance \( \eta_s \) of \( \partial \mathcal{S} \), where \( \eta_s \) is some positive constant depending on \( \mathcal{S} \)). Let the constants involved satisfy the inequality

\[
\frac{\gamma_s}{m_s^2} (n^2K_s+1),
\]

where \( K_s = \sup \{ |D_iu(x)| : x \in \mathcal{S} \} \).

We conclude that \( u \) cannot attain its maximum in the interior of \( \mathcal{G} \) unless \( u \) is constant.

PROOF. Suppose \( u \) is not constant on \( \mathcal{G} \) but \( u(x_0) = \max \{ u(x) : x \in \mathcal{G} \} \) for some \( x_0 \in \mathcal{G} \). Then, there are \( x_1 \) and \( x_2 \) in \( \mathcal{G} \) such that \( u(x_1) < u(x_2) = u(x_0) \) and \( |x_1 - x_2| < \text{dist} (x_1, \partial \mathcal{G}) \). There is an open sphere \( \mathcal{S}_1 \) about \( x_1 \) in which \( u(x) < u(x_0) \). Expand \( \mathcal{S}_1 \) if necessary, until its surface touches a point \( x_3 \) where \( u(x_3) = u(x_0) \) but \( u(x) < u(x_0) \) for \( x \in \mathcal{S}_1 \). Note that we have ensured \( x_3 \in \mathcal{G} \). Let \( \mathcal{S} \) be a subsphere of \( \mathcal{S}_1 \) with \( \partial \mathcal{S} \cap \partial \mathcal{S}_1 = \{ x_3 \} \). We may apply Theorem 3.1 to the sphere \( \mathcal{S} \) at \( x_3 \) because

\[
\beta_s = \liminf_{x \to x_3} \sum_{i,j=1}^{n} a_{ij}(x, u(x), 0)D_i\tau_s(x)D_j\tau_s(x) \geq \gamma_s |\text{grad} \tau_s(x_3)|^2 \geq m_s^2 \gamma > 0.
\]

Thus, \( D_iu(x_3) < 0 \) where \( \nu \) is the inner normal to \( \partial \mathcal{S} \) at \( x_3 \), contrary to the fact that \( x_3 \) is an interior maximum. This proves the theorem.

REMARK 5.1. We can weaken the uniform ellipticity condition \( IC \) (a) to

\[
\sum_{i,j=1}^{n} a_{ij}(x, u(x), 0)\lambda_i \lambda_j > 0
\]

for all \( x \in \mathcal{G} \) and \( \lambda \in \mathcal{R}^n \), provided that the coefficients \( a_{ij}(x, u(x), 0) \) are continuous in \( x \) on \( \mathcal{G} \) and provided that the inequality involving the constants is replaced by the stronger condition that for each \( \mathcal{S} \) there is a sequence \( c_{sk} \to \infty \) such that \( f_s \) satisfies 2.2 (iii) for each \( c_{sk} \), in this case, there will be a positive constant \( \gamma(x_3) \) and a neighbourhood of \( x_3 \) in which

\[
\sum a_{ij}(x, u(x), 0)D_i\tau_s(x)D_j\tau_s(x) > \gamma(x_3) > 0.
\]
In applying Theorem 3.1, we confine ourselves to this neighbourhood and take \( k_0 \) large enough that

\[
c_{sk_0} > \frac{M_s}{\gamma(x_0)} (n^2 K_s + 1).
\]

**Remark 5.2.** We may weaken the interior condition (IC) to the following. To each sphere \( S \) with \( S \subset G \) and each point \( y \in \partial S \), there correspond

(a) a neighbourhood \( N_{sy} \) of \( y \);

(b) a constant \( \gamma_{sy} \) such that

\[
\sum a_{ij}(x, u(x), 0)\lambda_i \lambda_j \geq \gamma_{sy} |\lambda|^2 > 0
\]

for all \( x \in S \cap N_{sy} \) and \( \lambda \in \mathbb{R}^n \); and

(c) functions \( f_{sy}, B_{sy}, \) and \( \tau_{sy} \) satisfying the conditions of 2.2, 2.4, and 2.5 except (iv), with constants \( M_{sy}, m_{sy}, c_{sy}, \) and so on, such that

\[
| a_{ij}(x, u(x), 0) - a_{ij}(x, u(x), \text{grad } u(x)) | \leq f_{sy}(\tau_{sy}(x), |\text{grad } u(x)|),
\]

\[
a(x, u(x), 0) - a(x, u(x), \text{grad } u(x)) \leq f_{sy}(\tau_{sy}(x), |\text{grad } u(x)|),
\]

and

\[
\sum a_{ij}(x, u(x), 0)D_{ij}\tau_{sy}(x) \geq -B_{sy}(\tau_{sy}(x))
\]

for all \( x \in S \cap N_{sy} \). Let the constants involved satisfy the inequality

\[
c_{sy} > \frac{M_{sy}}{m_{sy}^2 \gamma_{sy}} (n^2 K_{sy} + 1),
\]

where \( K_{sy} = \sup \{|D_{ij}u(x)| : x \in S \cap N_{sy}\} \).

**Remark 5.3.** There is a corresponding interior minimum principle. If we modify the hypothesis of Theorem 5.1 so that \( Eu \leq 0, a(x, u(x), 0) \leq 0, \) and

\[
a(x, u(x), 0) - a(x, u(x), \text{grad } u(x)) \geq -f_s(\tau_s, |\text{grad } \tau(x)|),
\]
while leaving the other conditions as they are, then \( u \) cannot attain its minimum in the interior of \( G \) unless \( u \) is constant.

**Remark 5.4.** Theorem 5.1 is a generalization of Theorem 4 of Redheffer [2]. If Redheffer's hypothesis holds then so does the hypothesis of Theorem 5.1. For a sphere \( S \) with \( \mathcal{S} \subseteq G \), let \( \gamma_i = \frac{1}{L_i} \); \( \tau_i(x) = r^2 - |x - \bar{x}|^2 \), where \( r \) and \( \bar{x} \) are the radius and center of \( S \);

\[
B_i(t) = (2n) \max \{|a_{ii}(x, u(x), 0)| : 0 \leq i \leq n, x \in \mathcal{S}\};
\]

and \( f(t, \varphi) = g(\varphi) \) for \( 0 < t < \infty \) and \( 0 < \varphi < \infty \), and \( f(t, 0) = 0 \).

**6. The boundary maximum principle.**

Before stating the main result of the section, Theorem 6.2, we modify Theorem 3.1, so that the hypothesis no longer requires \( u(x) < u(y) \) for points \( x \neq y \) on the boundary \( \partial G \).

**Theorem 6.1.** Suppose the hypothesis of Theorem 3.1 holds on \( G \) except that \( u(x) < u(y) \) on \( G \) instead of and condition (iii) is replaced by the two conditions

(i) \( D_y \tau_i \) is bounded on \( G \) for each \( i, j = 1, \ldots, n \) (at least in some neighbourhood of \( y \)) by \( B(\tau) \),

(ii) \( B \) is non-increasing and \( a_{ii}(x, u(x), 0) \) is continuous at \( y \) for all \( i, j = 1, \ldots, n \).

Suppose also

(iii) \( f \) is a non-increasing function of its first variable \( t \) (at least in some neighbourhood of \( t = 0 \)) and condition (iv) of 2.5 holds for \( \tau \).

Then the conclusion of Theorem 3.1 holds.

**Proof.** The proof consists in deforming \( G \) in a neighbourhood of \( y \) in such a way that \( u(x) < u(y) \) on the boundary of the deformed domain and Theorem 3.1 can be applied.

Since \( \left| \text{grad} \, \tau(x) \right| > 0 \) on \( \partial G \), we have \( D_i \tau_i(y) \neq 0 \) for some \( i \), say \( n \). Without loss of generality, let \( D_n \tau(y) > 0 \). Let \( N \) be a sphere centred
at \( y \) in which \( \tau \) is continuously differentiable, \( D_t \tau(x) > 0 \), and conditions (i)-(iii) of the hypothesis hold; for condition (iii) this means that \( f(t, \varphi) \) is a non-increasing function of \( t \) for all \( t \) with

\[
0 \leq t \leq \sup \{ \tau(x) : x \in N \cap G \}.
\]

Define the transformation \( g : R_\ell \to R_\ell \) by

\[
g(x) = (g_1(x), \ldots, g_n(x)) = (x_1, \ldots, x_{n-1}, x_n + \sum_{i=1}^{n-1} (y_i - x_i)^2).
\]

Let \( h \) be the inverse of \( g \), and let \( G_1 \) be the image of \( G \) under \( g \). The implicit functions theorem guarantees the existence of a sphere \( S \) in \( R_{n-1} \) with center \((y_1, \ldots, y_{n-1})\) and a unique continuous function \( s(x_1, \ldots, x_{n-1}) \) defined on \( S \) such that \( y_n = s(y_1, \ldots, y_{n-1}) \) and

\[
\tau(x_1, \ldots, x_{n-1}, s(x_1, \ldots, x_{n-1})) = 0 \text{ for } (x_1, \ldots, x_{n-1}) \in S;
\]

if one takes \( N \) small enough, the equation \( x_n = s(x_1, \ldots, x_{n-1}) \) represents \( \partial G \) in \( N \), and no other points of \( \partial G \) lie in \( N \). Since \( D_n \tau(x) > 0 \) in \( G \cap N \), \( G \cap N \) lies in the positive \( x_n \) direction from the graph of \( s \). The image \( s_1 \) of \( s \) under \( g \) is the boundary of \( G_1 \); the point \( y \) remains fixed and for any other point in \( S \) satisfying \( (x_1, \ldots, x_{n-1}) \neq (y_1, \ldots, y_{n-1}) \), we have \( s_1(x_1, \ldots, x_{n-1}) > s(x_1, \ldots, x_{n-1}) \). Define the function \( \tau_1 \) on \( \Omega_1 \cap N \) by \( \tau_1(x) = \tau(h(x)) \).
We verify that if $N$ is small enough, the hypothesis of Theorem 3.1 holds in $G_1 \cap N$. Obviously, $\tau_1$ satisfies conditions (i) and (ii) of 2.5. We check (iii). Choose $M_1 > M$ such that $\frac{M_1}{\beta_1} (n^2K+1) < c$, and let

$$\eta < \min \left\{ \frac{3m^2}{4}, M_1 - M \right\}.$$ 

Now,

$$|| \text{grad } \tau_1(x) ||^2 - || \text{grad } \tau(x) ||^2 \leq || \text{grad } \tau(h(x)) ||^2 - || \text{grad } \tau(x) ||^2 +$$

$$+ \left[ 4D_n \tau(h(x)) \sum_{i=1}^{n-1} D_i \tau(h(x))(y_i - x_i) + 4D_n \tau(h(x)) \right]^2 \sum_{i=1}^{n-1} (y_i - x_i)^2 \leq$$

$$\leq || \text{grad } \tau(h(x)) ||^2 - || \text{grad } \tau(x) ||^2 + 4M^2(n-1)(\text{diam } N) + 4M^2(\text{diam } N)^2.$$ 

We may take $N$ small enough that this expression is less than $\eta$ for $x \in N \cap \overline{G}_1$. Then $| \text{grad } \tau_1(x) | < M_1$ for $x \in N \cap \overline{G}_1$ and $| \text{grad } \tau_1(x) | > \frac{m}{2}$ for $x \in N \cap \partial G_1$. Now, we show that conditions (i)-(iii) of the hypothesis of Theorem 3.1 hold.

(i) $\lim \inf_{x \to y, x \in G_1} \Sigma a_{ij}(x, u(x), 0)D_j \tau_1(x)D_i \tau_1(x) = $ 

$$\Sigma a_{ij}(y, u(y), 0)D_j \tau(y)D_i \tau(y) = \beta_1.$$ 

(ii) The monotonicity of $f$ and the fact that $\tau_1(x) \leq \tau(x)$ in $N \cap G_1$ imply that the inequalities in (ii) hold.

(iii) Calculation of the second derivatives of $\tau_1$ and application of conditions (i) and (ii) of the hypothesis give us that

$$\Sigma a_{ij}(x, u(x), 0)D_j \tau_1(x) \geq -B(\tau_1(x)) \cdot T$$

on $N \cap G_1$, where $T$ is a constant. Thus $B_l(t) = B(t) \cdot T$ defines a function satisfying 2.4 and condition (iii) of Theorem 3.1.

We conclude that the hypothesis of Theorem 3.1 holds on $N \cap G_1$. Since $G_1$ contains an interval of a half ray $I$ with endpoint $y$ if the same is true of $G$, Theorem 6.1 is proved.

Theorem 6.2 (Boundary maximum principle). Let $G$, $E$ and $u$ satisfy 2.3 and suppose that
for all $x \in G$ in some neighbourhood of $\partial G$ and all $\lambda \in \mathbb{R}^n$. Suppose that part (b) of the interior condition (IC) of Theorem 5.1 holds with $\gamma$ for $\gamma$, and, in addition, the following boundary condition (BC) holds.

(BC) There are functions $f$, $B$ and $\tau$ satisfying 2.2, 2.4 and 2.5 such that

(a) $|a_{ij}(x, u(x), 0) - a_{ij}(x, u(x)), \text{grad } u(x)| \leq f(\tau(x), |\text{grad } u(x)|)$

and

(a) $|a_{ij}(x, u(x), 0) - a_{ij}(x, u(x)), \text{grad } u(x)| \leq f(\tau(x), |\text{grad } u(x)|)$

in $G$ in some neighbourhood of $\partial G$;

(b) $f$ is non-increasing in the first variable $t$, at least in some neighbourhood of $t = 0$;

(c) the constant $c$ associated with $f$ satisfies

$$c > \frac{M}{m^{2\gamma}}(n^2 K + 1)$$

(d) $D_{\tau^2}$ is bounded in some neighbourhood of $\partial G$ for each $i, j = 1, \ldots, n$ by $B(\tau)$;

(e) $B$ is non-increasing and $a_{ij}(x, u(x), 0)$ is continuous at $y$ for all $i, j = 1, \ldots, n$.

Then $u$ does not attain its maximum at any point $y$ of $\partial G$ unless either $u$ is constant in $G$ or

$$\limsup_{x \to y, x \neq y} \frac{u(x) - u(y)}{|x - y|} < 0,$$

where $l$ is any half ray of the type described in Theorem 3.1.

PROOF. This is a simple consequence of Theorems 5.1 and 6.1.
REMARK 6.1. Uniform ellipticity can be weakened to
\[ \sum a_{ij}(x, u(x), 0)\lambda_i\lambda_j > 0 \]
for all \( x \in \Omega \) and \( \lambda \in \mathbb{R}^n \) provided that the coefficients \( a_{ij}(x, u(x), 0) \) are continuous in \( x \) on \( \Omega \) and provided that the constants \( c \) and \( c_0 \), corresponding to \( f \) in (BC) and each \( f_s \) in (IC) can be chosen arbitrarily large, as described in Remark 5.1.

There is also a boundary minimum principle.

7. Application to a boundary value problem.

In the usual way, Theorems 5.1 and 6.2 give us the following uniqueness theorem.

THEOREM 7.1. Let \( u \in C^1(\partial G) \cap C^0(G) \) with \( |D_i u(x)| \leq K \) for all \( x \in G \). Let \( u \) be a solution of the boundary value problem

\[ Eu = 0 \text{ on } G, \]

\[ b(x) = \alpha(x, u(x), \nabla u(x)) + \beta(x, u(x), \nabla u(x))\cdot u(x) = 0 \]

on \( \partial G \),

where

\[ \alpha(x, u(x), \nabla u(x)) \geq 0, \quad \beta(x, u(x), \nabla u(x)) \leq 0, \]

and

\[ |\alpha(x, u(x), \nabla u(x))| + |\beta(x, u(x), \nabla u(x))| > 0; \]

\( l \) denotes a vector forming an acute angle with the inner normal to \( \partial G \) at \( x \) (\( l \) may vary with \( x \)). Suppose, also, that

\[ \sum a_{ij}(x, u(x), 0)\lambda_i\lambda_j \geq \gamma |\lambda|^2 > 0 \text{ for all } x \in G, \]

\( u(x) \cdot a(x, u(x), 0) \) on \( G \), and both the interior and boundary conditions, (IC) and (BC), hold with absolute value signs around

\[ a(x, u(x), 0) - a(x, u(x), \nabla u). \]
Then $u$ is constant in $G$.

**Proof.** It follows from Theorems 5.1 and 6.2 and their corresponding minimum principles (Remarks 5.3 and 6.1) that $u$ cannot attain a positive maximum or a negative minimum on $\partial G$ unless $u$ is constant.

**Remark 7.1.** If $\alpha=0$ at any point in $\partial G$, then $u \equiv 0$ in $G$.

**References**


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