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CONNECTIONS AND 1-JET FIBER BUNDLES

PEDRO L. GARCÍA *)

Introduction.

An important concept in modern Differential Geometry is the *k-jet fiber bundle* one. The use made of it in the recent theories on non-linear differential operators, global calculus of variations etc. is a well known fact (see [8], for instance). The case $k=1$ is especially interesting for the following reason: If $p : \overline{E} \to E$ is the 1-jet fiber bundle of the local sections of a differentiable fiber bundle $\pi : E \to X$, $q : T^v(E) \to E$ is the sub-bundle of the tangent vector bundle of E defined by the vectors tangent to the fibres of E, $\overline{A}$ is the algebra of differentiable functions on $\overline{E}$, and $\overline{M}$ is the $\overline{A}$-module of the differentiable fiber bundle morphisms $f : \overline{E} \to T^v(E)$ over E (with respect to: $(f+g)(\bar{e})=f(\bar{e})+g(\bar{e})$ and $(a \cdot f)(\bar{e})=a(\bar{e}) \cdot f(\bar{e})$; $f, g \in \overline{M}$, $a \in \overline{A}$), then we can introduce, in a canonical way, a special $\overline{M}$-valued 1-form θ on $\overline{E}$ (the *structure* 1-*form* of the 1-jet fiber bundle $\overline{E}$). The object defined by $(\overline{E}, \theta)$ is especially adecuate for the geometric-differential treatment of the variational problems that appear in the *theory of classical fields*. We have dealt this subject in several papers ([1], [2], [3], [4]). Our aim in this paper is to answer the following question: What can be said about the structure $(\overline{E}, \theta)$ when $\pi : E \to X$ is a principal G-bundle?.

We shall first show that the structure group G can be considered as acting on the 1-jet fiber bundle $\overline{E}$ on the right, freely and differen-

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tiably, in a natural way. If $\overline{E}/G$ is the set of orbits of G in $\overline{E}$, then a natural differentiable structure can be defined on $\overline{E}/G$ such that $\overline{E}$ becames a principal G-bundle over $\overline{E}/G$ and the projection $p : \overline{E} \to E$ becames a morphism of principal G-bundles (Theorem 1). So, « taking the 1-jet fiber bundle » allows us to associate a principal G-bundle $\overline{E}$ to each principal G-bundle E. This correspondence is a covariant functor from the category $\mathrm{Bun}_X(G)$ of principal G-bundles over X to the category $\mathrm{Bun}(G)$ of principal G-bundles. Theorem 2 characterizes, in terms of this functor, the connections on a principal fiber bundle. We prove that, for every principal G-bundle E, there exists a natural one to one correspondence between the set of connections on E and the set of sections of the canonical morphism $p : \overline{E} \to E$.

By means of the fundamental vector fields on the principal G-bundle E, the $\overline{A}$-module $\overline{M}$ can be canonically identified with the $\overline{A}$-module of differentiable mappings of $\overline{E}$ into the Lie algebra $\mathcal{G}$ of G. Hence, the structure 1-form θ of the 1-jet fiber bundle $\overline{E}$ is a $\mathcal{G}$-valued 1-form on $\overline{E}$. In Theorem 3 we prove that θ is a connection 1-form on $\overline{E}$ with the universal property following: if σ is a connection on E (regarded as a section of the morphism $p : \overline{E} \to E$ in the sense of Theorem 2) and θ^σ is its connection 1-form, then $\theta^\sigma = \sigma^* \theta$.

We belive this version of connection theory to be useful in all those situations that require the consideration of all connections at a time rather than a particular connection. A typical example of this is the Weil's theorem on characteristic classes. In section 4 we give, using the ideas developed in this paper, a simple proof of this theorem.

Concepts and notation not specified in this paper are the usual in modern texts of Differential Geometry. We always assume differentiability to be of class C^∞, all differentiable manifolds are assumed to be paracompacts and we often use the words « connection » and « connection 1-form » as synonomous.

This paper originates from talks with Proffessor J. Sancho Guimerá, some years ago. I want to thank Proffesor Sancho for his valuable advices, without them, this paper should not have, probably, been written.

1. Structure 1-form of the 1-jet fiber bundles.

Let $\pi : E \to X$ be a differentiable fiber bundle over X. Given $e \in E$ with $\pi(e)=x$, we say that the local sections s, s' of E defined near x with $s(x)=s'(x)=e$, have the same 1-jet at x when $(ds)_x=(ds')_x$. This defines an equivalence relation on the set of local sections s of E defined near x with $s(x)=e$. The set of equivalence classes is denoted by $\overline{E}_e$ and the equivalence class of s is denoted by $\bar{s}_x$ and called *1-jet at x of the section s*. Let $\overline{E}=\bigcup_{e \in E} \overline{E}_e$ and let $p : \overline{E} \to E$ be the function which maps $\overline{E}_e$ to e. We shall see that $p : \overline{E} \to E$ has a natural structure of differentiable fiber bundle.

Let $e_0 \in E$ with $\pi(e_0)=x_0$ and let U be an open neighborhood of x_0, with local coordinates (x_i), and such that a trivialisation $\varphi : E_U \to U \times E_{x_0}$ of the fiber bundle $E_U=\pi^{-1}(U)$ exists over U; finally, let V be a open neighborhood of e_0 in the fibre E_{x_0} with local coordinates (y_j). A natural bijection f from $\overline{E}_{\varphi^{-1}(U \times V)}=p^{-1}(\varphi^{-1}(U \times V))$ to the trivial fiber bundle $p' : U \times V \times R^{n \cdot m} \to U \times V$ can be established such that the following diagram is commutative:

$$\begin{tikzcd}
\overline{E}_{\varphi^{-1}(U \times V)} \arrow[r, "f"] \arrow[d, "p"'] & U \times V \times R^{n \cdot m} \arrow[d, "p'"] \\
\varphi^{-1}(U \times V) \arrow[r] & U \times V
\end{tikzcd}$$

We define f as follows:

$$f(\bar{s}_x)=\left(\varphi(s(x)), \left(\frac{\partial y_j}{\partial x_i}\right)_x\right) \in U \times V \times R^{n \cdot m},$$

where $y_j=y_j(x_i)$ are the equations of s with respect to the coordinate system defined by U, φ and V. As φ *is* a diffeomorphism, the differentiable fiber bundle structure of $U \times V \times R^{n \cdot m}$ defines, by f, a differentiable fiber bundle structure on $\overline{E}_{\varphi^{-1}(U \times V)}$. It is easily seen that this structure does not depend on the choice of U, φ and V. Thus, the differentiable fiber bundle structure of $p : \overline{E} \to E$ is well defined. The fiber bundle thus defined is called *1-jet fiber bundle* of the sections of the fiber bundle $\pi : E \to X$.

The functions $(x_i y_j p_{ij})$ defined on $\overline{E}_{\varphi^{-1}(U\times V)}$ by the rule:

$$x_i(\bar{s}_x)=x_i(x), \qquad y_j(\bar{s}_x)=y_j(s(x)), \qquad p_{ij}(\bar{s}_x)=\left(\frac{\partial y_j}{\partial x_i}\right)_x$$

are a local coordinate system of $\overline{E}$. Such coordinates will be called *natural coordinates* on the 1-jet fiber bundle.

$\overline{E}$ can also be regarded as a fiber bundle over X taking $\bar{p}=\pi\cdot p$ as projection. If $\pi : E\to X$ is a vector bundle, then $\bar{p} : \overline{E}\to X$ is a vector bundle too (of course, we must take vector bundle trivialisation $\varphi : E_U\to U\times E_{x_0}$ in the former construction).

If s is a section of $\pi : E\to X$ defined over a open set U, then the mapping $\bar{s} : x\in U\to \bar{s}_x\in\overline{E}$ is a section of $\bar{p} : \overline{E}\to X$. This section is called 1-*jet extension* of the section s. Obviously, the mapping $s\to\bar{s}$ is a inyection from the set Γ of sections of $\pi : E\to X$ to the set $\overline{\Gamma}$ of sections of $\bar{p} : \overline{E}\to X$.

Let s be a section of $\pi : E\to X$ defined near x, $T_{s(x)}(E)$ the tangent space of E at $s(x)$ and $T^v_{s(x)}(E)$ the subspace of $T_{s(x)}(E)$ defined by the vectors tangent to the fibre E_x of E.

DEFINITION 1. The linear mapping from $T_{s(x)}(E)$ to $T^v_{s(x)}(E)$ given by:

$$D\to D-d(s\circ\pi)_{s(x)}D$$

will be called the *vertical differential of s at x* and noted $(d^v s)_x$.

It follow inmediately that two sections s, s' of E defined near x with $s(x)=s'(x)$ have the same vertical differential at x if and only if they have the same 1-jet at x. This allows the following:

DEFINITION 2. The assignment to each $\bar{s}_x\in\overline{E}$ of the linear mapping $\theta_{\bar{s}_x} : T_{\bar{s}_x}(E)\to T^v_{s(x)}(E)$ given by: $\theta_{\bar{s}_x}(\overline{D})=(d^v s)_x p'_{\bar{s}_x}\overline{D}$ will be called *structure* 1-*form of the* 1-*jet fiber bundle* $p : \overline{E}\to E$ and denoted by θ.

The following Lemma furnishes an expression of the structure 1-form in terms of local coordinate systems.

LEMMA 1. If $(x_i y_j p_{ij})$ is a natural coordinate system of the 1-jet fiber bundle $p : \overline{E}\to E$ over a open neighborhood $\overline{U}$, then, for each $\bar{s}_x\in\overline{U}$, we have:

$$\theta_{\bar{s}_x}(\overline{D})=\sum_j\left[(dy_j-\sum_i p_{ij}dx_i)_{\bar{s}_x}\overline{D}\right]\left(\frac{\partial}{\partial y_j}\right)_{s(x)}$$

PROOF. It suffices to prove the equality when $\overline{D}$ is $\left(\frac{\partial}{\partial x_i}\right)_{\bar{s}_x}$, $\left(\frac{\partial}{\partial y_j}\right)_{\bar{s}_x}$ and $\left(\frac{\partial}{\partial p_{ij}}\right)_{\bar{s}_x}$. If $y_j=y_j(x_i)$ are the equations of s, we have:

$$\theta_{\bar{s}_x}\left(\frac{\partial}{\partial x_i}\right)_{\bar{s}_x}=(d^vs)_x\left(\frac{\partial}{\partial x_i}\right)_{s(x)}=\left(\frac{\partial}{\partial x_i}\right)_{s(x)}-(ds)_x\left(\frac{\partial}{\partial x_i}\right)_x=$$

$$=-\sum_j\left(\frac{\partial y_j}{\partial x_i}\right)_x\left(\frac{\partial}{\partial y_j}\right)_{s(x)}=-\sum_j p_{ij}(\bar{s}_x)\left(\frac{\partial}{\partial y_j}\right)_{s(x)}$$

$$\theta_{\bar{s}_x}\left(\frac{\partial}{\partial y_j}\right)_{\bar{s}_x}=(d^vs)_x\left(\frac{\partial}{\partial y_j}\right)_{s(x)}=\left(\frac{\partial}{\partial y_j}\right)_{s(x)};\ \theta_{\bar{s}_x}\left(\frac{\partial}{\partial p_{ij}}\right)_{\bar{s}_x}=0.$$

This proves the equality in the three cases //

By means of this Lemma we can see that the 1-jet extension $\bar{s}$ of a section s of the fiber bundle $\pi: E\to X$ defined over a open set U is characterized as follow: *it is the unique section $\bar{s}$ of the fiber bundle $\bar{p}:\overline{E}\to X$ defined over U such that $p\cdot\bar{s}=s$ and $\theta_{\bar{s}(U)}=0$* [1].

Let $q: T^v(E)\to E$ be the sub-bundle of the tangent vector bundle of E defined by the vectors tangent to the fibres of $\pi: E\to X$ and let $\overline{M}$ be the set of the differentiable fiber bundle morphisms $f:\overline{E}\to T^v(E)$ over E, i.e., the differentiable mappings f from $\overline{E}$ to $T^v(E)$ making the following diagram commutative:

$$\begin{tikzcd}
\overline{E} \arrow[rr, "f"] \arrow[dr, "p"'] & & T^v(E) \arrow[dl, "q"] \\
& E &
\end{tikzcd}$$

If $\overline{A}$ is the algebra of differentiable functions on $\overline{E}$, then $\overline{M}$ is a $\overline{A}$-module by the rules: $(f+f')(\bar{e})=f(\bar{e})+f'(\bar{e})$, $(a\cdot f)(\bar{e})=a(\bar{e})\cdot f(\bar{e})$, where $f, f'\in\overline{M}$ and $a\in\overline{A}$.

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[1]) To define the restriction $\theta_{\bar{s}(U)}$ of θ to $\bar{s}(U)$ we proceed as in the case of ordinary forms.

Using Lemma 1 we can see that the structure 1-form θ defines an $\overline{M}$-valued 1-form on $\overline{A}$ by the rule: $\theta(\overline{D})(\overline{e})=\theta_{\overline{e}}(\overline{D}_{\overline{e}})$. Thus, the methods of differential calculus on an algebra A with values in an A-module M can be applied to the structure 1-form (see [7]). In particular, if $(x_i y_j p_{ij})$ is a natural coordinate system of the 1-jet fiber bundle $p:\overline{E}\to E$ on a open neighborhood $\overline{U}=\overline{E}_{\varphi^{-1}(U\times V)}$, tacking $\overline{E}=\overline{E}_{\varphi^{-1}(U\times V)}$ $T^v(E)=T^v(\varphi^{-1}(U\times V))$ etc., a basis $(f_1 \dots f_n)$ of $\overline{M}$ can be defined as follows: $f_j(\overline{e})=\left(\frac{\partial}{\partial y_j}\right)_{p(\overline{e})}$. By Lemma 1, the 1-form θ has, with respect to this basis, the expression:

$$\theta=\sum_j \theta_j\cdot f_j$$

where θ_j is the ordinary 1-form $\theta_j=dy_j-\sum_i p_{ij}dx_i$ on $\overline{U}$.

2. 1-jet fiber bundle of the sections of a principal fiber bundle.

From now on, we shall assume $\pi: E\to X$ to be a differentiable principal G-bundle, where G is an arbitrary Lie group. Then, we have:

LEMMA 2. The Lie group G acts on the 1-jet fiber bundle $\overline{E}$ on the right, freely and differentiably, by the rule:

$$(\overline{s}_x\,,\ a)\in\overline{E}\times G\to\overline{s}_x\cdot a=\overline{(a\cdot s)}_x\in\overline{E}.$$

PROOF. The rule is well-defined, for if $\overline{s}_x=\overline{s}'_x$, then

$$(a\cdot s)(x)=s(x)\cdot a=s'(x)\cdot a=(a\cdot s')(x)$$

and

$$d(a\cdot s)_x=(da)_{s(x)}\cdot(ds)_x=(da)_{s'(x)}\cdot(ds')_x=d(a\cdot s')_x$$

whence

$$\overline{(a\cdot s)}_x=\overline{(a\cdot s')}_x\,.$$

G acts on the right and freely, for:

$$\overline{s}_x\cdot(a\cdot b)=\overline{((a\cdot b)\cdot s)}_x=\overline{(a\cdot s)}_x\cdot b=(\overline{s}_x\cdot a)\cdot b$$

and if $\bar{s}_x \cdot a = \bar{s}_x$, then $(\overline{a \cdot s})_x = \bar{s}_x$ and hence $(a \cdot s)(x) = s(x)$, i.e., $s(x) \cdot a = s(x)$ which implies $a = 1$.

Finally, let us prove that G acts differentiably, i.e., the mapping $(\bar{s}_x, a) \in \bar{E} \times G \to \bar{s}_x \cdot a \in \bar{E}$ is differentiable.

First we observe that the following diagram is commutative:

$$(*) \qquad \begin{array}{ccc} \bar{E} \times G & \xrightarrow{\bar{f}} & \bar{E} \\ {\scriptstyle p \times 1}\downarrow & & \downarrow{\scriptstyle p} \\ E \times G & \xrightarrow{f} & E \end{array}$$

where f and $\bar{f}$ are the corresponding product mappings.

Let now (e_0, a_0) be any point in $E \times G$. As f is continuous, we can find two open neighborhoods $W \times \bar{V}$ and W' of (e_0, a_0) in $E \times G$ and of $f(e_0, a_0) = e_0 \cdot a_0$ in E, respectively (W is an open neighborhood of e_0 in E and $\bar{V}$ is an open neighborhood of a_0 in G), such that $f(W \times \bar{V}) \subset W'$. Furthemore, we can assume W and W' to be of the special type we saw in section 1 (i.e., $W = \varphi^{-1}(U \times V)$, $W' = \varphi'^{-1}(U' \times V')$) and $\bar{V}$ to have local coordinates $(\bar{y}_j)$. Calling now $\bar{U} = p^{-1}(W)$ and $\bar{U}' = p^{-1}(W')$, we see, by the commutativity of the diagram (*), that $\bar{U} \times \bar{V}$ and $\bar{U}'$ are open neighborhoods in $\bar{E} \times G$ and $\bar{E}$, respectively, with local coordinates $(x_i y_j p_{ij} \bar{y}_j)$ and $(x'_i y'_j p'_{ij})$ such that $\bar{f}(\bar{U} \times \bar{V}) \subset \bar{U}'$.

The equations of $\bar{f}\,|_{\bar{U} \times \bar{V}}$ with respect to the former local coordinates are:

$$x'_i = x'_i(x_l), \qquad y'_j = y'_j(x_l, y_h, \bar{y}_k),$$

$$p'_{ij} = \sum_k \frac{\partial x_k}{\partial x'_i} \left(\frac{\partial y'_j}{\partial x_k} + \sum_h \frac{\partial y'_j}{\partial y_h} p_{kh} \right)$$

where $x'_i = x'_i(x_l)$, $y'_j = y'_j(x_l, y_h, \bar{y}_k)$ are the equations of $f\,|_{W \times \bar{V}}$ with respect to the local coordinate systems $(x_l, y_h, \bar{y}_k)$ and (x'_l, y'_h).

This proves the diferentiability of $\bar{f}\,|_{\bar{U} \times \bar{V}}$ and, consequently, the differentiability of $\bar{f}$ because the point $(e_0, a_0) \in E \times G$ is arbitrary //

The interest of having the Lie group G acting in this way on the 1-jet fiber bundle $\bar{E}$ is justified by the following:

THEOREM 1. Let $\bar{p} : \bar{E} \to X$ be the 1-jet fiber bundle of the sections of a differentiable principal G-bundle $\pi : E \to X$, $\bar{E}/G$ the set of orbits of G in $\bar{E}$ with respect to the operation defined by Lemma 2, and p_G the mapping from $\bar{E}/G$ to X induced by the projection $\bar{p}$. Then, we have:

a) $p_G : \bar{E}/G \to X$ has a natural structure of differentiable fiber bundle.

b) $\bar{E}$ is a differentiable principal G-bundle over $\bar{E}/G$.

c) $\bar{E}$ is canonically isomorphic with the induced bundle $p^*_G(E)$ of the principal G-bundle $\pi : E \to X$ under p_G, and the projection $p : \bar{E} \to E$ coincide with the canonical morphism of $p^*_G(E)$.

PROOF. *a*) Let $x_0 \in X$, and let U be an open neighborhood of x_0 such that there exists a section $\sigma : U \to E$ of E over U. if $\bar{E}_\sigma$ is the restriction of $\bar{E}$ to $\sigma(U)$, then a natural bijection f_σ between $\bar{E}_\sigma$ and $(\bar{E}/G)_U = p_G^{-1}(U)$ can be established such that the following diagram is commutative:

$$\begin{tikzcd}
\bar{E}_\sigma \arrow[r, "f_\sigma"] \arrow[d, "p"'] & (\bar{E}/G)_U \arrow[d, "p_G"] \\
\sigma(U) & U \arrow[l, "\sigma"']
\end{tikzcd}$$

f_σ is the mapping that assigns to each point $\bar{s}_x \in \bar{E}_\sigma$ the orbit $\bar{s}_x G$ of G in $\bar{E}$ passing through $\bar{s}_x$. Since $\sigma : U \to \sigma(U)$ is a diffeomorphism, the differentiable fiber bundle structure of $\bar{E}_\sigma$ defines, by f_σ, a differentiable fiber bundle structure on $(\bar{E}/G)_U$. Let us see that this structure does not depend of the choice of σ.

If σ and σ' are two sections of E over U, there exists a unique differentiable mapping $g : U \to G$ such that, for every $x \in U$, we have:

$$\sigma(x) \cdot g(x) = \sigma'(x).$$

We shall now see that the mapping $g : \bar{E}_\sigma \to \bar{E}_{\sigma'}$ given by $g(\bar{s}_x) = \bar{s}_x \cdot g(x)$ is a differentiable fiber bundle isomorphism such that the following diagram is commutative:

$$\begin{tikzcd} \overline{E}_\sigma \arrow[rr, "g"] \arrow[dr, "f_\sigma"'] & & \overline{E}_{\sigma'} \arrow[dl, "f_{\sigma'}"] \\ & (\overline{E}/G)_U & \end{tikzcd}$$

this will prove the independence of the choice and, consequently, that the differentiable fiber bundle structure of $p_G : \overline{E}/G \to X$ is well-defined.

Obviously, g maps fibres into fibres and its inverse is the mapping $g^{-1}(\bar{s}_x)=\bar{s}_x \cdot g(x)^{-1}$. The commutativity of the diagram is obvious, too. We only have to prove now that g and g^{-1} are differentiable.

g can be factorized as $g=\bar{f}\cdot j\cdot i$, where $i : \overline{E}_\sigma \to \overline{E}_\sigma \times U$ is the mapping $i(\bar{s}_x)=(\bar{s}_x\,,\ x)$, $j : \overline{E}_\sigma \times U \to \overline{E} \times G$ is the mapping $j(\bar{s}_x\,,\ y)= =(\bar{s}_x\,,\ g(y))$ and $\bar{f}$ is the mapping product from $\overline{E} \times G$ to $\overline{E}$. g is then differentiable because i, j, $\bar{f}$ are differentiable and $g(\overline{E}_\sigma)=\overline{E}_{\sigma'}$ is a submanifold of $\overline{E}$. We prove that g^{-1} is differentiable in a similar way.

b) In order to prove that $\overline{E}$ is a differentiable principal G-bundle over $\overline{E}/G$, we only need to prove that the canonical projection $\bar{\pi} : \overline{E} \to \overline{E}/G$ is differentiable and that $\overline{E}$ is locally trivial.

Let $x_0 \in X$, and let U be an open neighborhood of x_0 such that there exists a section $\sigma : U \to E$ of E over U. Then, there exists a unique differentiable mapping $g : E_U \to G$ such that, for every $e \in E_U$, we have:

$$e\cdot g(e)=\sigma(\pi(e)).$$

Note $\overline{E}_U=p^{-1}(E_U)$.

If $i : \overline{E}_U \to \overline{E}_U \times E_U$ is the mapping $i(\bar{s}_x)=(\bar{s}_x\,,\ s(x))$, $j : \overline{E}_U \times E_U \to \overline{E} \times G$ is the mapping $j(\bar{s}_x\,,\ e)=(\bar{s}_x\,,\ g(e))$, $\bar{f}$ is the mapping product from $\overline{E} \times G$ to $\overline{E}$, and f_σ is the bijection considered in *a*), then we have $f_\sigma^{-1}\cdot \bar{\pi}_{\overline{E}_U}=\bar{f}\cdot j\cdot i$, which proves $\bar{\pi}_{\overline{E}_U}$ is differentiable and, so, $\bar{\pi}$ is differentiable, since the point $x_0 \in X$ is arbitrary.

On the other hand, the mapping:

$$\bar{\varphi} : \bar{s}_x \in \overline{E}_U \to (\bar{\pi}(\bar{s}_x),\ g(s(x))^{-1}) \in (\overline{E}/G)_U \times G$$

is a diffeomorphism such that, for every $\bar{s}_x \in \overline{E}_U$, $a \in G$, we have:

$$\bar{\varphi}(\bar{s}_x\cdot a)=(\bar{\pi}(\bar{s}_x\cdot a),\ g(s(x)\cdot a)^{-1}=(\bar{\pi}(\bar{s}_x),\ g(s(x))^{-1}\cdot a)=\bar{\varphi}(\bar{s}_x)\cdot a.$$

This proves that $\overline{E}$ is locally trivial.

c) Finally, it is not hard now to show that the mapping:

$$\bar{s}_x \in \overline{E} \to (\bar{s}_x G,\ s(x)) \in p^*_G(E)$$

is a isomorphism of principal G-bundles and that the projection $p : \overline{E} \to E$ coincide with the canonical morphism of $p^*_G(E)$. This completes the proof of Theorem 1. //

According to this, « taking the 1-jet fiber bundle » allows us to associate to each principal G-bundle $\pi : E \to X$ another principal G-bundle $\overline{\pi} : \overline{E} \to \overline{X}$. If $\pi : E \to X$ and $\pi' : E' \to X$ are two principal G-bundles over X and $f : E \to E'$ is a morphism of principal G-bundles over X, then the mapping $\bar{f} : \overline{E} \to \overline{E}'$ given by $\bar{f}(\bar{s}_x) = \overline{(f \cdot s)}_x$ is a morphism of principal G-bundles. So, « *to take the* 1-*jet fiber bundle* » *is a covariant functor from the category* $\mathrm{Bun}_X(G)$ *of differentiable principal* G-*bundles over* X *to the category* $\mathrm{Bun}(G)$ *of differentiable principal* G-*bundles.*

In terms of this functor, the connections on a principal fiber bundle can be characterized as follows:

THEOREM 2. If $J : \mathrm{Bun}_X(G) \to \mathrm{Bun}(G)$ is the functor « to take the 1-jet fiber bundle » from the category $\mathrm{Bun}_X(G)$ of differentiable principal G-bundles over X to the category $\mathrm{Bun}(G)$ of differentiable principal G-bundles, then: for every $E \in \mathrm{Bun}_X(G)$ there exists a natural one to one correspondence between the set of connections on E and the set of sections of the canonical morphism $p_E : \bar{s}_x \in J(E) \to s(x) \in E$.

PROOF. If σ is a section of the morphism p_E: $J(E) \to E$, let Γ^σ be the distribution on E that assigns to each point $e \in E$ the subspace $\Gamma^\sigma_e = \mathrm{img}\,(ds^e)_x$ of $T_e(E)$, where $\bar{s}^e_x = \sigma(e)$. We shall see that Γ^σ is a connection on E. In fact, for every $e \in E$, Γ^σ_e is transversal to the fibre E_e of E passing through e, since s^e is a cross section of $\pi : E \to X$.

For every $e \in E$ and $a \in G$, we have $(da)_e \Gamma^\sigma_e = \Gamma^\sigma_{e.a}$, because $\sigma(e.a) = \sigma(e) \cdot a$, since σ is a section of p_E in $\mathrm{Bun}(G)$. Finally, Γ^σ is differentiable because σ is.

Thus, we have defined a natural correspondence Γ from the set of sections of the morphism p_E to the set of connections on E. Let us see that it is one to one.

If σ and σ' are two sections of the morphism p_E such that $\Gamma^\sigma=\Gamma^{\sigma'}$, then for every $e\in E$ we have $\Gamma^\sigma_e=\Gamma^{\sigma'}_e$, that is, $\operatorname{img}(ds^e)_x=\operatorname{img}(ds'^e)_x$, where $\bar{s}^e_x=\sigma(e)$ and $\bar{s}'^e_x=\sigma'(e)$. This implies $s^e(x)=s'^e(x)$ and $(ds^e)_x=(ds'^e)_x$, so $\bar{s}^e_x=\bar{s}'^e_x$, i.e. $\sigma(e)=\sigma'(e)$. Consecuently $\sigma=\sigma'$, as e is arbitrary. Γ is, then, injective. If Γ' is a connection on E, the mapping σ: $e\in E\to\to\bar{s}^e_x\in J(E)$, where $\bar{s}^e_x$ is the only point of $J(E)$ such that $\operatorname{img}(ds^e)_x=\Gamma'_e$ is a section of the morphism p_E (in fact, we only need invert the above process) such that $\Gamma^\sigma=\Gamma'$. Thus, Γ is surjective.

REMARK. Because $\bar{E}$ is the induced bundle $p^*_G(E)$ of $\pi: E\to X$ under p_G: $\bar{E}/G\to X$ and the projection p: $\bar{E}\to E$ is the canonical morphism of $p^*_G(E)$ (Theorem 1), then, the mapping that assigns to each section of p its projection to X, allows us to set up a natural one to one correspondence between the set of connections on E and the set of sections of p_G. In particular, if E is the bundle of linear frames over an n-dimensional manifold X, this result means that *we can construct canonically a fiber bundle $f: B\to X$ over X, with fibres of dimension n^3, such that the set of linear connections on X is in a natural one to one correspondence with the set of sections of f: $B\to X$.* If (x_i, y^i_j) is a (natural) local coordinate system in the bundle of linear frames over X and (x_i, p^i_{jk}) is the local coordinate system induced by (x_i, y^i_j) in B, then, we can prove that the local equations of the section s of f, which corresponds to a given linear connection ∇ over X, are $p^i_{jk}=-\Gamma^i_{kj}(x_1\dots x_n)$, where Γ^i_{kj} are the coefficients of ∇ with respect to the local coordinate system (x_i).

3. Canonical connection on the 1-jet fiber bundle of the sections of a principal fiber bundle.

Let $\pi: E\to X$ be a principal G-bundle, $\bar{E}$ the 1-jet fiber bundle of the sections of E (which is also a principal G-bundle by Theorem 1), and θ the structure 1-form of $\bar{E}$. As we saw in section 1, θ is a 1-form on $\bar{E}$ with values in the $\bar{A}$-module $\bar{M}$ of the fiber bundle morphisms $f: \bar{E}\to T^v(E)$ over E ($\bar{A}=$ algebra of differentiable functions on $\bar{E}$, $T^v(E)=$ sub-bundle of the tangent vector bundle of E defined by the vectors tangent to the fibres of E).

If $\mathfrak{G}$ is the Lie algebra of G, $T^v(E)$ can be canonically identified with the trivial vector bundle $E\times\mathfrak{G}$ by means of the vector bundle

isomorphism that assigns to each point $(e, A) \in E \times \mathcal{G}$ the tangent vector $A^*_e \in T^v(E)$, where A^* is the fundamental vector field corresponding to $A \in \mathcal{G}$. This allows to identify the $\overline{A}$-module $\overline{M}$ with the $\overline{A}$-module of differentiable mappings from $\overline{E}$ to $\mathcal{G}$. Thus we have, in particular, that *the structure* 1-*form* θ *is a* 1-*form on* $\overline{E}$ *with values in the Lie algebra* $\mathcal{G}$.

In these conditions we shall prove the following:

THEOREM 3. The structure 1-form θ of the 1-jet fiber bundle $\overline{E}$ is a connection 1-form on $\overline{E}$ (this connection will be called *canonical connection* on the 1-jet fiber bundle). If σ is a connection on E (regarded as a section of the morphism p: $\overline{E} \to E$ in the sense of Theorem 2) and θ^σ it its connection 1-form, then $\theta^\sigma = \sigma^*\theta$. If Ω and Ω^σ are the curvature 2-forms of connections θ and θ^σ, respectively, then $\Omega^\sigma = \sigma^*\Omega$.

To prove this Theorem, we shall use the following property of the fundamental vector fields:

LEMMA 3. If $f : E \to \overline{E}$ is a morphism of principal G-bundles, then, for every $A \in \mathcal{G}$ and every $e \in E$, we have:

$$(df)_e A^*_e = \overline{A}^*_{f(e)}$$

where A^* and $\overline{A}^*$ are the fundamental vector fields defined by A on E and $\overline{E}$, respectively.

PROOF. If $\{a_t\}$ is the 1-parameter subgroup of G defined by A and $\overline{g}$ is any differentiable function on $\overline{E}$, we have:

$$((df)_e A^*_e)\overline{g} = A^*_e(\overline{g} \cdot f) = \lim_{t \to 0} 1/t[(\overline{g} \cdot f)(e \cdot a_t) - (\overline{g} \cdot f)(e)] =$$

$$= \lim_{t \to 0} 1/t[\overline{g}(f(e \cdot a_t)) - \overline{g}(f(e))] = \lim_{t \to 0} 1/t[\overline{g}(f(e) \cdot a_t) - \overline{g}(f(e))] =$$

$$= \overline{A}^*_{f(e)}\overline{g} \quad //$$

PROOF OF THEOREM 3.

a) Let us see that the distribution $\overline{s}_x \in \overline{E} \to \ker \theta_{\overline{s}_x}$ is a connection on $\overline{E}$ and thus, the first part of the theorem will be proved.

For every $\overline{s}_x \in \overline{E}$, $\ker \theta_{\overline{s}_x}$ is transversal to the fibre $\overline{E}_{\overline{s}_x}$ of $\pi : \overline{E} \to \overline{E}/G$ passing through $\overline{s}_x$. In fact, if $T_{\overline{s}_x}(\overline{E})$ is the tangent space of $\overline{E}$ at $\overline{s}_x$

and $T^v_{\bar{s}_x}(\bar{E})$ is the subspace of $T_{\bar{s}x}(\bar{E})$ defined by the vectors tangent to the fibre of $p:\bar{E}\to E$ passing through $\bar{s}_x$, then, it is easy to see that:

$$T_{\bar{s}_x}(\bar{E})=T^v_{\bar{s}_x}(\bar{E})\oplus \operatorname{img}(d\bar{s})_x\oplus\{\bar{A}^*_{\bar{s}_x}\}$$

where $\bar{s}$ is the 1-jet extension of the section s and $\{\bar{A}^*_{\bar{s}_x}\}$ are the fundamental vector fields of $\bar{E}$ at $\bar{s}_x$.

If $D_{\bar{s}_x}=D^v_{\bar{s}_x}+(d\bar{s})_xD_x+\bar{A}^*_{\bar{s}_x}\in\ker\theta_{\bar{s}_x}$, then we have $o=(d^vs)_x(p'_{\bar{s}_x}D_{\bar{s}_x})=$ $=(d^vs)_x(p'_{\bar{s}_x}\bar{A}^*_{\bar{s}_x})$. But p is a morphism of principal G-bundles, so, by Lemma 3, we have $p'_{\bar{s}_x}\bar{A}^*_{\bar{s}_x}=A^*_{s(x)}$ and hence, $o=(d^vs)_xA^*_{s(x)}=A^*_{s(x)}$. This implies $A=o$ and, so, $\bar{A}^*_{\bar{s}_x}=o$, i.e., $D_{\bar{s}_x}\in T^v_{\bar{s}_x}(\bar{E})\oplus\operatorname{img}(d\bar{s})_x$. In a similar way we prove that if $D_{\bar{s}_x}\in T^v_{\bar{s}_x}(\bar{E})\oplus\operatorname{img}(d\bar{s})_x$ then $D_{\bar{s}_x}\in\ker\theta_{\bar{s}_x}$. So, $\ker\theta_{\bar{s}_x}=T^v_{\bar{s}_x}(\bar{E})\oplus\operatorname{img}(d\bar{s})_x$, which proves our statement because $\{\bar{A}^*_{\bar{s}_x}\}$ is the tangent space to the fibre $\bar{E}_{\bar{s}_x}$ at $\bar{s}_x$.

The distribution $\bar{s}_x\in\bar{E}\to\ker\theta_{\bar{s}_x}$ is G-invariant. In fact, if $D_{\bar{s}_x}\in\ker\theta_{\bar{s}_x}$ (i.e., $D_{\bar{s}_x}=D^v_{\bar{s}_x}+(d\bar{s})_xD_x$) and $a\in G$, then, from the fact of p being a morphism of principal G-bundles follows that p commutes with the acting of G and, so:

$$d^v(a\cdot s)_x(p'_{\bar{s}_x}\cdot a'_{\bar{s}_x}D_{\bar{s}_x})=d^v(a\cdot s)_x(a'_{s(x)}\cdot p'_{\bar{s}_x}D_{\bar{s}_x})=$$
$$=d^v(a\cdot s)_x(a'_{s(x)}\cdot s'_xD_x)=d^v(a\cdot s)_x((a\cdot s)'_xD_x)=o$$

that is, $a'_{\bar{s}_x}D_{\bar{s}_x}\in\ker\theta_{\bar{s}_{x.a}}$. So, $a'_{\bar{s}_x}(\ker\theta_{\bar{s}_x})=\ker\theta_{\bar{s}_{x.a}}$, as we want to prove.

Finally, the distribution $\bar{s}_x\in\bar{E}\to\ker\theta_{\bar{s}_x}$ is differentiable because θ is.

b) Given $e\in E$, if $D_e\in\Gamma^\sigma_e$, then, by the definition of Γ^σ_e (proof of Theorem 2), we have $D_e=(ds^e)_xD_x$, where $\bar{s}^e_x=\sigma(e)$ and $D_x\in T_x(X)$. Hence:

$$(\sigma^*\theta)_eD_e=\theta_{\bar{s}^e_x}(\sigma'_eD_e)=\varphi(d^vs^e)_x(p'_{\bar{s}^e_x}\cdot\sigma'_e\cdot s^{e\prime}_xD_x)=\varphi(d^vs^e)_x(s^{e\prime}_xD_x)=o$$

where φ is the canonical isomorphism between $T^v(E)$ and $E\times\mathfrak{g}$ we mentioned at the beginning of this section.

If $D_e = A^*_{e'}$, $A \in \mathcal{G}$, we have:

$$(\sigma^*\theta)_e D_e = \theta_{\bar{s}^e_x}(\sigma'_e A^*_e) = \theta_{\bar{s}^e_x}(\overline{A}^*_{s^e_x}) = \varphi(d^v s^e)_x (p'_{\bar{s}^e_x} \overline{A}^*_{s^e_x}) =$$

$$= \varphi(d^v s^e)_x A^*_e = \varphi(A^*_e) = A.$$

In this calculus we have used that $\sigma'_e A^*_e = \overline{A}^*_{\sigma(e)}$ and $p'_{\sigma(e)} \overline{A}^*_{\sigma(e)} = A^*_e$, which follow from Lemma 3, σ and p being morphisms of principal G-bundles.

This proves that $\sigma^*\theta$ coincide with the connection 1-form θ^σ of Γ^σ.

c) If h and h^σ are the « projectors to the horizontal subspace » of the connections θ and θ^σ, respectively, we have: $\Omega(D, D') = d\theta(hD, hD')$ and $\Omega^\sigma(D, D') = d\theta^\sigma(h^\sigma D, h^\sigma D')$. From $\theta^\sigma = \sigma^*\theta$ it follows $\sigma'_e \cdot h^\sigma = h \cdot \sigma'_e$ for every $e \in E$ and hence:

$$\Omega^\sigma_e(D_e, D'_e) = (d\theta^\sigma)_e(h^\sigma D_e, h^\sigma D'_e) = (d\sigma^*\theta)_e(h^\sigma D_e, h^\sigma D'_e) =$$

$$= (\sigma^* d\theta)_e(h^\sigma D_e, h^\sigma D'_e) = (d\theta)_{\sigma(e)}(\sigma'_e \cdot h^\sigma D_e, \sigma'_e \cdot h^\sigma D'_e) =$$

$$= (d\theta)_{\sigma(e)}(h \cdot \sigma'_e D_e, h \cdot \sigma'_e D'_e) = \Omega_{\sigma(e)}(\sigma'_e D_e, \sigma'_e D'_e) =$$

$$= (\sigma^*\Omega)_e(D_e, D'_e).$$

This completes the proof of Theorem 3. //

Theorem 2 and a Nomizu's result ([5], pag. 90, Th. 8.2) allows us to prove the following property of the canonical connection θ:

THEOREM 4. If $\pi : E \to X$ is a principal G-bundle with E connected and $\dim X \geq 2$, then, all the holonomy bundles $\overline{E}(\bar{e})$, $\bar{e} \in \overline{E}$, with respect to the canonical connection θ on $\overline{E}$, coincide with $\overline{E}$. In particular, the holonomy group (with respect to any point) of the canonical connection θ coincide with G.

PROOF. By ([5], pag. 90, Th. 8.2), there exists a connection σ on E such that all the holonomy bundles $E(e)$, $e \in E$, with respect to σ, coincide with E. Given a point $\bar{s}^0_{x_0} \in \overline{E}$, if $\bar{s}_x$ is any other point in $\overline{E}$, then we can find: a horizontal path γ (with respect to σ) from $s^0(x_0)$ to $s(x)$, a path δ_0 on the fibre $p^{-1}(s^0(x_0))$ from $\bar{s}^0_{x_0}$ to $\sigma(x_0)$ and, finally, a path δ on the fibre $p^{-1}(s(x))$ from $\sigma(x)$ to $\bar{s}_x$. It follow inmediately

that the path $(\delta_0, \sigma\cdot\gamma, \delta)$ on $\overline{E}$ goes from $\bar{s}^0_{x_0}$ to $\bar{s}_x$ and is horizontal with respect ot the connection θ. //

4. Application to Weil's Theorem on characteristic classes.

As an application of the ideas developed in the former sections we shall give a simple proof of Weil's Theorem on characteristic classes.

It is well known (see, [6], Chap. 12) that this theorem refers to the following: if G is a Lie group with Lie algebra $\mathfrak{G}$ and we note $I^k(G)$ the vector space of $(ad\ G)$-invariant symmetric k-linear mappings of $\mathfrak{G}$ into R, then a real commutative algebra structure can be defined on $I(G)=\sum_{k=0}^{\infty} I^k(G)$ by the rule:

$$(f\cdot g)(A_1 \dots A_{k+l})=\frac{1}{(k+l)!}\sum_{\varepsilon} f(A_{\varepsilon_1} \dots A_{\varepsilon_k})\cdot g(A_{\varepsilon_{k+1}} \dots A_{\varepsilon_{k+l}})$$

where $f\in I^k(G)$, $g\in I^l(G)$ and the sum runs (over all permutations ε of $(1 \dots k+l)$. Let now $\pi : E\rightarrow X$ be a principal G-bundle and σ a connection on E with curvature 2-form Ω^σ. We can define a homomorphism from the algebra $I(G)$ to the cohomology algebra $H^*(X, R)$ as follows: for each $f\in I^k(G)$, let $f(\Omega^\sigma)$ be the $2k$-form on E given by:

$$f(\Omega^\sigma)_e(D_1 \dots D_{2k})=\frac{1}{(2k)!}\sum_{\varepsilon}(\operatorname{sig}\varepsilon)f(\Omega^\sigma(D_{\varepsilon_1}D_{\varepsilon_2}) \dots \Omega^\sigma(D_{\varepsilon_{2k-1}}D_{\varepsilon_{2k}}))$$

where $e\in E$ and $D_1 \dots D_{2k}\in T_e(E)$. From the $(ad\ G)$-invariance of f and the Bianchi's identity, it follows that $f(\Omega^\sigma)$ is π-projectable and closed. Noting $\overline{f(\Omega^\sigma)}$ the element of the De Rham cohomology group $H^{2k}(X, R)$ defined by $f(\Omega^\sigma)$, we have finally established a mapping from $I(G)$ to $H^*(X, R)$ that is, trivially, a homomorphism of R-algebras. This is the so called *Weil's homomorphism* with respect to the given connection. Weil's theorem states that this homomorphism does not depend on the choice of the connection.

Using the ideas developed in the former sections, we can prove this theorem in the following way:

Let θ be the canonical connection on the 1-jet fiber bundle $\overline{E}$ of the sections of $\pi : E \to X$ and Ω its curvature 2-form. Let $f \in I^k(G)$ and two different connections σ and σ' on E be given; then, by Theorem 3, we have $f(\Omega^\sigma) = \sigma^* f(\Omega)$ and $f(\Omega^{\sigma'}) = \sigma'^* f(\Omega)$. But σ and σ' are differentiably homotopic mappings (in fact, we only need to consider the 1-parameter family of connections $\theta^t = \theta^\sigma + t(\theta^{\sigma'} - \theta^\sigma)$, $0 \leq t \leq 1$, where θ^σ and $\theta^{\sigma'}$ are the connection 1-forms of σ and σ', respectively) and, thus, we have $\overline{f(\Omega^\sigma)} = \overline{f(\Omega^{\sigma'})}$, as we want to prove.

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