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SU UN PROBLEMA ESTERNO RELATIVO ALLE EQUAZIONI DI NAVIER-STOKES

PAOLO MURATORI *)

Introduzione.

Sia Ω il complementare di un aperto limitato Ω_0 dello spazio euclideo $\boldsymbol{R}^3$ con frontiera $\partial\Omega$ sufficientemente regolare. Fissato $T \in \boldsymbol{R}^+$ (insieme dei reali positivi), sia

$$Q = \Omega \times [-T, 0].$$

Consideriamo il sistema di Navier-Stokes ($u = (u_1, u_2, u_3)$)

$$(\alpha) \qquad \frac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla) u = -\nabla p + F$$

$$(\beta) \qquad \nabla \cdot u = 0$$

con $\nabla \times F = 0$, e consideriamo il problema dell'unicità di una soluzione u, p di (α)-(β) in Q con assegnati valori su $\partial\Omega \times [-T, 0]$ e $\Omega \times \{0\}$. I risultati conseguiti, espressi dai teoremi 1 e 2 del § 3, sono da porre a confronto con analoghi risultati di D. E. Edmunds [1] (vedi anche [2]); le ipotesi in cui mi sono posto sono meno restrittive di quelle del citato Autore. Precisamente se u, q e $u+v$, $q+p$ sono due soluzioni di (α)-(β), l'ipotesi qui fatta sul comportamento asintotico di p è meno forte di quella fatta in [1]; inoltre non faccio ipotesi di limita-

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*) Indirizzo dell'A.: Istituto Matematico dell'Università di Bologna.

Lavoro eseguito nell'ambito dei gruppi di ricerca matematici del C.N.R.

tezza delle derivate di u e v rispetto al tempo e di sommabilità su Ω dei quadrati delle derivate spaziali di v.

1. Sia Ω il complementare di un aperto limitato Ω_0 dello spazio euclideo $\boldsymbol{R}^3$ con frontiera $\partial\Omega$ sufficientemente regolare. Fissato $T \in \boldsymbol{R}^+$ (insieme dei reali positivi), sia

$$Q = \Omega \times [-T, 0].$$

Se $R \in \boldsymbol{R}^+$ poniamo

$$\omega'(R) = \{x = (x_1, x_2, x_3) \in \boldsymbol{R}^3;\ x_1^2 + x_2^2 + x_3^2 \leqslant R^2\}$$

$$\sigma(R) = \{x = (x_1, x_2, x_3) \in \boldsymbol{R}^3;\ x_1^2 + x_2^2 + x_3^2 = R^2\}$$

$\forall R \geqslant$ di un certo R_0 che supponiamo inoltre >1 per cui risulta $\overline{\boldsymbol{R}^3 - \omega'(R_0)} \subset \Omega$

$$\omega_{R_1}(R) = \{x = (x_1, x_2, x_3) \in \boldsymbol{R}^3;\ R_1^2 \leqslant x_1^2 + x_2^2 + x_3^2 \leqslant R^2\}$$

$\forall R$, R_1 con $R_0 \leqslant R_1 \leqslant R$,

$$\omega_R = \{x = (x_1, x_2, x_3) \in \boldsymbol{R}^3;\ R^2 \leqslant x_1^2 + x_2^2 + x_3^2\}.$$

Useremo costantemente nel seguito le seguenti notazioni

$$L_1 = \left\{ g : Q \to \boldsymbol{R};\ g \text{ continua in } Q,\ \frac{\partial g}{\partial x_i} \text{ continua in } \operatorname{Int}(Q) \text{ per } 1 \leqslant i \leqslant 3 \right\}$$

$$L_2 = \left\{ g : Q \to \boldsymbol{R};\ g,\ \frac{\partial g}{\partial x_i} \text{ continue in } Q \text{ per } 1 \leqslant i \leqslant 3,\ \frac{\partial^2 g}{\partial x_i \partial x_j} \text{ continue in } \operatorname{Int}(Q) \text{ per } 1 \leqslant i, j \leqslant 3 \right\}$$

$$L_3=\Big\{g:Q\to \boldsymbol{R};\ g,\ \frac{\partial g}{\partial t},\ \frac{\partial g}{\partial x_i},\ \frac{\partial^2 g}{\partial x_i\partial x_j},$$

$$\frac{\partial^2 g}{\partial t\partial x_i}\ \text{continue in } Q \text{ per } 1\leqslant i,\ j\leqslant 3,$$

$$\frac{\partial^3 g}{\partial x_k\partial x_i\partial x_j}\ \text{continue in Int}\,(Q) \text{ per } 1\leqslant i,\ j,\ k\leqslant 3\Big\};$$

inoltre se $u=(u_1\,,\ u_2\,,\ u_3)$, $v=(v_1\,,\ v_2\,,\ v_3)$ con $u_i\,,\ v_i\in L_2$ e $f\in L_2$ poniamo

$$(u,\ v)_{\omega_{R_1}(R)}=\int_{\omega_{R_1}(R)}\Big(\sum_{i,\,j=1}^{3}u_iv_i\Big)dx,\qquad \|\,u\,\|^2_{\omega_{R_1}(R)}=(u,\ u)_{\omega_{R_1}(R)}$$

$$(u,\ v)_{\sigma(R)}=\int_{\sigma(R)}\Big(\sum_{i,\,j=1}^{3}u_iv_i\Big)d\sigma,\qquad \|\,n\,\|^2_{\sigma(R)}=(u,\ u)_{\sigma(R)}$$

$$|||\,u\,|||^2_{\omega_{R_1}(R)}=\int_{\omega_{R_1}(R)}\Big(\sum_{i,\,j=1}^{3}\Big(\frac{\partial u_i}{\partial x_j}\Big)^2\Big)dx,\qquad |||\,u\,|||^2_{\sigma(R)}=\int_{\sigma(R)}\Big(\sum_{i,\,j=1}^{3}\Big(\frac{\partial u_i}{\partial x_j}\Big)^2\Big)d\sigma$$

$$|||\,f,\ u\,|||^2_{\omega_{R_1}(R)}=\int_{\omega_{R_1}(R)}f^2\Big(\sum_{i,\,j=1}^{3}\Big(\frac{\partial u_i}{\partial x_j}\Big)^2\Big)dx,$$

$$|||\,f,\ u\,|||^2_{\sigma(R)}=\int_{\sigma(R)}f^2\Big(\sum_{i,\,j=1}^{3}\Big(\frac{\partial u_i}{\partial x_j}\Big)^2\Big)d\sigma\qquad \forall R\geqslant R_1\geqslant R_0\,.$$

Analogo significato per

$$\|\,u\,\|^2_{\omega(R)}\,,\ |||\,u\,|||^2_{\omega(R)}\,,\ \|\,u\,\|^2_{\Omega}\,,\ |||\,u\,|||^2_{\Omega}\,.$$

Gli stessi simboli saranno anche usati, col significato usuale, se al posto del vettore u si pone una funzione.

Riportiamo per comodità le seguenti note formule

(a) $\quad u\cdot v=u_1v_1+u_2v_2+u_3v_3\,,\ u^2=u\cdot u,\ |\,u\,|=(u\cdot u)^{1/2}$

(b) $\quad u\times v=(u_2v_3-u_3v_2\,,\ u_3v_1-u_1v_3\,,\ u_1v_2-u_2v_1)$

(c) $$\nabla f=\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$$

(d) $$\nabla \cdot u=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}$$

(e) $$\nabla \times u=\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)$$

(f) $$(u \cdot \nabla) v=\left(\sum_{i=1}^{3} \frac{\partial v_1}{\partial x_i} u_i, \sum_{i=1}^{3} \frac{\partial v_2}{\partial x_i} u_i, \sum_{i=1}^{3} \frac{\partial v_3}{\partial x_i} u_i\right)$$

(g) $$\nabla^2 u=\Delta u=\left(\sum_{i=1}^{3} \frac{\partial^2 u_1}{\partial x_i^2}, \sum_{i=1}^{3} \frac{\partial^2 u_2}{\partial x_i^2}, \sum_{i=1}^{3} \frac{\partial^2 u_3}{\partial x_i^2}\right)$$

(h) $$\nabla u=\left\|\begin{matrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{matrix}\right\|$$

(i) $$(\nabla u) v=\left(\sum_{i=1}^{3} \frac{\partial u_i}{\partial x_1} v_i, \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_2} v_i, \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_3} v_i\right)$$

in particolare

(i) $$(\nabla u) u=\frac{1}{2} \nabla(u^2)$$

(l) $$\nabla(u \cdot v)=u \times(\nabla \times v)+(u \cdot \nabla) v+v \times(\nabla \times u)+(v \cdot \nabla) u$$

(m) $$\nabla \cdot(u \times v)=v \cdot(\nabla \times u)-u \cdot(\nabla \times v)$$

(n) $$\nabla \times(\nabla \times u)=\nabla(\nabla \cdot u)-\nabla^2 u=\nabla(\nabla \cdot u)-\Delta u$$

(o) $$\nabla \cdot(f u)=f \nabla \cdot u+u \cdot \nabla f$$

(p) $$\nabla \times(\nabla f)=0$$

(q) $$\nabla \times(u \times v)=(v \cdot \nabla) u-(\nabla \cdot u) v+(\nabla \cdot v) u-(u \cdot \nabla) v$$

(r) $\nabla\cdot(\nabla\times u)=0$

(s) $u\cdot\Delta u=-\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)^2+\frac{1}{2}\nabla\cdot(\nabla(u^2))$

(t) $u\cdot\Delta u=-(\nabla\times u)^2-\nabla\cdot((\nabla\times u)\times u)$

(u) $\frac{\partial u}{\partial t}\cdot\nabla u=-\frac{1}{2}\frac{\partial}{\partial t}\left(\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)^2\right)+\nabla\cdot\left((\nabla u)\frac{\partial u}{\partial t}\right)$

(v) $\frac{\partial u}{\partial t}\cdot\nabla u=-\frac{1}{2}\frac{\partial}{\partial t}(\nabla\times u)^2-\nabla\cdot\left((\nabla\times u)\times\frac{\partial u}{\partial t}\right).$

Nel seguito useremo una serie di costanti, che denoteremo con C_1, C_2, ..., C'_1, ..., C_α, C_β ..., senza specificare volta per volta la dipendenza da certe funzioni limitate che intervengono nei calcoli.

2. LEMMA 1. *Sia $a\in\boldsymbol{R}^+$ e sia $\Phi:[a,+\infty[\to\boldsymbol{R}^+$ di classe $C^{(1)}$ tale che $\frac{d\Phi(R)}{dR}=\Phi'(R)\geqslant\delta$ $\forall R\geqslant a$ per un certo $\delta\in\boldsymbol{R}^+$. Allora, se esistono $C\in\boldsymbol{R}^+$ e $m\in N$ tali che*

$$\Phi(R)\leqslant C(R^m+R^{m+1}(\Phi'(R))^{1/2})\qquad\forall R\geqslant a,$$

esiste di conseguenza $R_1\in\boldsymbol{R}^+$ tale che

$$\Phi(R)\leqslant R^{3m+3}\qquad\forall R\geqslant R_1.$$

Per la dimostrazione vedi [3].

LEMMA 2. *Se w_1, w_2, $w_3\in L_3$ e f, $p\in L_2$, f è costante in t, $\nabla\cdot w=0$, allora, posto*

$$Lw=\Delta w-\frac{\partial w}{\partial t},\ \lambda=\tau-t,\ C_{\tau,\gamma,\delta}=\left(2\tau+\frac{1}{2}\gamma^{-1}+\delta+4\tau^2\right),$$

si ha

$$\left(\frac{\gamma\delta}{4\tau^2}+\gamma\right)\int_{-\tau}^{0}\lambda^{-2\gamma}\,\|fw(t-h)\|^2_{\omega_{R_1}(R)}dt+\int_{-\tau}^{0}\lambda^{-2\gamma}\,|||f,\ w(t-h)|||^2_{\omega_{R_1}(R)}dt+$$

$$+C_{\tau,\gamma,\delta}\int_{-\tau}^{0}\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\omega_{R_1}(R)}dt\leqslant$$

$$\leqslant-\frac{1}{2}\int_{-\tau}^{0}\lambda^{-2\gamma}(\nabla(w^2(t-h)),\ \nabla(f^2))_{\omega_{R_1}(R)}dt-$$

$$-2C_{\tau,\gamma,\delta}\int_{-\tau}^{0}\lambda^{-\gamma}((\nabla w(t-h))\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h)),\ \nabla(f^2))_{\omega_{R_1}(R)}dt+$$

$$+\int_{-\tau}^{0}\lambda^{-2\gamma}(p(t-h)w(t-h),\ \nabla(f^2))_{\omega_{R_1}(R)}dt+$$

$$+2C_{\tau,\gamma,\delta}\int_{-\tau}^{0}\lambda^{-2\gamma}\,\|fLw(t-h)-f\nabla p(t-h)\|^2_{\omega_{R_1}(R)}dt+$$

$$+2C_{\tau,\gamma,\delta}\int_{-\tau}^{0}\lambda^{-\gamma}\left(\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h)),\ p(t-h)\nabla(f^2)\right)_{\omega_{R_1}(R)}dt+$$

$$+4\int_{-\tau}^{0}\lambda^{-2\gamma}(\|fw(t-h)\|^2_{\sigma(R_1)}+|||f,\ w(t-h)|||^2_{\sigma(R_1)})dt+$$

$$+4\int_{-\tau}^{0}\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\sigma(R_1)}dt+$$

$$+4\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\,\|fp(t-h)\|^2_{\sigma(R_1)}dt\right)^{1/2}\left(\int_{-\tau}^{0}\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\sigma(R_1)}dt\right)^{1/2}+$$

$$+4\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\,\|fp(t-h)\|^2_{\sigma(R_1)}dt\right)^{1/2}\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\,\|fw(t-h)\|^2_{\sigma(R_1)}dt\right)^{1/2}+$$

$$+4\int_{-\tau}^{0}\lambda^{-2\gamma}(\| fw(t-h)\|^2_{\sigma(R)}+|||f,\ w(t-h)|||^2_{\sigma(R)})dt+$$

$$+4\int_{-\tau}^{0}\left\| f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\sigma(R)}dt+$$

$$+4\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\| fp(t-h)\|^2_{\sigma(R)}dt\right)^{1/2}\left(\int_{-\tau}^{0}\left\| f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\sigma(R)}dt\right)^{1/2}+$$

$$+4\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\| fp(t-h)\|^2_{\sigma(R)}dt\right)^{1/2}\left(\int_{-\tau}^{0}\| fw(t-h)\|^2_{\sigma(R)}dt\right)^{1/2}+$$

$$+C_{\tau,\gamma,\delta}(2\tau)^{-2\gamma}|||f,\ w(-t-h)|||^2_{\omega_{R_1}(R)}+4\gamma(\tau)^{-2\gamma-1}\|fw(-h)\|^2_{\omega_{R_1}(R)}-$$

$$-\tau\gamma(2\tau)^{-2\gamma-1}\| fw(-\tau-h)\|^2_{\omega_{R_1}(R)}$$

$$\forall\tau,\ R,\ R_1,\ \gamma,\ \delta\in\boldsymbol{R}^+,\ \forall h\geqslant 0 \text{ con } R\geqslant R_1\geqslant R_0,\ \gamma\geqslant 1,\ \tau\leqslant\frac{1}{3},$$

$$0\leqslant h+\tau\leqslant T,\ 0<\delta\leqslant\frac{1}{4}.$$

DIMOSTRAZIONE. Cominciamo con l'osservare che

$$\int_{-\tau}^{0}\lambda^{-2\gamma}\| fLw(t-h)-f\nabla p(t-h)\|^2_{\omega_{R_1}(R)}dt= \tag{1}$$

$$=\int_{-\tau}^{0}\Big[\| f\lambda^{-\gamma}\Delta w(t-h)+f\gamma\lambda^{-\gamma-1}w(t-h)\|^2_{\omega_{R_1}(R)}+$$

$$+\left\| f\frac{\partial}{dt}(\lambda^{-\gamma}w(t-h))\right\|^2_{\omega_{R_1}(R)}-$$

$$-2\left(f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h)),\ f\lambda^{-\gamma}\Delta w(t-h)\right)_{\omega_{R_1}(R)}-$$

$$-2\gamma\left(f\lambda^{-\gamma-1}w(t-h),\ f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right)_{\omega_{R_1}(R)}+$$

$$+2\left(f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h)),\ f\lambda^{-\gamma}\nabla p(t-h)\right)_{\omega_{R_1}(R)}-$$

$$-2(f\lambda^{-\gamma}\Delta w(t-h)+f\gamma\lambda^{-1-\gamma}w(t-h),\ f\lambda^{-\gamma}\nabla p(t-h))_{\omega_{R_1}(R)}+$$

$$+\lambda^{-2\gamma}\|\nabla p(t-h)\|_{\omega_{R_1}(R)}\Big]dt.$$

Infatti

$$\lambda^{-2\gamma}\left\|f\Delta w(t-h)-f\frac{\partial}{\partial t}(w(t-h))-f\nabla p(t-h)\right\|^2_{\omega_{R_1}(R)}=$$

$$=\lambda^{-2\gamma}\Big[\|f\Delta w(t-h)\|^2_{\omega_{R_1}(R)}-2\left(f\Delta w(t-h),\ f\frac{\partial}{\partial t}(w(t-h))\right)_{\omega_{R_1}(R)}+$$

$$+\left\|f\frac{\partial}{\partial t}(w(t-h))\right\|^2_{\omega_{R_1}(R)}+\|f\nabla p(t-h)\|^2_{\omega_{R_1}(R)}-$$

$$-2(f\Delta w(t-h),\ f\nabla p(t-h))_{\omega_{R_1}(R)}+$$

$$+2\left(f\frac{\partial}{\partial t}(w(t-h)),\ f\nabla p(t-h)\right)_{\omega_{R_1}(R)}\Big],$$

$$\|f\lambda^{-\gamma}\Delta w(t-h)+f\gamma\lambda^{-\gamma-1}w(t-h)\|^2_{\omega_{R_1}(R)}=\lambda^{-2\gamma}[\|f\Delta w(t-h)\|^2_{\omega_{R_1}(R)}+$$

$$+\gamma^2\lambda^{-2}\|fw(t-h)\|^2_{\omega_{R_1}(R)}+2\gamma\lambda^{-1}(f\Delta w(t-h),\ fw(t-h))_{\omega_{R_1}(R)}],$$

$$\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\omega_{R_1}(R)}=\lambda^{-2\gamma}\Big[\gamma^2\lambda^{-2}\|fw(t-h)\|^2_{\omega_{R_1}(R)}+$$

$$+\left\|f\frac{\partial}{\partial t}(w(t-h))\right\|^2_{\omega_{R_1}(R)}+2\gamma\lambda^{-1}\left(fw(t-h),\ f\frac{\partial}{\partial t}(w(t-h))\right)_{\omega_{R_1}(R)}\Big],$$

$$-2\left(f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h)),\ \lambda^{-\gamma}f\Delta w(t-h)\right)_{\omega_{R_1}(R)}=$$

$$=\lambda^{-2\gamma}\Big[-2\gamma\lambda^{-1}(fw(t-h),\ f\Delta w(t-h))_{\omega_{R_1}(R)}-$$

$$-2\left(f\frac{\partial}{\partial t}(w(t-h)),\ f\Delta w(t-h)\right)_{\omega_{R_1}(R)}\Big],$$

$$-2\gamma\left(f\lambda^{-\gamma-1}w(t-h),\ f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right)_{\omega_{R_1}(R)}=$$

$$=\lambda^{-2\gamma}\Big[-2\gamma^2\lambda^{-2}\,\|fw(t-h)\|^2_{\omega_{R_1}(R)}-$$

$$-2\gamma\lambda^{-1}\left(fw(t-h),\ f\frac{\partial}{\partial t}(w(t-h))\right)_{\omega_{R_1}(R)}\Big],$$

$$2\left(f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h)),\ f\lambda^{-\gamma}\nabla p(t-h)\right)_{\omega_{R_1}(R)}=$$

$$=\lambda^{-2\gamma}\Big[2\gamma\lambda^{-1}(fw(t-h),\ f\nabla p(t-h))_{\omega_{R_1}(R)}+$$

$$+2\left(f\frac{\partial}{\partial t}(w(t-h)),\ f\nabla p(t-h)\right)_{\omega_{R_1}(R)}\Big],$$

$$-2(f\lambda^{-\gamma}\Delta w(t-h)+f\gamma\lambda^{-1-\gamma}w(t-h),\ f\lambda^{-\gamma}\nabla p(t-h))_{\omega_{R_1}(R)}=$$

$$=\lambda^{-2\gamma}[-2(f\Delta w(t-h),\ f\nabla p(t-h))_{\omega_{R_1}(R)}-$$

$$-\ \gamma\lambda^{-1}(fw(t-h),\ f\nabla p(t-h))_{\omega_{R_1}(R)}].$$

Dalla (1) segue che

$$\text{(2)}\qquad \int_{-\tau}^{0}\lambda^{-2\gamma}\,\|fLw(t-h)-f\nabla p(t-h)\|^2_{\omega_{R_1}(R)}\geqslant$$

$$\geqslant\int_{-\tau}^{0}\Big[\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}(t-h))\right\|^2_{\omega_{R_1}(R)}+$$

$$+\gamma\lambda^{-2\gamma-2}\,\|fw(t-h)\|^2_{\omega_{R_1}(R)}\Big]\,dt-$$

$$-2\int_{-\tau}^{0}dt\int_{\sigma(R)}f^2\lambda^{-\gamma}(\nabla w(t-h))\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\Big)\cdot n\,d\sigma+$$

$$+2\int_{-\tau}^{0}dt\int_{\sigma(R_1)}f^2\lambda^{-\gamma}\left((\nabla w(t-h))\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right)\cdot n\,d\sigma-$$

$$-(2\tau)^{-2\gamma}\,|||\,f,\ w(-t-h)\,|||^2_{\omega_{R_1}(R)}+$$
$$+\gamma(2\tau)^{-2\gamma-1}\,\|\,fw(-t-h)\,\|^2_{\omega_{R_1}(R)}+$$
$$+2\int_{-\tau}^{0}dt\int_{\omega_{R_1}(R)}\lambda^{-\gamma}\Big(\,(\nabla w(t-h))\,\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\Big)\cdot\nabla(f^2)dx+$$
$$+(\tau)^{-2\gamma}\,|||\,f,\ w(-h)\,|||^2_{\omega_{R_1}(R)}-\gamma(\tau)^{-2\gamma-1}\,\|\,fw(-h)\,\|^2_{\omega_{R_1}(R)}+$$
$$+2\int_{-\tau}^{0}dt\int_{\sigma(R)}\lambda^{-\gamma}f^2p(t-h)\,\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\cdot n\,d\sigma$$
$$-2\int_{-\tau}^{0}dt\int_{\sigma(R_1)}\lambda^{-\gamma}f^2p(t-h)\,\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\cdot n\,d\sigma-$$
$$-2\int_{-\tau}^{0}dt\int_{\omega_{R_1}(R)}\lambda^{-\gamma}p(t-h)\,\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\cdot\nabla(f^2)dx$$

dove n è il vettore unitario normale alla superficie $\sigma(R)$ ($\sigma(R_1)$) diretto verso l'esterno; infatti

$$\|\,\lambda^{-\gamma}f\Delta w(t-h)+f\gamma\lambda^{-\gamma-1}w(t-h)\,\|^2_{\omega_{R_1}(R)}+\lambda^{-2\gamma}\,\|\,f\nabla p(t-h)\,\|^2_{\omega_{R_1}(R)}-$$
$$-2(f\lambda^{-\gamma}\Delta w(t-h)+f\gamma\lambda^{-\gamma-1}w(t-h),\ f\lambda^{-\gamma}\nabla p(t-h))_{\omega_{R_1}(R)}\geqslant 0,$$

$$2\int_{-\tau}^{0}\Big(f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h)),\ \lambda^{-\gamma}f\nabla p(t-h)\Big)_{\omega_{R_1}(R)}dt=$$
$$=2\int_{-\tau}^{0}dt\int_{\sigma(R)}f^2\lambda^{-\gamma}p(t-h)\,\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\cdot n\,d\sigma-$$
$$-2\int_{-\tau}^{0}dt\int_{\sigma(R_1)}f^2\lambda^{-\gamma}p(t-h)\,\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\cdot n\,d\sigma-$$
$$-2\int_{-\tau}^{0}dt\int_{\omega_{R_1}(R)}\lambda^{-\gamma}p(t-h)\,\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\cdot\nabla(f^2)dx,$$

$$-\int_{-\tau}^{0} 2\gamma dt \int_{\omega_{R_1}(R)} f^2 \lambda^{-\gamma-1} w(t-h) \cdot \frac{\partial}{\partial t}(\lambda^{-\gamma} w(t-h)) dx =$$

$$= \int_{-\tau}^{0} \gamma \lambda^{-2\gamma-2} \| f w(t-h) \|^2_{\omega_{R_1}(R)} dt +$$

$$+ \gamma (2\tau)^{-2\gamma-1} \| f w(-\tau-h) \|^2_{\omega_{R_1}(R)} - \gamma (\tau)^{-2\gamma-1} \| f w(-h) \|^2_{\omega_{R_1}(R)},$$

$$-2\int_{-\tau}^{0} \left(f \frac{\partial}{\partial t}(\lambda^{-\gamma} w(t-h)),\ f \lambda^{-\gamma} \Delta w(t-h) \right)_{\omega_{R_1}(R)} dt =$$

$$= -2\int_{-\tau}^{0} dt \int_{\sigma(R)} \lambda^{-\gamma} f^2 (\nabla w(t-h)) \frac{\partial}{\partial t}(\lambda^{-\gamma} w(t-h)) \cdot n d\sigma +$$

$$+ 2\int_{-\tau}^{0} dt \int_{\sigma(R_1)} \lambda^{-\gamma} f^2 \left((\nabla w(t-h)) \frac{\partial}{\partial t}(\lambda^{-\gamma} w(t-h)) \right) \cdot n d\sigma -$$

$$- (2\tau)^{-2\gamma} |||\, f,\ w(-\tau-h) \,|||^2_{\omega_{R_1}(R)} +$$

$$+ 2\int_{-\tau}^{0} dt \int_{\omega_{R_1}(R)} \lambda^{-\gamma} \left((\nabla w(t-h)) \frac{\partial}{\partial t}(\lambda^{-\gamma} w(t-h)) \right) \cdot \nabla (f^2) dx +$$

$$+ (\tau)^{-2\gamma} |||\, f,\ w(-h) \,|||^2_{\omega_{R_1}(R)}.$$

D'altra parte

$$2\int_{-\tau}^{0} dt \int_{\sigma(R)} \lambda^{-\gamma} f^2 \left((\nabla w(t-h)) \frac{\partial}{\partial t}(\lambda^{-\gamma} w(t-h)) \right) \cdot n d\sigma \leqslant$$

$$\leqslant \int_{-\tau}^{0} \left\| f \frac{\partial}{\partial t}(\lambda^{-\gamma} w(t-h)) \right\|^2_{\sigma(R)} dt +$$

$$+ \int_{-\tau}^{0} \lambda^{-2\gamma} |||\, f,\ w(t-h) \,|||^2_{\sigma(R)} dt,$$

$$2\left|\int_{-\tau}^{0} dt \int_{\sigma(R)} f^2\lambda^{-\gamma}p(t-h)\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\cdot n d\sigma\right| \leqslant$$

$$\leqslant 2\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\,\|fp(t-h)\|^2_{\sigma(R)}dt\right)^{1/2}\left(\int_{-\tau}^{0}\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\sigma(R)}dt\right)^{1/2}$$

e quindi dalla (2) segue che

$$(3)\qquad \int_{-\tau}^{0}\left[\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\omega_{R_1}(R)}+\gamma\lambda^{-2\gamma-2}\,\|fw(t-h)\|^2_{\omega_{R_1}(R)}\right]dt\leqslant$$

$$\leqslant\int_{-\tau}^{0}\lambda^{-2\gamma}\,\|fLw(t-h)-f\nabla p(t-h)\|^2_{\omega_{R_1}(R)}dt+$$

$$+\int_{-\tau}^{0}\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\sigma(R)}dt+\int_{-\tau}^{0}\lambda^{-2\gamma}\,|||f,\ w(t-h)|||^2_{\sigma(R)}dt+$$

$$+\int_{-\tau}^{0}\left[\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\sigma(R_1)}+\lambda^{-2\gamma}(|||f,\ w(t-h)|||^2_{\sigma(R_1)}\right]dt+$$

$$+(2\tau)^{-2\gamma}\,|||f,\ w(-\tau-h)|||^2_{\omega_{R_1}(R)}-\gamma(2\tau)^{-2\gamma-1}\,\|fw(-\tau-h)\|^2_{\omega_{R_1}(R)}+$$

$$+\gamma(\tau)^{-2\gamma-1}\,\|fw(-h)\|^2_{\omega_{R_1}(R)}-$$

$$-2\int_{-\tau}^{0}dt\int_{\omega_{R_1}(R)}\lambda^{-\gamma}\left((\nabla w(t-h))\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right)\cdot\nabla(f^2)dx+$$

$$+2\int_{-\tau}^{0}dt\int_{\omega_{R_1}(R)}\lambda^{-\gamma}p(t-h)\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\cdot\nabla(f^2)dx+$$

$$+2\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\,\|fp(t-h)\|^2_{\sigma(R)}dt\right)^{1/2}\left(\int_{\tau}^{0}\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\sigma(R)}dt\right)^{1/2}+$$

$$+2\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\,\|fp(t-h)\|^2_{\sigma(R_1)}dt\right)^{1/2}\left(\int_{-\tau}^{0}\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\sigma(R_1)}dt\right)^{1/2}.$$

È

$$(4)\qquad -\int_{-\tau}^{0}(\lambda^{-\gamma-1}fw(t-h),\ \lambda^{-\gamma+1}f[Lw(t-h)-\nabla p(t-h)])_{\omega_{R_1}(R)}dt=$$

$$=\int_{-\tau}^{0}\lambda^{-2\gamma}(|||\,f,\ w(t-h)\,|||^2_{\omega_{R_1}(R)}-\gamma\lambda^{-1}\,\|\,fw(t-h)\,\|^2_{\omega_{R_1}(R)})dt-$$

$$-\int^{0}dt\int_{\sigma(R)}\lambda^{-2\gamma}f^2((\nabla w(t-h))w(t-h))\cdot nd\sigma-$$

$$-\frac{1}{2}(2\tau)^{-2\gamma}\,\|\,fw(-\tau-h)\,\|^2_{\omega_{R_1}(R)}+\int_{-\tau}^{0}dt\int_{\sigma(R_1)}\lambda^{-2\gamma}f^2((\nabla w(t-h))w(t-h))\cdot nd\sigma+$$

$$+\frac{1}{2}(\tau)^{-2\gamma}\,\|\,fw(-h)\,\|^2_{\omega_{R_1}(R)}+$$

$$+\int_{-\tau}^{0}\lambda^{-2\gamma}dt\int_{\omega_{R_1}(R)}((\nabla w(t-h))w(t-h))\cdot\nabla(f^2)dx-$$

$$-\int_{-\tau}^{0}\lambda^{-2\gamma}dt\int_{\omega_{R_1}(R)}p(t-h)w(t-h)\cdot\nabla(f^2)dx+\int_{-\tau}^{0}\lambda^{-2\gamma}dt\int_{\sigma(R)}f^2p(t-h)w(t-h)\cdot nd\sigma-$$

$$-\int_{-\tau}^{0}\lambda^{-2\gamma}dt\int_{\sigma(R_1)}f^2p(t-h)w(t-h)\cdot nd\sigma;$$

infatti

$$-\int_{-\tau}^{0}\left(f\lambda^{-\gamma-1}w(t-h),\ -f\lambda^{-\gamma+1}\frac{\partial}{\partial t}(w(t-h))\right)_{\omega_{R_1}(R)}dt=$$

$$=-\frac{1}{2}(2\tau)^{-2\gamma}\,\|\,fw(-\tau-h)\,\|^2_{\omega_{R_1}(R)}+$$

$$+\frac{1}{2}(\tau)^{-2\gamma}\,\|\,fw(-h)\,\|^2_{\omega_{R_1}(R)}-\gamma\int_{-\tau}^{0}dt\int_{\omega_{R_1}(R)}f^2\lambda^{-2\gamma-1}w^2(t-h)dx,$$

$$-\int_{-\tau}^{0}(\lambda^{-\gamma-1}fw(t-h),\ \lambda^{-\gamma+1}f\Delta w(t-h))_{\omega_{R_1}(R)}dt=$$

$$=-\int_{-\tau}^{0}dt\int_{\sigma(R)}\lambda^{-2\gamma}f^2((\nabla w(t-h)w(t-h))\cdot nd\sigma+$$

$$+\int_{-\tau}^{0}dt\int_{\sigma(R_1)}\lambda^{-2\gamma}f^2((\nabla w(t-h)w(t-h))\cdot nd\sigma+$$

$$+\int_{-\tau}^{0}\lambda^{-2\gamma}|||f,\ w(t-h)|||^2_{\omega_{R_1}(R)}dt+$$

$$+\int_{-\tau}^{0}\lambda^{-2\gamma}dt\int_{\omega_{R_1}(R)}((\nabla w(t-h))w(t-h))\cdot\nabla(f^2)dx,$$

$$\int_{-\tau}^{0}dt\int_{\omega_{R_1}(R)}(\lambda^{-\gamma-1}fw(t-h),\ f\lambda^{-\gamma+1}f\nabla p(t-h))_{\omega_{R_1}(R)}dt=$$

$$=-\int_{-\tau}^{0}\lambda^{-2\gamma}dt\int_{\omega_{R_1}(R)}p(t-h)w(t-h)\cdot\nabla(f^2)dx+$$

$$+\int_{-\tau}^{0}\lambda^{-2\gamma}dt\int_{\sigma(R)}f^2p(t-h)w(t-h)\cdot nd\sigma-$$

$$-\int_{-\tau}^{0}\lambda^{-2\gamma}dt\int_{\sigma(R_1)}f^2p(t-h)w(t-h)\cdot nd\sigma.$$

Dalla (4) segue che

$$\int_{-\tau}^{0}\lambda^{-2\gamma}|||f,\ w(t-h)|||^2_{\omega_{R_1}(R)}dt\leqslant \tag{5}$$

$$\leqslant 2\tau^2\int_{-\tau}^{0}\lambda^{-2\gamma}\|fLw(t-h)-f\nabla p(t-h)\|^2_{\omega_{R_1}(R)}dt+$$

$$+\frac{1}{2}\int_{-\tau}^{0}\lambda^{-2\gamma}[\| fw(t-h)\|^2_{\sigma(R)}+|||f,\ w(t-h)|||^2_{\sigma(R)}]\,dt+$$

$$+\frac{1}{2}\int_{-\tau}^{0}\lambda^{-2\gamma}[\| fw(t-h)\|^2_{\sigma(R_1)}+|||f,\ w(t-h)|||^2_{\sigma(R_1)}]\,dt+$$

$$+\left(2\gamma\tau+\frac{1}{2}\right)\int_{-\tau}^{0}\lambda^{-2-2\gamma}\| fw(t-h)\|^2_{\omega_{R_1}(R)}dt+$$

$$+\frac{1}{2}(2\tau)^{-2\gamma}\| fw(-\tau-h)\|^2_{\omega_{R_1}(R)}-$$

$$-\int_{-\tau}^{0}\lambda^{-2\gamma}dt\int_{\omega_{R_1}(R)}((\nabla w(-\tau-h))w(-\tau-h))\cdot\nabla(f^2)dx+$$

$$+\int_{-\tau}^{0}\lambda^{-2\gamma}dt\int_{\omega_{R_1}(R)}p(t-h)w(t-h)\cdot\nabla(f^2)dx+$$

$$+\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\| fp(t-h)\|^2_{\sigma(R)}dt\right)^{1/2}\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\| fw(t-h)\|^2_{\sigma(R)}dt\right)^{1/2}+$$

$$+\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\| fp(t-h)\|^2_{\sigma(R_1)}dt\right)^{1/2}\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\| fw(t-h)\|^2_{\sigma(R_1)}dt\right)^{1/2},$$

perchè

$$\left|\int_{-\tau}^{0}(\lambda^{-\gamma-1}fw(t-h),\ f\lambda^{-\gamma+1}[Lw(t-h)-\nabla p(t-h)])_{\omega_{R_1}(R)}dt\right|\leqslant$$

$$\leqslant\frac{1}{2}\int_{-\tau}^{0}\lambda^{-2\gamma-2}\| fw(t-h)\|^2_{\omega_{R_1}(R)}dt+$$

$$+2\tau^2\int_{-\tau}^{0}\lambda^{-2\gamma}\| fLw(t-h)-f\nabla p(t-h)\|^2_{\omega_{R_1}(R)}dt,$$

$$\gamma\int_{-\tau}^{0}\lambda^{-2\gamma-1}\,\|\,fw(t-h)\,\|^2_{\omega_{R_1}(R)}dt\leqslant 2\tau\gamma\int_{-\tau}^{0}\lambda^{-2\gamma-2}\,\|\,fw(t-h)\,\|^2_{\omega_{R_1}(R)}dt.$$

Sommando alla (5) la (3) moltiplicata per $2\tau+\frac{1}{2}\gamma^{-1}+\delta+4\tau^2$ si ottiene

$$(\delta\gamma+4\tau^2\gamma)\int_{-\tau}^{0}\lambda^{-2\gamma-2}\,\|\,fw(t-h)\,\|^2_{\omega_{R_1}(R)}dt+$$

$$+\left(2\tau+\frac{1}{2}\gamma^{-1}+\delta+4\tau^2\right)\int_{-\tau}^{0}\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}(t-h))\right\|^2_{\omega_{R_1}(R)}dt+$$

$$+\int_{-\tau}^{0}\lambda^{-2\gamma}\,|||\,f,\ w(t-h)\,|||^2_{\omega_{R_1}(R)}dt\leqslant$$

$$\leqslant\left(2\tau+\frac{1}{2}\gamma^{-1}+\delta+6\tau^2\right)\int_{-\tau}^{0}\lambda^{-2\gamma}\,\|\,fLw(t-h)-f\nabla p(t-h)\,\|^2_{\omega_{R_1}(R)}dt+$$

$$+\frac{1}{2}\int_{-\tau}^{0}\lambda^{-2\gamma}\,\|\,fw(t-h)\,\|^2_{\sigma(R)}dt+$$

$$+\left(2\tau+\frac{1}{2}\gamma^{-1}+\delta+4\tau^2+\frac{1}{2}\right)\int_{-\tau}^{0}\lambda^{-2\gamma}\,|||\,f,\ w(t-h)\,|||^2_{\sigma(R)}dt+$$

$$+\left(2\tau+\frac{1}{2}\gamma^{-1}+\delta+4\tau^2\right)\int_{-\tau}^{0}\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\sigma(R)}dt+$$

$$+\frac{1}{2}\int_{-\tau}^{0}\lambda^{-2\gamma}\,\|\,fw(t-h)\,\|^2_{\sigma(R_1)}dt+$$

$$+\left(2\tau+\frac{1}{2}\gamma^{-1}+\delta+4\tau^2+\frac{1}{2}\right)\int_{-\tau}^{0}\lambda^{-2}\,|||\,f,\ w(t-h)\,|||^2_{\sigma(R_1)}dt+$$

$$+\left(2\tau+\frac{1}{2}\gamma^{-1}+\delta+4\tau^2\right)\int_{-\tau}^{0}\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\sigma(R_1)}dt+$$

$$+\left[\frac{1}{2}(2\tau)^{-2\gamma}-\gamma(2\tau)^{-2\gamma-1}\left(2\tau+\frac{1}{2}\gamma^{-1}+\delta+4\tau^2\right)\right]\| fw(-\tau-h)\|^2_{\omega_{R_1}(R)}+$$

$$+\left(2\tau+\frac{1}{2}\gamma^{-1}+\delta+4\tau^2\right)(2\tau)^{-2\gamma}\,|||\, f,\ w(-\tau-h)\,|||^2_{\omega_{R_1}(R)}-$$

$$-\int_{-\tau}^{0}\lambda^{-2\gamma}dt\int_{\omega_{R_1}(R)}((\nabla w(t-h))w(t-h))\cdot\nabla(f^2)dx-$$

$$-2\left(2\tau+\frac{1}{2}\gamma^{-1}+\delta+4\tau^2\right)\int_{-\tau}^{0}dt\int_{\omega_{R_1}(R)}\lambda^{-\gamma}\left((\nabla w(t-h))\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right)\cdot\nabla(f^2)dx+$$

$$+\gamma(\tau)^{-2\gamma-1}\left(2\tau+\frac{1}{2}\gamma^{-1}+\delta+4\tau^2\right)\| fw(-h)\|^2_{\omega_{R_1}(R)}+$$

$$+\int_{-\tau}^{0}\lambda^{-2\gamma}dt\int_{\omega_{R_1}(R)}p(t-h)w(t-h)\cdot\nabla(f^2)dx+$$

$$+\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\| fp(t-h)\|^2_{\sigma(R)}dt\right)^{1/2}\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\| fw(t-h)\|^2_{\sigma(R)}dt\right)^{1/2}+$$

$$+\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\| fp(t-h)\|^2_{\sigma(R_1)}dt\right)^{1/2}\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\| fw(t-h)\|^2_{\sigma(R_1)}dt\right)^{1/2}+$$

$$+2\left(2\tau+\frac{1}{2}\gamma^{-1}+\delta+4\tau^2\right)\int_{-\tau}^{0}dt\int_{\omega_{R_1}(R)}\lambda^{-\gamma}p(t-h)\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\cdot\nabla(f^2)dx+$$

$$+2\left(2\tau+\frac{1}{2}\gamma^{-1}+\delta+4\tau^2\right)\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\| fp(t-h)\|^2_{\sigma(R)}dt\right)^{1/2}\cdot$$

$$\cdot\left(\int_{-\tau}^{0}\left\| f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\sigma(R)}dt\right)^{1/2}+2\left(2\tau+\frac{1}{2}\gamma^{-1}+\delta+4\tau^2\right)\cdot$$

$$\cdot\left(\int_{-\tau}^{0}\lambda^{-2\gamma}\| fp(t-h)\|^2_{\sigma(R_1)}dt\right)^{1/2}\left(\int_{-\tau}^{0}\left\|\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\sigma(R_1)}dt\right)^{1/2},$$

da cui, tenendo presente che

$$\int_{-\tau}^{0} \lambda^{-2\gamma} \| fw(t-h) \|^2_{\omega_{R_1}(R)} dt \leqslant 4\tau^2 \int_{-\tau}^{0} \lambda^{-2\gamma-2} \| fw(t-h) \|^2_{\omega_{R_1}(R)} dt,$$

si ottiene il lemma.

OSSERVAZIONE. Nel caso in cui $w=0$ su $\partial\Omega \times [-T, 0]$, $f=1$ in Q e $| p(x, t) | \leqslant C/(1+| x |)^{1/2}$ $\forall (x, t) \in Q$, se si integra su $\omega(R)$ anzichè su $\omega_{R_1}(R)$ si ottiene

$$\int_{-\tau}^{0} \lambda^{-2\gamma} \| w(t-h) \|^2_{\omega(R)} dt + \int_{-\tau}^{0} \lambda^{-2\gamma} ||| w(t-h) |||^2_{\omega(R)} dt +$$

$$+ C_{\tau,\gamma,\delta} \int_{-\tau}^{0} \left\| \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|^2_{\omega(R)} dt \leqslant$$

$$\leqslant 2C_{\tau,\gamma,\delta} \int_{-\tau}^{0} \lambda^{-2\gamma} \| Lw(t-h) - \nabla p(t-h) \|^2_{\omega(R)} dt +$$

$$+ 4 \int_{-\tau}^{0} \left\| \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|^2_{\sigma(R)} dt +$$

$$+ 4 \int_{-\tau}^{0} \lambda^{-2\gamma} (\| w(t-h) \|^2_{\sigma(R)} + ||| w(t-h) |||^2_{\sigma(R)}) dt +$$

$$+ C(\tau, \gamma) R^{1/2} \left(\int_{-\tau}^{0} \left\| \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|^2_{\sigma(R)} dt \right)^{1/2} +$$

$$+ C(\tau, \gamma) R^{1/2} \left(\int_{-\tau}^{0} \lambda^{-2\gamma} \| w(t-h) \|^2_{\sigma(R)} dt \right)^{1/2} +$$

$$+ C_{\tau,\gamma,\delta} (2\tau)^{-2\gamma} ||| w(-\tau-h) |||^2_{\omega(R)} +$$

$$+ (4\gamma)(\tau)^{-2\gamma-1} \| w(-h) \|^2_{\omega(R)} - \tau\gamma (2\tau)^{-2\gamma-1} \| w(-\tau-h) \|^2_{\omega(R)}$$

$$\forall \tau,\ R,\ R_1,\ \gamma,\ \delta \in \boldsymbol{R}^+,\ \forall h \geqslant 0 \text{ con } R \geqslant R_1 \geqslant R_0,\ \gamma \geqslant 1,\ \tau \leqslant \frac{1}{3},$$

$$0 \leqslant h+\tau \leqslant T,\ 0<\delta \leqslant \frac{1}{4},$$

essendo $C(\tau, \gamma)$ una costante dipendente da τ e da γ.

LEMMA 3. *Sia $a \in \boldsymbol{R}^+$ e sia $\Phi : [a, +\infty[\to \boldsymbol{R}^+$ di classe $C^{(1)}$ tale che $\Phi'(R) \geqslant 0$ $\forall R \geqslant a$. Se esistono un $\gamma \in \boldsymbol{R}^+$ e una successione $(\beta_n)_{n \in N}$ divergente positivamente tali che*

$$\Phi(\beta_n) \leqslant \beta_n^{\gamma} \qquad \forall n \in N,$$

allora, fissato δ, $0<\delta \leqslant 1$, esiste di conseguenza una successione $(\alpha_n)_{n \in N}$ divergente positivamente tale che

$$\Phi'(\alpha_n) \leqslant \alpha_n^{\gamma+\delta-1} \qquad \forall n \in N.$$

DIMOSTRAZIONE. Supponiamo per assurdo che non esista una tale successione; allora esiste $R_1 \geqslant a$ tale che

$$\Phi'(R) \geqslant R^{\gamma+\delta-1} \qquad \forall R \geqslant R_1,$$

da cui, integrando sull'intervallo $[R_1, R]$ $(R>R_1)$ si ottiene

$$\frac{R^{\gamma+\delta}-R_1^{\gamma+\delta}}{\gamma+\delta} \leqslant \Phi(R)-\Phi(R_1) \qquad \forall R \geqslant R_1$$

e quindi

$$\beta_n^{\gamma+\delta} \frac{1}{\gamma+\delta} \leqslant R_1^{\gamma+\delta} \frac{1}{\gamma+\delta} + \Phi(\beta_n) \leqslant R_1^{\gamma+\delta} \frac{1}{\gamma+\delta} + \beta_n^{\gamma}$$

$\forall \beta_n \geqslant R_1$, che è un assurdo.

LEMMA 4. *Sia u, p soluzione di (α)-(β) in Q con*

i) $u_i \in L_3$, $F_i \in L_1$ per $1 \leqslant i \leqslant 3$, $p \in L_2$

ii) u_i, $\dfrac{\partial u_i}{\partial x_j}$ *limitate in Q per $1 \leqslant i, j \leqslant 3$*

iii) u_1, u_2, u_3 *nulle su $\Omega \times \{0\}$.*

Allora esistono $h_0, h_1, h_2, \ldots, h_s$ *con* $0=h_0<h_1<h_2<\ldots<h_s=T$ *tali che*

$$\int_{h_j}^{h_{j+1}} d\tau \int_{-\tau}^{-h_j} dt \int_{\omega_{R_0}} w^2 dx < +\infty \qquad \text{per } 0 \leqslant j \leqslant s.$$

DIMOSTRAZIONE. Per provare il lemma, procediamo per tappe.

I. Posto

$$w = \nabla \times u \tag{6}$$

$$\Phi(R) = \int_{-T}^{0} dt \int_{R_0}^{R} |||\, W \,|||^2_{\omega_{R_0}(r)} dr \tag{7}$$

$$\Psi(R) = \int_{-T}^{0} dt \int_{t}^{0} d\xi \int_{R_0}^{R} d\rho \int_{R_0}^{\rho} \left\| \frac{\partial w}{\partial t} \right\|^2_{\omega_{R_0}(r)} dr \tag{8}$$

proviamo che esiste $\overline{R}_0>0$ tale che

$$\int_{R_0}^{R} \Phi(\rho) d\rho \leqslant R^{16} \qquad \forall R \geqslant \overline{R}_0 \tag{9}$$

$$\Psi(R) \leqslant R^{51} \qquad \forall R \geqslant \overline{R}_0 . \tag{10}$$

Dalla (α) e dalla (6), ricordando che per ipotesi $\nabla \times F = 0$, si ottiene

$$\frac{\partial w}{\partial t} - \Delta w + (u \cdot \nabla) w - (w \cdot \nabla) u = 0. \tag{11}$$

Moltiplicando scalarmente la (11) per w si ottiene

$$\frac{1}{2} \frac{\partial (w^2)}{\partial t} - w \cdot \Delta w + ((u \cdot \nabla) w) \cdot w - ((w \cdot \nabla) u \cdot w = 0 \tag{12}$$

da cui, tenendo presente la (s) e che

$$((u\cdot\nabla)w)\cdot w=\nabla\cdot\left(\frac{1}{2}w^2u\right)$$

$$|((w\cdot\nabla)u)\cdot w|\leqslant C_1w^2$$

si ottiene

$$\sum_{i,j=1}^{3}\left(\frac{\partial w_i}{\partial x_j}\right)^2\leqslant\frac{1}{2}\nabla\cdot(\nabla(w^2)-w^2u)+C_1w^2-\frac{1}{2}\frac{\partial(w^2)}{\partial t}. \tag{13}$$

Dalla (13) eseguendo l'integrazione

$$\int_{-T}^{0}dt\int_{R_0}^{R}dr\int_{\omega_{R_0}(r)}(\ldots)\,dx$$

e tenendo presente la (ii) segue

$$\int_{-T}^{0}dt\int_{R_0}^{R}|||\,w\,|||^2_{\omega_{R_0}(r)}dr\leqslant C_2R^4+C_2R^5\left(\int_{-T}^{0}|||\,w\,|||^2_{\omega_{R_0}(R)}dt\right)^{1/2}\qquad\forall R\geqslant R_0$$

e quindi tenendo presente la (7) si ottiene

$$\Phi(R)\leqslant C_2R^4+C_2R^5(\Phi'(R))^{1/2}\qquad\forall R\geqslant R_0\,,$$

da cui, per il lemma 1, segue che

$$\Phi(R_1)\leqslant R^{15}\qquad\forall R\geqslant R_1 \tag{14}$$

per un certo $R_1>0$ da ciò segue la (9). Moltiplicando ora la (11) scalarmente per $\dfrac{\partial w}{\partial t}$ otteniamo

$$\left(\frac{\partial w}{\partial t}\right)^2-\frac{\partial w}{\partial t}\cdot\Delta w+((u\cdot\nabla)w)\cdot\frac{\partial w}{\partial t}-((w\cdot\nabla)u)\cdot\frac{\partial w}{\partial t}=0,$$

da cui, tenendo presente la (u) e che

$$\left|((w\cdot\nabla)u)\cdot\frac{\partial w}{\partial t}\right|\leqslant C_3+\frac{1}{8}\left(\frac{\partial w}{\partial t}\right)^2$$

$$\left|((u\cdot\nabla)w)\cdot\frac{\partial w}{\partial t}\right|\leqslant C_3\sum_{i,j=1}^{3}\left(\frac{\partial w_i}{\partial x_j}\right)^2+\frac{1}{8}\left(\frac{\partial w}{\partial t}\right)^2$$

si ottiene

$$\frac{1}{4}\left(\frac{\partial w}{\partial t}\right)\leqslant\nabla\cdot\left((\nabla w)\frac{\partial w}{\partial t}\right)-\frac{1}{2}\frac{\partial}{\partial t}\left(\sum_{i,j=1}^{3}\left(\frac{\partial w_i}{\partial x_j}\right)^2\right)+C_3\left(\sum_{i,j=1}^{3}\left(\frac{\partial w_i}{\partial x_j}\right)^2\right)+C_3\,,$$

da cui, ricordando (14) e (7),

$$\int_{R_0}^{R}d\rho\int_{-T}^{0}dt\int_{t}^{0}d\xi\int_{R_0}^{\rho}\left\|\frac{\partial w}{\partial t}\right\|^2_{\omega_{R_0}(r)}dr\leqslant\int_{R_0}^{R}d\rho\int_{-T}^{0}dt\int_{t}^{0}d\xi\int_{\omega_{R_0}(\rho)}(\nabla w)\frac{\partial w}{\partial t}dx+$$

$$+C_4R^{16}\leqslant C_5R^8\left(\int_{-T}^{0}dt\int_{t}^{0}d\xi\int_{R_0}^{R}\left\|\frac{\partial w}{\partial t}\right\|^2_{\omega_{R_0}(r)}dr\right)^{1/2}+C_5R^{16}\,;$$

quindi, ricordando la (8), si ottiene

$$\psi(R)\leqslant C_5R^{16}+C_5R^{17}(\psi'(R))^{1/2}\qquad\qquad\forall R\geqslant R_1\,;$$

di qui, per il lemma 1, segue la (10).

II. Poniamo ora

$$\begin{aligned}(15)\quad \wedge_{R_1}(R,\,m,\,h)=&\int_0^{\tau_0}d\tau\int_{-\tau}^{0}\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha}m}(\|fw(t-h)\|^2_{\omega_{R_1}(R)}+\\&+\tau^2\,|||f,\,w(t-h)|||^2_{\omega_{R_1}(R)})dt+\\&+\delta\int_0^{\tau_0}\tau^2(2\tau)^{R_1^{\alpha}m}d\tau\int_{-\tau}^{0}\left\|f\frac{\partial}{\partial t}((t-\tau)^{-R_1^{\alpha}m/2}w(t-h))\right\|^2_{\omega_{R_1}(R)}dt+\\&+\int_{-\tau_0}^{0}t^2\,\|fw(t-h)\|^2_{\omega_{R_1}(R)}dt+\frac{1}{2}\int_{-\tau_0}^{0}t^2\,|||f,\,w(t-h)|||^2_{\omega_{R_1}(R)}dt\end{aligned}$$

dove $f(x, t)=\exp(-a(1+x_1^2+x_2^2+x_3^2)^{1/20})$ e $\alpha_m=1\Big/\left(9-\dfrac{1}{uu}\right)$.

Proviamo che esistono δ, τ_0, a, R_2, $C_{10}\in \boldsymbol{R}^+$ tali che

$$\Lambda_{R_1}(R, m, h)\leqslant C_{10}\Lambda'(R_1, m, h)+C_{10}+R_1^{\alpha_m}2^{R_1^{\alpha_m}}\,\|fw(-h)\|^2_{\omega_{R_1}} \tag{16}$$

$$\forall R,\ R_1,\ h,\ m \text{ con } R\geqslant R_1\geqslant R_2,\ 0\leqslant h+\tau_0\leqslant T,\ m\in N$$

dove $\Lambda'(R_1, m, h)=\left(\dfrac{d\Lambda_{R_1}(R, m, h)}{dR}\right)_{R=R_1}$.

Cominciamo con l'osservare che

$$\int_{-\tau}^{0}\lambda^{-2\gamma}dt\int_{\omega_{R_1}(R)}(fLw(t-h))^2dx=$$

$$=\int_{-\tau}^{0}\lambda^{-2\gamma}dt\int_{\omega_{R_1}(R)}[f(w(t-h)\cdot\nabla)u(t-h)-f(u(t-h)\cdot\nabla)w(t-h)]^2dx\leqslant$$

$$\leqslant C_6\int_{-\tau}^{0}\lambda^{-2\gamma}(\|fw(t-h)\|^2_{\omega_{R_1}(R)}+|||f,\ w(t-h)|||^2_{\omega_{R_1}(R)})dt,$$

$$\left|\frac{1}{2}\int_{-\tau}^{0}\lambda^{-2\gamma}dt\int_{\omega_{R_1}(R)}\nabla(w^2(t-h))\cdot\nabla(f^2)dx\right|\leqslant$$

$$\leqslant aC_7\int_{-\tau}^{0}\lambda^{-2\gamma}(\|fw(t-h)\|^2_{\omega_{R_1}(R)}+|||f,\ w(t-h)|||^2_{\omega_{R_1}(R)})dt,$$

$$\left|\int_{-\tau}^{0}dt\int_{\omega_{R_1}(R)}\lambda^{-\gamma}\left((\nabla w(t-h))\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right)\cdot\nabla(f^2)dx\leqslant\right.$$

$$\leqslant aC_8\int_{-\tau}^{0}\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\omega_{R_1}(R)}dt+$$

$$+aC_8\int_{-\tau}^{0}\lambda^{-2\gamma}|||f,\ w(t-h)|||^2_{\omega_{R_1}(R)}dt.$$

Allora, detta K la più grande delle costanti C_6, C_7, C_8 e scelti a, δ, τ_0, $\gamma_0 \in \boldsymbol{R}^+$ soddisfacenti le condizioni

$$aK < \frac{1}{4}, \quad \gamma_0 > 2\left(a + 2\left(2\tau_0 + \frac{1}{2\gamma_0} + \delta + 4\tau_0^2\right)\right)K,$$

$$1 > 2K\left(a + 2(a+1)\left(2\tau_0 + \frac{1}{2\gamma_0} + \delta + 4\tau_0^2\right)\right),$$

oltre alle condizioni già imposte $\gamma_0 \geqslant 1$, $0 < \tau_0 \leqslant \frac{1}{3}$, $0 \leqslant h + \tau_0 \leqslant T$, $0 < \delta \leqslant \frac{1}{4}$, per il lemma 2 se si pone $p = 0$ si ha

$$\left(\gamma + \frac{\delta\gamma}{4\tau^2}\right)\int_{-\tau}^{0} \lambda^{-2\gamma} \| fw(t-h) \|^2_{\omega_{R_1}(R)} dt + \int_{-\tau}^{0} \lambda^{-2\gamma} |||\, f,\ w(t-h) \,|||^2_{\omega_{R_1}(R)} dt +$$

$$+\delta \int_{-\tau}^{0} \left\| f \frac{\partial}{\partial t}(\lambda^{-\gamma} w(t-h)) \right\|^2_{\omega_{R_1}(R)} dt \leqslant$$

$$\leqslant 8 \int_{-\tau}^{0} \lambda^{-2\gamma} (\| fw(t-h) \|^2_{\sigma(R)} + |||\, f,\ w(t-h) \,|||^2_{\sigma(R)}) dt +$$

$$+8 \int_{-\tau}^{0} \left\| f \frac{\partial}{\partial t}(\lambda^{-\gamma} w(t-h)) \right\|^2_{\sigma(R)} dt +$$

$$+8 \int_{-\tau}^{0} \lambda^{-2\gamma} (\| fw(t-h) \|^2_{\sigma(R_1)} + |||\, f,\ w(t-h) \,|||^2_{\sigma(R_1)}) dt +$$

$$+8 \int_{-\tau}^{0} \left\| f \frac{\partial}{\partial t}(\lambda^{-\gamma} w(t-h)) \right\|^2_{\sigma(R_1)} dt +$$

$$+2C_{\tau,\gamma,\delta}(2\tau)^{-2\gamma} |||\, f,\ w(-\tau-h) \,|||^2_{\omega_{R_1}(R)} -$$

$$-2\tau\gamma(2\tau)^{-2\gamma-1} \| fw(-\tau-h) \|^2_{\omega_{R_1}(R)} + 4\gamma(\tau)^{-2\gamma-1} \| fw(-h) \|^2_{\omega_{R_1}(R)}$$

$\forall \tau$, γ, R, R_1, h con $0 < \tau \leqslant \tau_0$, $\gamma \geqslant \gamma_0$, $R \geqslant R_1 \geqslant R_0$, $0 \leqslant h + \tau_0 \leqslant T$,

da cui, moltiplicando ambo i membri per $(2\tau)^{2\gamma+2}$ e integrando su $[0, \tau_0]$ si ottiene per $\gamma\delta \geqslant 1$

$$(17)\quad \int_0^{\tau_0} (2\tau)^{2\gamma} d\tau \int_{-\tau}^{0} \lambda^{-2\gamma}(\| fw(t-h) \|^2_{\omega_{R_1}(R)} + \tau^2 \| | f,\ w(t-h) \| |^2_{\omega_{R_1}(R)}) dt +$$

$$+ \int_{-\tau_0}^{0} t^2 \| | f,\ w(t-h) \| |^2_{\omega_{R_1}(R)} dt +$$

$$+ \delta \int_0^{\tau_0} (2\tau)^{2+2\gamma} d\tau \int_{-\tau}^{0} \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|^2_{\omega_{R_1}(R)} dt +$$

$$+ 4\gamma \int_{-\tau_0}^{0} t^2 \| fw(t-h \|^2_{\omega_{R_1}(R)} dt \leqslant$$

$$\leqslant 8 \int_0^{\tau_0} (2\tau)^{2\gamma+2} d\tau \int_{-\tau}^{0} \lambda^{-2\gamma}(\| fw(t-h) \|^2_{\sigma(R_1)} + \| | f,\ w(t-h) \| |^2_{\sigma(R_1)}) dt +$$

$$+ 8 \int_0^{\tau_0} (2\tau)^{2+2\gamma} d\tau \int_{-\tau}^{0} \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|^2_{\sigma(R_1)} dt +$$

$$+ 8 \int_0^{-\tau_0} (2\tau)^{2\gamma+2} d\tau \int_{-\tau}^{0} \lambda^{-2\gamma}(\| fw(t-h) \|^2_{\sigma(R)} + \| | f,\ w(t-h) \| |^2_{\sigma(R)}) dt +$$

$$+ 8 \int_0^{\tau_0} (2\tau)^{2+2\gamma} d\tau \int_{-\tau}^{0} \left\| f \frac{\partial}{\partial t} (\lambda^{-\gamma} w(t-h)) \right\|^2_{\sigma(R)} dt +$$

$$+ \gamma 2^{2\gamma} \| fw(-h) \|^2_{\omega_{R_1}(R)} + 5 \int_{-\tau_0}^{0} t^2 \| | f,\ w(t-h) \| |^2_{\omega_{R_1}(R)} dt.$$

Ora dalla (12) si ottiene

$$f^2 t^2 \sum_{i,j=1}^{3} \left(\frac{\partial w_i}{\partial x_j} \right)^2 (t-h) \leqslant -\frac{1}{2} t^2 f^2 \frac{\partial}{\partial t} (w^2(t-h)) +$$

$$+\frac{1}{2}\nabla\cdot(t^2f^2(\nabla(w^2(t-h))-w^2(t-h)u(t-h)))+$$

$$+C_1t^2f^2w^2(t-h)+\frac{1}{2}|(\nabla(w^2(t-h))-w^2(t-h)u(t-h))\cdot\nabla(f^2)|\,t^2$$

da cui

$$\int_{-\tau}^{0}dt\int_{\omega_{R_1}(R)}t^2f^2\sum_{i,j=1}^{3}\left(\frac{\partial w_i}{\partial x_j}\right)^2(t-h)dx\leqslant \tag{18}$$

$$\leqslant\frac{1}{2}\int_{-\tau_0}^{0}t^2dt\int_{\sigma(R_1)}f^2\,|\nabla(w^2(t-h))-w^2(t-h)u(t-h)|\,d\sigma+$$

$$+\frac{1}{2}\int_{-\tau_0}^{0}t^2dt\int_{\sigma(R)}f^2\,|\nabla(w^2(t-h))-w^2(t-h)u(t-h)|\,d\sigma+$$

$$+\frac{1}{2}\int_{-\tau_0}^{0}t^2dt\int_{\omega_{R_1}(R)}|\nabla(w^2(t-h))-w^2(t-h)u(t-h)|\;|\nabla(f^2)|\,dx+$$

$$+C_1\int_{-\tau_0}^{0}t^2dt\int_{\omega_{R_1}(R)}f^2w^2(t-h)dx+\frac{1}{2}\tau_0^2\int_{\omega_{R_1}(R)}f^2w^2(-\tau_0-h)dx+$$

$$+\int_{-\tau_0}^{0}t\,\|fw(t-h)\|^2_{\omega_{R_1}(R)}dt\leqslant C_9\int_{-\tau_0}^{0}t^2(\|fw(t-h)\|^2_{\sigma(R_1)}+$$

$$+|||f,\ w(t-h)|||^2_{\sigma(R_1)})dt+C_9\int_{-\tau_0}^{0}t^2(\|fw(t-h)\|^2_{\sigma(R)}+$$

$$+|||f,\ w(t-h)|||^2_{\sigma(R)})dt+(C_1+aC_9)\int_{-\tau_0}^{0}t^2\,\|fw(t-h)\|^2_{\omega_{R_1}(R)}dt+$$

$$+aC_9\int_{-\tau_0}^{0}t^2\,|||f,\ w(t-h)|||^2_{\omega_{R_1}(R)}dt+$$

$$+\frac{1}{2}\tau_0^2\int_{\omega_{R_1}(R)}f^2w^2(-\tau_0-h)dx+\int_{-\tau_0}^{0}t\,\|fw(t-h)\|^2_{\omega_{R_1}(R)}dt.$$

Maggiorando mediante la (18) l'ultimo addendo nel secondo membro della (17), si ottiene

$$
\begin{aligned}
(19)\qquad & \int_0^{\tau_0}(2\tau)^{2\gamma}d\tau\int_{-\tau}^{0}\lambda^{-2\gamma}(\| fw(t-h)\|^2_{\omega_{R_1}(R)}+\tau^2\,|||f,\ w(t-h)|||^2_{\omega_{R_1}(R)})dt+\\
&+\delta\int_0^{\tau_0}(2\tau)^{2\gamma}\tau^2d\tau\int_{-\tau}^{0}\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\omega_{R_1}(R)}dt+\int_{-\tau_0}^{0}t^2\,\| fw(t-h)\|^2_{\omega_{R_1}(R)}dt+\\
&+\frac{1}{2}\int_{-\tau_0}^{0}t^2\,|||f,\ w(t-h)|||^2_{\omega_{R_1}(R)}dt\leqslant\\
&\leqslant 8\int_0^{\tau_0}(2\tau)^{2\gamma+2}d\tau\int_{-\tau}^{0}\lambda^{-\gamma 2}(\| fw(t-h)\|^2_{\sigma(R_1)}+|||f,\ w(t-h)|||^2_{\sigma(R_1)})dt+\\
&+8\int_0^{\tau_0}(2\tau)^{2+2\gamma}\int_{-\tau}^{0}\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\sigma(R_1)}dt+\\
&+8\int_0^{\tau_0}(2\tau)^{2\gamma+2}d\tau\int_{-\tau}^{0}\lambda^{-2\gamma}(\| fw(t-h)\|^2_{\sigma(R)}+|||f,\ w(t-h)|||^2_{\sigma(R)})dt+\\
&+8\int_0^{\tau_0}(2\tau)^{2+2\gamma}d\tau\int_{-\tau}^{0}\left\|f\frac{\partial}{\partial t}(\lambda^{-\gamma}w(t-h))\right\|^2_{\sigma(R)}dt+\\
&+5C_9\int_{-\tau_0}^{0}t^2(\| fw(t-h)\|^2_{\sigma(R_1)}+|||f,\ w(t-h)|||^2_{\sigma(R_1)})dt+\\
&+5C_9\int_{-\tau_0}^{0}t^2(\| fw(t-h)\|^2_{\sigma(R)}+|||f,\ w(t-h)|||^2_{\sigma(R)})dt+\\
&+\frac{5}{2}\tau_0^2\int_{\omega_{R_1}(R)}f^2w^2(-\tau_0-h)dx+5\int_{-\tau_0}^{0}t\,\| fw(t-h)\|^2_{\omega_{R_1}(R)}dt+\\
&+\gamma 2^{2\gamma}\,\| fw(-h)\|^2_{\omega_{R_1}(R)}
\end{aligned}
$$

$\forall\gamma, h$ con $\gamma\geqslant\gamma_1\geqslant\gamma_0$, $4\gamma_1\geqslant 1+5(C_1+aC_9)$, $5aC_9\leqslant\frac{1}{2}$ e $0\leqslant h+\tau_0\leqslant T$.

Dalla (19) posto $2\gamma = R_1^{\alpha_m}$ e tenendo presente la (ii) si ottiene

$$\Lambda_{R_1}(R, m, h) \leqslant C_{10}\Lambda'(R_1, m, h) + C_{10}\Lambda'(R, m, h) + \tag{20}$$
$$+ C_{10} + R_1^{\alpha_m} 2^{R_1^{\alpha_m}} \| fw(-h) \|^2_{\omega_{R_1}}$$

$\forall R, R_1, h, m$ con $m \in N$, $h + \tau_0 \leqslant T$, $R \geqslant R_1 \geqslant R_2$, per un certo $R_2 > 0$.

Fissato $R_1 \geqslant R_2$ supponiamo che esista $R_3 \geqslant R_1$ tale che sia

$$\Lambda_{R_1}(R_3, m, h) > C_{10}\Lambda'(R_1, m, h) + C_{10} + R_1^{\alpha_m} 2^{R_1^{\alpha_m}} \| fw(-h) \|^2_{\omega_{R_1}};$$

allora essendo

$$\Lambda_{R_1}(R_3, m, h) \leqslant \Lambda_{R_1}(R, m, h) \text{ per } R \geqslant R_3 \text{ perchè } \Lambda'(R, m, h) \geqslant 0,$$

si ha

$$0 < \beta = 1 - \frac{C_{10}\Lambda'(R_1, m, h) + C_{10} + R_1^{\alpha_m} 2^{R_1^{\alpha_m}} \| fw(-h) \|^2_{\omega_{R_1}}}{\Lambda_{R_1}(R_3, m, h)} \leqslant$$
$$\leqslant 1 - \frac{C_{10}\Lambda'(R_1, m, h) + C_{10} + R_1^{\alpha_m} 2^{R_1^{\alpha_m}} \| fw(-h) \|^2_{\omega_{R_1}}}{\Lambda_{R_1}(R, m, h)} \leqslant$$
$$\leqslant C_{10} \frac{\Lambda'(R, m, h)}{\Lambda_{R_1}(R, m, h)} \qquad \forall R \geqslant R_3,$$

da cui, integrando sull'intervallo $[R_3, R]$ $(R > R_3)$

$$\beta(R - R_3) \leqslant C_{10} \log \left(\frac{\Lambda_{R_1}(R_3, m, h)}{\Lambda_{R_1}(R, m, h)} \right) \qquad \forall R \geqslant R_3. \tag{21}$$

Inoltre dalla (15), per le (9), (10) e la (ii), segue che esiste R_4 tale che

$$\int_{R_1}^{R} d\rho \int_{R_1}^{\rho} \Lambda_{R_1}(r, m, h) dr \leqslant R^{52} \qquad \forall R \geqslant R_4;$$

da ciò, applicando il lemma 3 due volte, discende l'esistenza di una successione $(\alpha_n)_{n\in N}$ divergente positivamente tale che

$$\Lambda_{R_1}(\alpha_n\,,\ m,\ h)\leqslant\alpha_n^{52} \qquad \forall n\in N$$

quest'ultima contraddice la (21); da ciò segue l'affermazione (16).

III. Posto

$$\Gamma(R_1\,,\ m,\ h)=\lim_{R\to+\infty}\Lambda_{R_1}(R,\ m,\ h),\ \Gamma'(R,\ m,\ h)=\frac{d\Gamma(R,\ m,\ h)}{dR} \tag{22}$$

proviamo che esiste R_5 tale che

$$\frac{1}{2}\Gamma(R_1\,,\ m,\ h)\leqslant -C_{10}\,\Gamma'(R_1\,,\ m,\ h)+C_{10}+R_1^{\alpha_m}2^{R_1^{\alpha_m}}\|\,fw(-h)\,\|^2_{\omega_{R_1}} \tag{23}$$

$$\forall R_1\,,\ h,\ m \text{ con } R_1\geqslant R_5\,,\ m\in N,\ h+\tau_0\leqslant T.$$

Infatti, posto

$$A=\int_0^{\tau_0}d\tau\int_{-\tau}^{0}\alpha_m R_1^{\alpha_m-1}\log\left(\frac{2\tau}{\tau-t}\right)\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}(\|\,fw(t-h)\,\|^2_{\omega_{R_1}}+$$
$$+\tau^2\,|||\,f,\ w(t-h)\,|||^2_{\omega_{R_1}})dt,$$

$$B=\int_0^{\tau_0}d\tau\int_{-\tau}^{0}\left[\frac{1}{2}\alpha_m R_1^{2\alpha_m-1}\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\frac{\tau^2}{(\tau-t)^2}\,\|\,fw(t-h)\,\|^2_{\omega_{R_1}}+\right.$$
$$+\frac{\alpha_m}{4}R_1^{3\alpha_m-1}\log\left(\frac{2\tau}{\tau-t}\right)\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\frac{\tau^2}{(\tau-t)^2}\,\|\,fw(t-h)\,\|^2_{\omega_{R_1}}+$$
$$+\alpha_m R_1^{\alpha_m-1}\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\frac{\tau^2}{\tau-t}\left(fw(t-h),\ f\frac{\partial}{\partial t}(w(t-h))\right)_{\omega_{R_1}}+$$
$$+\alpha_m\tau^2R_1^{\alpha_m-1}\log\left(\frac{2\tau}{\tau-t}\right)\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\left\|f\frac{\partial}{\partial t}(w(t-h))\right\|_{\omega_{R_1}}+$$
$$\left.+\alpha_m R_1^{2\alpha_m-1}\log\left(\frac{2\tau}{\tau-t}\right)\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\frac{\tau^2}{\tau-t}\left(fw(t-h),\ f\frac{\partial}{\partial t}(w(t-h))\right)_{\omega_{R_1}}\right]dt,$$

tenendo presente che

$$(24)\qquad \tau^2(2\tau)^{R_1^{\alpha_m}}\left[\frac{\partial}{\partial t}(\lambda^{-R_1^{\alpha_m}/2}w(t-h))\right]=\frac{1}{4}R_1^{2\alpha_m}\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\frac{\tau^2}{(\tau-t)^2}w^2(t-h)+$$
$$+\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\left(\frac{\partial}{\partial t}(w(t-h))\right)^2\tau^2+\frac{1}{2}R_1^{\alpha_m}\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\frac{\tau^2}{\tau-t}\frac{\partial}{\partial t}(w^2(t-\tau)),$$

si ha

$$(25)\qquad -\Gamma'(R_1,\ m,\ h)=\Lambda'(R_1,\ m,\ h)-A-\delta B.$$

Dalla (24) si deduce

$$\tau^2\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\left(\frac{\partial}{\partial t}(w(t-h))\right)^2\leqslant$$
$$\leqslant\frac{1}{2}R_1^{2\alpha_m}\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\frac{\tau^2}{\tau-t}(w(t-h))^2+2\tau^2(2\tau)^{R_1^{\alpha_m}}\left(\frac{\partial}{\partial t}(\lambda^{-R_1^{\alpha_m}/2}w(t-h))\right)$$

e quindi

$$\alpha_m R_1^{\alpha_m-1}\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\frac{\tau^2}{\tau-t}\left(fw(t-h),\ \frac{\partial}{\partial t}(w(t-h))\right)^2_{\omega_{R_1}}\leqslant$$
$$\leqslant\alpha_m R_1^{\alpha_m-1}\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\|fw(t-h)\|^2_{\omega_{R_1}}+$$
$$+\alpha_m R_1^{\alpha_m-1}\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\tau^2\left\|f\frac{\partial}{\partial t}(w(t-h))\right\|^2_{\omega_{R_1}}\leqslant$$
$$\leqslant\alpha_m R_1^{\alpha_m-1}\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\|fw(t-h)\|^2_{\omega_{R_1}}+$$
$$+\frac{1}{2}\alpha_m R_1^{3\alpha_m-1}\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\|fw(t-h)\|^2_{\omega_{R_1}}+$$
$$+2\alpha_m R_1^{\alpha_m-1}(2\tau)^{R_1^{\alpha_m}}\tau^2\left\|f\frac{\partial}{\partial t}(\lambda^{-R_1^{\alpha_m}/2}w(t-h))\right\|^2_{\omega_{R_1}},$$

$$R_1^{2\alpha_m-1}\alpha_m \log\left(\frac{2\tau}{\tau-t}\right)\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\frac{\tau^2}{\tau-t}\left(fw(t-h),\ f\frac{\partial}{\partial t}(w(t-h))\right)_{\omega_{R_1}}\leqslant$$

$$\leqslant 2\alpha_m R_1^{2\alpha_m-1}\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\|fw(t-h)\|^2_{\omega_{R_1}}+$$

$$+\alpha_m R_1^{4\alpha_m-1}\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\|fw(t-h)\|^2_{\omega_{R_1}}+$$

$$+4\alpha_m R_1^{2\alpha_m-1}(2\tau)^{R_1^{\alpha_m}}\tau^2\left\|f\frac{\partial}{\partial t}(\lambda^{-R_1^{\alpha_m}/2}w(t-h))\right\|^2_{\omega_{R_1}};$$

perciò si ha

$$\delta C_{10}B\leqslant C_{11}R_1^{-\alpha_m}\int_0^{\tau_0}d\tau\int_{-\tau}^{0}\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}\|fw(t-h)\|^2_{\omega_{R_1}}dt+ \tag{26}$$

$$+C_{11}R_1^{-\alpha_m}\int_0^{\tau_0}d\tau\int_{-\tau}^{0}\tau^2(2\tau)^{R_1^{\alpha_m}}\left\|f\frac{\partial}{\partial t}(\lambda^{-R_1^{\alpha_m}/2}w(t-h))\right\|^2_{\omega_{R_1}}dt$$

ed inoltre

$$A\leqslant 2\alpha_m R_1^{-\alpha_m}\int_0^{\tau_0}d\tau\int_{-\tau}^{0}\left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_m}}(\|fw(t-h)\|^2_{\omega_{R_1}}+ \tag{27}$$

$$+\tau^2|||f,\ w(t-h)|||^2_{\omega_{R_1}})dt.$$

Dalla (25), per le (26) e (27), si deduce

$$-C_{10}\Gamma'(R_1,\ m,\ h)\geqslant C_{10}\Lambda'(R_1,\ m,\ h)-\frac{1}{2}\Gamma(R_1,\ m,\ h)\qquad \forall R_1\geqslant R_5$$

per un certo $R_5>0$, e combinando quest'ultima con la (16) si ottiene la (23).

IV. Dimostriamo ora il lemma. Dalla (23) preso $m=1$ e $h=0$ si ottiene

$$\frac{1}{2}\Gamma(R_1,\ 1,\ 0)\leqslant -C_{10}\Gamma'(R_1,\ 1,\ 0)+C_{10}\qquad \forall R_1\geqslant R_5$$

da cui tenendo presente che $\Gamma(R_1, m, h) \geqslant 0$ si ha

$$\Gamma'(R_1, 1, 0) \leqslant 1 \qquad \forall R_1 \geqslant R_5$$

e quindi

$$\Gamma(R_1, 1, 0) \leqslant C_{12} R_1 \qquad \forall R_1 \geqslant R_5 . \tag{28}$$

Dalle (28), (22), (15) segue che

$$\int_0^{\tau_0/2} d\tau \int_{-\tau}^{0} dt \int_{\omega_{R_1}} f^2 w^2(t) dx \leqslant C_{12} R_1 \left(\frac{4}{3}\right)^{-R_1^{\alpha_1}} \qquad \forall R \geqslant R_1 . \tag{29}$$

perchè

$$C_{12} R_1 \geqslant \int_0^{\tau_0} d\tau \int_{-\tau}^{0} \left(\frac{2\tau}{\tau - t}\right)^{R_1^{\alpha_1}} dt \int_{\omega_{R_1}} f^2 w^2(t) dx \geqslant$$

$$\geqslant \int_0^{\tau_0} d\tau \int_{-\tau/2}^{0} \left(\frac{2\tau}{\tau - t}\right)^{R_1^{\alpha_1}} dt \int_{\omega_{R_1}} f^2 w^2(t) dx \geqslant$$

$$\geqslant \int_0^{\tau_0} d\tau \int_{-\tau/2}^{0} \left(\frac{4}{3}\right)^{R_1^{\alpha_1}} dt \int_{\omega_{R_1}} f^2 w^2(t) dx.$$

Per la (29) esistono $R_6 \geqslant R_5$ e $\theta_1 > 0$ tali che

$$\int_0^{\tau_0/2} d\tau \int_{-\tau}^{0} dt \int_{\omega_{R_6+K}(R_6+K+1)} w^2(t) dx \leqslant e^{-\theta_1 (R_6+K)^{\alpha_1}} \qquad \forall K \in N \tag{30}$$

infatti

$$\int_0^{\tau_0/2} d\tau \int_{-\tau}^{0} dt \int_{\omega_{R_6+K}(R_6+K+1)} f^2 w^2(t) dx \geqslant$$

$$\geqslant \exp\left(-2a(R_6+K+1)^{1/20}\right) \int_0^{\tau_0/2} d\tau \int_{-\tau}^{0} dt \int_{\omega_{R_6+K}(R_6+K+1)} w^2(t) dx.$$

Proviamo ora che

$$\int_{R_6}^{+\infty} R_1^{\alpha_2} 2^{R_1^{\alpha_2}} dR_1 \int_0^{\tau_0/2} d\tau \int_{-\tau}^{0} dt \int_{\omega_{R_1}} w^2(t)dx < +\infty. \tag{31}$$

Infatti per la (30) si ha

$$\int_0^{\tau_0/2} d\tau \int_{-\tau}^{0} dt \int_{\omega_{R_6}} |x|^{2+\alpha_2} 2^{|x|^{\alpha_2}} w^2(t)dx \leqslant$$

$$\leqslant \sum_{K=0}^{\infty} (R_6+K+1)^{2\alpha_2} 2^{(R_1+K+1)^{\alpha_2}} \int_0^{\tau_0/2} d\tau \int_{-\tau}^{0} dt \int_{\omega_{R_6+K}(R_6+K+1)} w^2(t)dx \leqslant$$

$$\leqslant \sum_{K=0}^{\infty} (R_6+K+1)^{2\alpha_2} 2^{(R_6+K+1)^{\alpha_2}} e^{-\theta_1(R_6+K)^{\alpha_1}} = C_{14}$$

e perciò

$$R_1^{2+\alpha_2} 2^{R_1^{\alpha_2}} \int_0^{\tau_0/2} d\tau \int_{-\tau}^{0} dt \int_{\omega_{R_1}} w^2(t)dx \leqslant \tag{32}$$

$$\leqslant \int_0^{\tau_0/2} d\tau \int_{-\tau}^{0} dt \int_{\omega_{R_1}} |x|^{2+\alpha} 2^{|x|^{\alpha_2}} w^2(t)dx \leqslant C_{14}$$

$\forall R_1 \geqslant R_6$. Dalla (32) segue subito la (31).

Dalla (31) per il teorema di Fubini discende l'esistenza di $h_1 > 0$ tale che $(\tau_0/2)\frac{5}{6} \leqslant h_1 \leqslant \tau_0/2$ e

$$\int_{R_6}^{+\infty} R_1^{\alpha_2} 2^{R_1^{\alpha_2}} \| fw(-h_1) \|^2_{\omega_{R_1}} dR_1 < +\infty. \tag{33}$$

Inoltre dalla (32) si ottiene

$$\int_{h_0}^{h_1} d\tau \int_{-\tau}^{-h_0} dt \int_{\omega_{R_0}} w^2 dx < +\infty.$$

Dalla (23) preso $m=2$, $h=h_1$ si ottiene

$$C_{10}\Gamma'(R_1\,,\ 2,\ h_1)\leqslant C_{10}+R_1^{\alpha_2}2^{R_1^{\alpha_2}}\|\,fw(-h_1)\,\|^2_{\omega_{R_1}} \qquad \forall R\geqslant R_6$$

da cui, per la (33), si ottiene

$$\Gamma(R_1\,,\ 2,\ h_1)\leqslant C_{15}R_1 \qquad \forall R_1\geqslant R_6\,. \tag{34}$$

Dalle (34), (22), (15) segue che

$$\int_0^{\tau_0/2} d\tau \int_{-\tau}^{0} dt \int_{\omega_{R_1}} f^2w^2(t-h_1)dx\leqslant C_{15}R_1\left(\frac{4}{3}\right)^{-R_1^{\alpha_2}}$$

perchè

$$C_{15}R_1\geqslant \int_0^{\tau_0} d\tau \int_{-\tau}^{0} \left(\frac{2\tau}{\tau-t}\right)^{R_1^{\alpha_2}} dt \int_{\omega_{R_1}} f^2w^2(t-h_1)dx\geqslant$$

$$\geqslant \int_0^{\tau_0} d\tau \int_{-\tau/2}^{0} \left(\frac{4}{3}\right)^{R_1^{\alpha_2}} dt \int_{\omega_{R_1}} f^2w^2(t-h_1)dx;$$

perciò esistono $R_7\geqslant R_6$ e $\theta_2>0$ tali che

$$\int_0^{\tau_0/2} d\tau \int_{-\tau}^{0} dt \int_{\omega_{R_6+K}(R_6+K+1)} w^2(t-h_1)dx\leqslant e^{-\theta_2(R_6+K)^{\alpha_2}} \qquad \forall K\in N;$$

e quindi come prima

$$\int_{R_7}^{+\infty} R_1^{\alpha_3}2^{R_1^{\alpha_3}}dR_1 \int_0^{\tau_0/2} d\tau \int_{-\tau}^{0} dt \int_{\omega_{R_1}} w^2(t-h_1)dx<+\infty;$$

di conseguenza per il teorema di Fubini esiste $h_2>h_1$ tale che h_1+ $+\dfrac{\tau_0}{2}\dfrac{5}{6}\leqslant h_2\leqslant h_1+\dfrac{\tau_0}{2}$ e

$$\int_{R_7}^{+\infty} R_1^{\alpha_3}2^{R_1^{\alpha_3}}\,\|\,fw(-h_2)\,\|^2_{\omega_{R_1}}dR_1<+\infty. \tag{35}$$

Inoltre si ha

$$\int_{h_1}^{h_2} d\tau \int_{-\tau}^{-h_1} dt \int_{\omega_{R_0}} w^2 dx < +\infty .$$

Dalla (23) preso $m=3$ e $h=h_2$ si ottiene

$$C_{10}\Gamma'(R_1\,,\ 3,\ h_2) \leqslant C_{10} + R_1^{\alpha_3} 2^{R_1^{\alpha_3}} \| fw(-h_2) \|^2_{\omega_{R_1}} \qquad \forall R_1 \geqslant R_7$$

da cui, per la (35), si ottiene

$$\Gamma(R_1\,,\ 3,\ h_2) \leqslant C_{16} R_1 \qquad \forall R_1 \geqslant R_7 .$$

Ripetendo questo ragionamento un numero finito di volte si dimostra il lemma.

LEMMA 5. *Siano u, q e $u+v$, $q+p$ due soluzioni di* (α), (β) *in Q, relative alla stessa F e con*

(i) $u_i\,,\ v_i \in L_3\,,\ F_i \in L_1$ *per* $1 \leqslant i \leqslant 3,\ q,\ p \in L_2$

(ii) $u_i\,,\ \dfrac{\partial u_i}{\partial x_j}\,,\ v_i\,,\ \dfrac{\partial v_i}{\partial x_j}$ *limitate in Q per* $1 \leqslant i,\ j \leqslant 3$ *e*

$$|\, p(x,\ t)\,| \leqslant C/(1+|\, x\,|)^{1/2} \qquad \forall (x,\ t) \in Q$$

(iii) $u_i\,,\ v_i$ *nulle su* $\Omega \times \{0\}$ *per* $1 \leqslant i \leqslant 3$ *e* v_i *nulle su* $\partial\Omega \times \times [-T,\ 0]$ *per* $1 \leqslant i \leqslant 3$.

Allora esiste $\tau_2 \in]0,\ T]$ *tale che* $\left(\dfrac{\partial v_i}{\partial x_j}\right)^2$ *è sommabile su* Ω *per quasi ogni* $t \in [-\tau_2\,,\ 0]$ *e per* $1 \leqslant i,\ j \leqslant 3$.

DIMOSTRAZIONE. Per la (α) si ha

$$\frac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla) u = -\nabla q + F$$

$$\frac{\partial (u+v)}{\partial t} - \Delta (u+v) + ((u+v) \cdot \nabla)(u+v) = -\nabla (q+p) + F$$

da cui

$$\frac{\partial v}{\partial t}-\Delta v+(v\cdot\nabla)u+((u+v)\cdot\nabla)v=-\nabla p. \tag{36}$$

Moltiplicando scalarmente la (36) per v e tenendo presente la (t) e che

$$((v\cdot\nabla)u)\cdot v+(((u+v)\cdot\nabla)v)\cdot v\leqslant C_\alpha v^2+\frac{1}{2}\nabla\cdot(v^2(v+u)),$$

si ottiene

$$-\frac{1}{2}\frac{\partial(v^2)}{\partial t}\leqslant C_\alpha v^2+(\nabla\times v^2)+\nabla\cdot\left(v\times(\nabla\times v)+\frac{1}{2}v^2(v+u)+pv\right)$$

da cui segue $(0<\tau_1\leqslant T)$

$$\begin{aligned}\frac{1}{2}\int_{-\tau_1}^{0}dt\int_{\omega(R)}v^2dx\leqslant C_\beta\int_0^{\tau_1}d\tau\int_{-\tau}^{0}dt\int_{\sigma(R)}v^2d\sigma+\\ +\int_0^{\tau_1}d\tau\int_{-\tau}^{0}dt\int_{\omega(R)}(\nabla\times v)^2dx+\int_0^{\tau_1}d\tau\int_{-\tau}^{0}dt\int_{\sigma(R)}(\nabla\times v)^2d\sigma+\\ +\int_0^{\tau_1}d\tau\int_{-\tau}^{0}dt\int_{\sigma(R)}|pv|\,d\sigma+\tau_1C_\alpha\int_{-\tau_1}^{0}dt\int_{\omega(R)}v^2dx.\end{aligned} \tag{37}$$

Poichè $|\nabla\times v|\leqslant|\nabla\times u|+|\nabla\times(u+v)|$ e poichè per il lemma 4 esiste C_γ tale che

$$\left.\begin{aligned}&\int_0^{h_1}d\tau\int_{-\tau}^{0}dt\int_{\omega(R)}(\nabla\times u)^2dx\leqslant\frac{1}{2}C_\gamma,\\ &\int_{-\tau}^{h_1}d\tau\int_0^{0}dt\int_{\omega(R)}(\nabla\times(u+v))^2dx\leqslant\frac{1}{2}C_\gamma\end{aligned}\right\}\quad\forall R\geqslant R_0,$$

si ha

$$\int_0^{\tau_1} d\tau \int_{-\tau}^{0} dt \int_{\omega(R)} (\nabla \times v)^2 dx \leqslant C_\gamma \qquad \forall R \geqslant R_0\,,\ \forall \tau_1 \leqslant h_1\,; \tag{38}$$

Posto

$$M(R) = \int_{-\tau_1}^{0} d\tau \int_{\omega(R)} v^2 dx \tag{39}$$

dalla (37) per la (38) e la (ii) si ottiene, per $C_\alpha \tau_1 \leqslant \frac{1}{4}$ e $\tau_1 \leqslant h_1$,

$$\begin{aligned} M(R) &\leqslant C_\delta M'(R) + C_\delta (RM'(R))^{1/2} + C_\delta \int_0^{\tau_1} d\tau \int_{-\tau_1}^{0} dt \int_{\sigma(R)} (\nabla \times v)^2 dx + C_\delta \leqslant \\ &\leqslant \frac{1}{20} RM(R') + C_\delta \int_0^{\tau_1} d\tau \int_{-\tau}^{0} dt \int_{\sigma(R)} (\nabla \times v)^2 d\sigma + C_\lambda \qquad \forall R \geqslant R_8\,. \end{aligned} \tag{40}$$

Proviamo che

$$M(R) \leqslant 2C_\lambda \qquad \forall R \geqslant R_8\,. \tag{41}$$

Infatti se esiste $R_9 > R_8$ tale che $M(R_2) > 2C_\lambda$, tenendo presente che $M'(R) \geqslant 0$, si ha

$$M(R) \leqslant \frac{1}{10} RM'(R) + 2C_\delta \int_0^{\tau_1} d\tau \int_{-\tau}^{0} dt \int_{\sigma(R)} (\nabla \times v)^2 dx \qquad \forall R \geqslant R_9$$

e quindi anche

$$\frac{1}{R} \leqslant \frac{1}{10} \frac{M'(R)}{M(R)} + C_\mu \int_0^{\tau_1} d\tau \int_{-\tau}^{0} dt \int_{\sigma(R)} (\nabla \times v)^2 dx,$$

da cui, integrando sull'intervallo $[R_9\,, R]$ $(R > R_9)$ e tenendo presente la (38) si ottiene

$$10 \log \left(\frac{R}{R_9} \right) \leqslant \log \left(\frac{M(R)}{M(R_9)} \right) + C_\nu \qquad \forall R \geqslant R_9\,,$$

da cui

$$R^{10}[e^{-C_\nu}R_2^{-10}M(R_2)]\leqslant M(R)\leqslant C_\xi R^3 \qquad \forall R\geqslant R_9\,,$$

che è assurdo.

Moltiplicando scalarmente la (36) per v, tenendo presente la (s) e che

$$-((v\cdot\nabla)u)\cdot v-((u+v)\cdot\nabla)v)\cdot v\leqslant C_\alpha v^2-\frac{1}{2}\nabla\cdot(v^2(v+u))$$

si ottiene

$$\frac{1}{2}\frac{\partial(v^2)}{\partial t}+\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)^2\leqslant\nabla\cdot\left(\frac{1}{2}\nabla(v^2)-\frac{1}{2}v^2(v+u)-pv\right)+C_\alpha v^2 \tag{42}$$

da cui, tenendo presente la (ii), si ottiene

$$\begin{aligned}\frac{1}{2}\int_{-\tau_2}^{0}dt\int_{\omega(R)}v^2dx+\int_{-\tau_2}^{0}dt\int_{-\tau_2}^{t}d\xi\int_{\omega(R)}\sum_{i,j=1}^{3}\left(\frac{\partial v_i}{\partial x_j}\right)^2dx\leqslant C_\zeta\int_{-\tau_2}^{0}dt\int_{\sigma(R)}v^2d\sigma+\\ +C_\zeta\int_{-\tau_2}^{0}dt\int_{-\tau_2}^{t}d\xi\int_{\sigma(R)}\sum_{i,j=1}^{3}\left(\frac{\partial v_i}{\partial x_j}\right)^2d\sigma+C_\zeta R^{1/2}\left(\int_{-\tau_2}^{0}dt\int_{\sigma(R)}v^2d\sigma\right)^{1/2}+\\ +\frac{\tau_2}{2}\int_{\omega(R)}v^2(-\tau_2)dx+C_\alpha\tau_2\int_{-\tau_2}^{0}dt\int_{\omega(R)}v^2dx.\end{aligned}$$

Dalla (41) e dal teorema di Fubini segue che $v^2(\cdot,\,t)$ è sommabile su Ω per quasi $t\in[-\tau_1\,,\,0]$ scegliamo $\tau_2\in[0,\,\tau_1]$ vale che $v^2(\cdot,\,-\tau_2)$ sia sommabile su Ω e risulti $\tau_2C_\alpha<1/4$.

Allora posto

$$N(R)=\frac{1}{4}\int_{-\tau_2}^{0}dt\int_{\omega(R)}v^2dx+\int_{-\tau_2}^{0}dt\int_{-\tau_2}^{t}d\xi\int_{\omega(R)}\sum_{i,j=1}^{3}\left(\frac{\partial v_i}{\partial x_j}\right)^2dx, \tag{43}$$

si ha

$$\begin{aligned}N(R)&\leqslant 4C_\zeta N'(R)+2C_\zeta(RN'(R))^{1/2}+C_\pi\leqslant\\ &\leqslant\frac{1}{20}RN'(R)+C_\chi \qquad \forall R\geqslant R_{10}\,;\end{aligned}$$

di qui ragionando come prima risulta

$$N(R) \leqslant 2C_\chi \qquad \forall R \geqslant R_{10}.$$

Da ciò per il teorema di Fubini segue il lemma.

3. TEOREMA 1. *Siano u, q e $u+v$, $q+p$ due soluzioni di* $(\alpha)-(\beta)$ *in Q, relative alla stessa F e con*

(i) u_i, $v_i \in L_3$, $F_i \in L_1$ *per* $1 \leqslant i \leqslant 3$, p, $q \in L_2$

(ii) u_i, $\dfrac{\partial u_i}{\partial x_j}$, v_i, $\dfrac{\partial v_i}{\partial x_j}$ *limitate in Q per* $1 \leqslant i, j \leqslant 3$ *e*

$$|p(x, t)| \leqslant C/(1+|x|)^{1/2} \qquad \forall (x, t) \in Q$$

(iii) u_i, v_i *nulle su* $\Omega \times \{0\}$ *per* $1 \leqslant i \leqslant 3$ *e* v_i *nulle su* $\partial\Omega \times \times [-T, 0]$ *per* $1 \leqslant i \leqslant 3$.

Allora $p=0$, $v=0$.

DIMOSTRAZIONE. Cominciamo col provare che esiste una successione $(\alpha_n)_{N \in n}$ divergente positivamente tale che

$$\int_{-T}^{0} dt \int_{\omega_{R_0}(\alpha_n)} \left(\frac{\partial v}{\partial t}\right)^2 dx \leqslant \alpha_n^{15} \qquad \forall n \in N. \tag{44}$$

Allo scopo, moltiplicando scalarmente la (36) per $\dfrac{\partial v}{\partial t}$; si ottiene

$$\left(\frac{\partial v}{\partial t}\right)^2 - \Delta v \cdot \frac{\partial v}{\partial t} + \frac{\partial v}{\partial t} \cdot (v \cdot \nabla) u + \frac{\partial v}{\partial t} \cdot ((u+v) \cdot \nabla) v = -\frac{\partial v}{\partial t} \cdot \nabla p, \tag{45}$$

da cui, tenendo presente la (v) e che

$$\left| \frac{\partial v}{\partial t} \cdot [(v \cdot \nabla) u + ((u+v) \cdot \nabla) v] \right| \leqslant \frac{1}{4} \left(\frac{\partial v}{\partial t}\right)^2 + C'_1 \tag{46}$$

si ha

$$\frac{1}{2} \left(\frac{\partial v}{\partial t}\right)^2 \leqslant C'_1 + \nabla \cdot \left(\frac{\partial v}{\partial t} \times (\nabla \times v) - p \frac{\partial v}{\partial t} \right) - \frac{1}{2} \frac{\partial}{\partial t} ((\nabla \times v)^2),$$

e quindi

$$\int_{-T}^{0} dt \int_{R_0}^{R} dr \int_{\omega_{R_0}(r)} \left(\frac{\partial v}{\partial t}\right)^2 dx \leqslant C'_2 R^4 + C'_2 R^2 \left(\int_{-T}^{0} dt \int_{\omega_{R_0}(r)} \left(\frac{\partial v}{\partial t}\right)^2 dx \right)^{1/2} \quad \forall R \geqslant R_0 .$$

Perciò, posto

$$P(R) = \int_{-T}^{0} dt \int_{R_0}^{R} dr \int_{\omega_{R_0}(r)} \left(\frac{\partial v}{\partial t}\right)^2 dx$$

si ottiene

$$P(R) \leqslant C'_2 R^4 + C'_2 R^5 (P'(R))^{1/2} \qquad \forall R \geqslant R_0 ;$$

di qui, per i lemmi 1 e 3, segue la (44).

Dalla (36) e dalla (ii) segue

$$(47) \qquad (Lv - \nabla p)^2 = ((v \cdot \nabla) u + ((u+v) \cdot \nabla) v)^2 \leqslant C'_3 \left(v^2 + \sum_{i,j=1}^{3} \left(\frac{\partial v_i}{\partial x_j}\right)^2 \right).$$

Dall'osservazione al lemma 2 e tenendo presente il lemma 5, presi $h=0$, $w=v$ e R', γ', τ_3, $\delta'>0$ tali che

$$||| v(-\tau_3) |||^2_\Omega < +\infty , \quad || v(-\tau_3) ||^2_{\omega(R')} > 0, \quad \tau_3 \leqslant \tau_2 , \quad 2C_{\tau_3, \gamma', \delta'} C'_3 \leqslant \frac{1}{2}$$

$$\tau_3 \gamma' (2\tau_3)^{-2\gamma'-1} || v(-\tau_3) ||^2_{\omega(R')} \geqslant C_{\tau_3, \gamma', \delta'} (2\tau_3)^{-2\gamma'} ||| v(-\tau_3) |||^2_\Omega$$

e posto

$$Q(R) = \int_{-\tau_3}^{0} \lambda^{-2\gamma'} || v ||^2_{\omega(R)} dt + \int_{-\tau_3}^{0} \lambda^{-2\gamma'} ||| v |||^2_{\omega(R)} dt +$$

$$+ C_{\tau_3, \gamma', \delta'} \int_{-\tau_3}^{0} \left\| \frac{\partial}{\partial t} ((\lambda^{-\gamma'} v) \right\|^2_{\omega(R)} dt,$$

tenendo presente la (47), si ottiene

$$(48) \qquad Q(R) \leqslant C'_4 Q'(R) + C'_4 (R Q'(R))^{1/2} \qquad \forall R \geqslant R' \geqslant R_0 .$$

Proviamo che $Q(R)=0$ e quindi $v=0$.

Infatti, ragionando per assurdo, supponiamo che esista $R_{11}>R'$ tale che sia $Q(R_{11})>0$.

Dalla (48) si ottiene

$$Q(R_{11})\leqslant C'_4Q'(R)+C'_4(RQ'(R))^{1/2} \qquad \forall R\geqslant R_{11}$$

e quindi

$$(Q'(R))^{1/2}\geqslant \frac{2Q(R_{11})}{C'_4(R^{1/2}+(R+4Q(R_{11})/C'_4)^{1/2})}\geqslant \frac{2Q(R_{11})}{C'_5R^{1/2}}$$

$\forall R\geqslant R_{12}\geqslant R_{11}$ con R_{12} tale che $\dfrac{R}{Q(R)}\geqslant \dfrac{4}{C'_4}$ per $R\geqslant R_{12}$. Introducendo questa valutazione nella (48) si ha

$$Q(R)\leqslant C'_6RQ'(R) \qquad \forall R\geqslant R_{13}\geqslant R_{12},$$

da cui,

$$((R^{1/C'_6}Q(R_{13}))/R_{13}^{1/C'_6})\leqslant Q(R) \qquad \forall R\geqslant R_{13}.$$

Da questa segue che esiste $R_{14}\geqslant R_{13}$ tale che

$$(Q'(R))^{1/2}\geqslant \frac{2Q(R_{14})}{C'_4(R^{1/2}+(R+4Q(R_{14})/C'_4)^{1/2})}\geqslant \frac{20C'_4}{R^{1/2}} \qquad \forall R\geqslant R_{15}\geqslant R_{14}.$$

Introducendo questa valutazione nella (48) si ottiene

$$Q(R)\leqslant C'_4Q'(R)+\frac{1}{20}RQ'(R)\leqslant \frac{R}{19}Q'(R)$$

$\forall R\geqslant R_{16}\geqslant R_{15}$, da cui

$$C'_7R^{19}\leqslant Q(R) \qquad \forall R\geqslant R_{16}. \tag{49}$$

Dalla (ii) e dalla (44) segue che

$$\int_{-\tau_3}^{0}\lambda^{-2\gamma'}\,\|v\|^2_{\omega(\alpha_n)}dt+\int_{-\tau_3}^{0}\lambda^{-2\gamma'}\,|||v|||^2_{\omega(\alpha_n)}dt\leqslant C'_8\alpha_n^3 \qquad \forall n\in N$$

$$\int_{-\tau_3}^{0} \left\| \frac{\partial}{\partial t}(\lambda^{-\gamma} v) \right\|^2_{\omega(\alpha_n)} dt = \int_{-\tau_3}^{0} \left\| \gamma' \lambda^{-\gamma'-1} v + \lambda^{-\gamma'} \frac{\partial v}{\partial t} \right\|^2_{\omega(\alpha_n)} dt \leqslant C'_8 \alpha_n^{15} \qquad \forall n \in N$$

e quindi

$$Q(\alpha_n) \leqslant C'_9 \alpha_n^{15} \qquad \forall n \in N$$

che contraddice la (49).

Dunque $Q(R)=0$ e quindi $v=0$, $p=0$ *su* $\Omega \times [-\tau_3, 0]$.

Dopo di ciò riapplicando il lemma 5 e il ragionamento ora fatto si ottiene $v=0$, $p=0$ su $\Omega \times [-\tau_4, -\tau_3]$ con un τ_4 che si può ritenere prossimo tanto quanto si vuole a $2\tau_3$. Ecc.

Ciò conclude la dimostrazione.

TEOREMA 2. *Siano u, q e $u+v$, $q+p$ due soluzioni di* $(\alpha)-(\beta)$ *in Q relative alla stessa F e con*

(i) u_i, $v_i \in L_3$, $F_i \in L_1$ *per* $1 \leqslant i \leqslant 3$, p, $q \in L_2$

(ii) u_i, $\dfrac{\partial u_i}{\partial x_j}$, v_i, $\dfrac{\partial v_i}{\partial x_j}$ *limitate in Q per* $1 \leqslant i, j \leqslant 3$ *e*

$$|p(x, t)| \leqslant C/(1+|x|)^{1/2} \qquad \forall (x, t) \in Q$$

(iii) v_i *nulle su* $\Omega \times \{0\}$ *e su* $\partial\Omega \times [-T, 0]$ per $1 \leqslant i \leqslant 3$

(iv) $\left(\dfrac{\partial v_i}{\partial x_j}\right)^2(-T)$ *e* $\left(\dfrac{\partial u_j}{\partial x_j}\right)^2(-T)$ *sommabile su* Ω *per* $1 \leqslant i, j \leqslant 3$.

Allora $v=0$, $p=0$.

DIMOSTRAZIONE. Cominciamo col provare che

$$\int_{-T}^{0} dt \int_{\Omega} (\nabla \times v)^2 dx < +\infty. \tag{50}$$

Per $w = \nabla \times u$ oppure $w = \nabla \times (u+v)$ ricordiamo che

$$\frac{1}{2} \frac{\partial (w^2)}{\partial t} + \sum_{i,j=1}^{3} \left(\frac{\partial w_i}{\partial x_j}\right)^2 \leqslant \frac{1}{2} \nabla \cdot (\nabla (w^2) - w^2 u) + C_1 w^2. \tag{13}$$

Da essa si deduce per $0<h\leqslant T$

$$\frac{1}{2}\int_{-T}^{-T+h} dt \int_{\omega(R)} w^2 dx + \int_{-T}^{-T+h} dt \int_{-T}^{t} d\xi \int_{\omega(R)} \sum_{i,j=1}^{3} \left(\frac{\partial w_i}{\partial x_j}\right)^2 dx \leqslant$$

$$\leqslant C'_{10} \int_{-T}^{-T+h} dt \int_{-T}^{t} d\xi \int_{\sigma(R)} \sum_{i,j=1}^{3} \left(\frac{\partial w_i}{\partial x_j}\right)^2 dx + C'_{10} \int_{-T}^{-T+h} dt \int_{\sigma(R)} w^2 d\sigma +$$

$$+ hC_1 \int_{-T}^{-T+h} dt \int_{\omega(R)} w^2 dx + \frac{h}{2} \int_{\omega(R)} w^2(-T) dx;$$

ponendo

$$S(R) = \int_{-T}^{-T+h} dt \int_{\omega(R)} w^2 dx + \int_{-T}^{-T+h} dt \int_{-T}^{t} d\xi \int_{\omega(R)} \sum_{i,j=1}^{3} \left(\frac{\partial w_i}{\partial x_j}\right)^2 dx, \tag{51}$$

se si prende $hC_1 \leqslant \frac{1}{4}$ ricordando la (iv), si ha

$$S(R) \leqslant C'_{11} S'(R) \leqslant C'_{11} \qquad \forall R \geqslant R_0 . \tag{52}$$

Proviamo che dalla (52) segue che

$$S(R) \leqslant 2C'_{11} \qquad \forall R \geqslant R_0 ; \tag{53}$$

infatti in caso contrario, esisterebbe $R_{17} \geqslant R_0$ tale che $S(R_{17}) > 2C'_{11}$ e quindi dalla (52) seguirebbe

$$S(R) \leqslant 2C'_{11} S'(R) \qquad \forall R \geqslant R_{17} ,$$

da cui

$$e^{R/(2C'_{11})} (e^{-R_{17}/(2C'_{11})} S(R_{17})) \leqslant S(R) \qquad \forall R \geqslant R_{17} \tag{54}$$

Dalla (13), tenendo presente la (ii), si ottiene

$$\int_{-T}^{0} dt \int_{R_0}^{R} ||| w |||^2_{\omega_{R_0}(r)} dr \leqslant C'_{12} R^4 + C'_{12} R^5 \left(\int_{-T}^{0} ||| w |||^2_{\omega_{R_0}(R)} dt \right)^{1/2} \qquad \forall R \geqslant R_0$$

e quindi ricordando la (7) si ottiene

$$\Phi(R) \leqslant C'_{12} R^4 + C'_{12} R^5 (\Phi'(R))^{1/2} \qquad \forall R \geqslant R_0 ;$$

di qui, per il lemma 1, segue che

$$\Phi(R) \leqslant R^{15} \qquad \forall R \geqslant R_1 . \tag{55}$$

Per la (ii) e le (55), (51) si ha

$$\int_{R_0}^{R} S(r) dr \leqslant R^{16} \qquad \forall R \geqslant R_{18} ;$$

quindi, per il lemma 3, esiste una successione $(\alpha_n)_{n \in N}$ divergente positivamente tale che

$$S(\alpha_n) \leqslant \alpha_n^{16}$$

Ciò contraddice la (54).

Dalla (53) in particolare discende che

$$\int_{-T}^{-T+h} dt \int_{\omega(R)} w^2 dx \leqslant 2C'_{11} \qquad \forall R \geqslant R_0 ,$$

perciò il teorema di Fubini segue che w^2 è sommabile su Ω per quasi ogni $t \in [-T, -T+h]$; non è restrittivo supporre che tra questi valori ci sia anche $-T+h$. Ripetendo allora il ragionamento fatto si vede che

$$\int_{-T+h}^{-T+2h} dt \int_{\omega(R)} w^2 dx \leqslant 2C'_{11} \qquad \forall R \geqslant R_0$$

e che w^2 è sommabile per quasi ogni $t \in [-T+h, -T+2h]$. Così procedendo e tenendo che $|\nabla \times v| \leqslant |\nabla \times u| + |\nabla \times (u+v)|$ si prova la (50).

Proviamo ora che esiste h, $0<h\leqslant T$, tale che

$$\int_{-h}^{0} dt \int_{\Omega} v^2 dx < +\infty. \tag{56}$$

Allo scopo cominciamo con l'osservare che dalla (α) segue che

$$\frac{\partial u}{\partial t} - \Delta u + \frac{1}{2}\nabla(u^2) - u \times (\nabla \times u) = -\nabla q + F$$

$$\frac{\partial(u+v)}{\partial t} - \Delta(u+v) + \frac{1}{2}\nabla((u+v)^2) - (u+v) \times ((\nabla \times (u+v)) =$$
$$= -\nabla(q+p) + F$$

e quindi

$$\frac{\partial v}{\partial t} - \Delta v + \frac{1}{2}\nabla(v^2 + 2u\cdot v) - (u+v) \times (\nabla \times v) - v \times (\nabla \times u) = -\nabla p.$$

Moltiplicando scalarmente quest'ultima per $\frac{\partial v}{\partial t}$ tenendo presente la (v), e osservando che

$$\left| \frac{\partial v}{\partial t} \cdot [(u+v) \times (\nabla \times v) - v \times (\nabla \times u)] \right| \leqslant$$
$$\leqslant \frac{1}{2}\left(\frac{\partial v}{\partial t}\right)^2 + C'_{12}(\nabla \times v)^2 + C'_{12} v^2,$$

si ottiene

$$\frac{1}{2}\left(\frac{\partial v}{\partial t}\right)^2 + \frac{1}{2}\frac{\partial}{\partial t}(\nabla \times v)^2 \leqslant -\nabla \cdot \left(\left(p + \frac{v^2}{2} + u\cdot v\right)\frac{\partial v}{\partial t} + (\nabla \times v) \times \frac{\partial v}{\partial t}\right) +$$
$$+ C'_{12}(\nabla \times v)^2 + C'_{12} v^2.$$

Ne segue che

$$\frac{1}{2}\int_{-h}^{0} dt \int_{\omega(R)} \left(\frac{\partial v}{\partial t}\right) dx \leqslant \tag{57}$$

$$\leqslant \frac{1}{2}\int_{-h}^{0} dt \int_{\sigma(R)} (\nabla\times v)^2 d\sigma + \int_{-h}^{0} dt \int_{\sigma(R)} \left(\frac{\partial v}{\partial t}\right)^2 d\sigma + C'_{13}\int_{-h}^{0} dt \int_{\sigma(R)} v^2 d\sigma +$$

$$+C'_{13}R^{12}\left(\int_{-h}^{0} dt \int_{\sigma(R)} \left(\frac{\partial v}{\partial t}\right)^2 d\sigma\right)^{1/2} + \frac{1}{2}\int_{\omega(R)} (\nabla\times v)^2(-h)dx +$$

$$+C'_{12}\int_{-h}^{0} dt \int_{\omega(R)} (\nabla\times v)^2 dx + C'_{12}\int_{-h}^{0} dt \int_{\omega(R)} v^2 dx.$$

Proviamo che

$$\int_{-h}^{0} dt \int_{\omega(R)} v^2 dx \leqslant h^2 \int_{-h}^{0} dt \int_{\omega(R)} \left(\frac{\partial v}{\partial t}\right)^2 dx \qquad \forall R \geqslant R_0 \tag{58}$$

$$\int_{-h}^{0} dt \int_{\sigma(R)} v^2 d\sigma \leqslant h^2 \int_{-h}^{0} dt \int_{\sigma(R)} \left(\frac{\partial v}{\partial t}\right)^2 d\sigma \qquad \forall R \geqslant R_0 . \tag{59}$$

Infatti

$$|v_i(t)| = \left|\int_0^t \left(\frac{\partial v_i}{\partial t}\right) d\xi\right| \leqslant |t|^{1/2} \left|\int_0^t \left(\frac{\partial v_i}{\partial t}\right)^2 d\xi\right|^{1/2} \quad \text{per } 1 \leqslant i \leqslant 3$$

e quindi

$$\int_{-h}^{0} v^2(t)dt \leqslant h^2 \int_{-h}^{0} \left(\frac{\partial v}{\partial t}\right)^2 d\xi .$$

Ciò prova la (58) e la (59).

Dalla (57) preso h tale che $(\nabla\times v)^2(-h)$ sia sommabile su Ω (il che è possibile per quasi ogni h, $0 \leqslant h \leqslant T$, per la (50) e per il teorema di Fubini) e tale che sia $c'_{12}h^2 < \frac{1}{4}$, ricordando ancora la (50) e le (58), (59), se si pone

$$T(R) = \int_{-h}^{0} dt \int_{\omega(R)} \left(\frac{\partial v}{\partial t}\right)^2 dx \tag{60}$$

si ottiene

$$T(R) \leqslant C'_{14} T'(R) + C'_{14} R^{1/2} (T'(R))^{1/2} + C'_{14} \int_{-h}^{0} dt \int_{\sigma(R)} (\nabla \times v)^2 d\sigma + C'_{14} \leqslant$$

$$\leqslant \frac{1}{40} R T'(R) + C'_{14} \int_{-h}^{0} dt \int_{\sigma(R)} (\nabla \times v)^2 d\sigma + C'_{15} \qquad \forall R \geqslant R_{19}.$$

Proviamo ora che

$$T(R) \leqslant 2C'_{15} \qquad \forall R \geqslant R_{19}. \tag{61}$$

Infatti se esiste $R_{20} \geqslant R_{19}$ tale che $T(R_{20}) > 2C'_{15}$, tenendo presente che $T'(R) \geqslant 0$, si ha

$$\frac{20}{R} \leqslant \frac{T(R)}{T'(R)} + C'_{16} \int_{-h}^{0} dt \int_{\sigma(R)} (\nabla \times v)^2 dx, \qquad \forall R \geqslant R_{20}$$

da cui, integrando sull'intervallo $[R_{20}, R]$ $(R > R_{20})$ e tenendo presente la (48) si ottiene

$$20 \log \left(\frac{R}{R_{20}} \right) \leqslant \log \left(\frac{T(R)}{T(R_{20})} \right) + C'_{17} \qquad \forall R \geqslant R_{20},$$

e quindi

$$R^{20} [e^{-C'_{17}} R_{20}^{-20} T(R_{20})] \leqslant T(R) \qquad \forall R \geqslant R_{20}$$

che è una contraddizione con la (44), che è valida anche nelle ipotesi del presente teorema. Dalla (60) e dalla (61) per la (58), segue la (56).

Proviamo ora che

$$\int_{-h}^{0} dt \int_{-h}^{t} d\xi \int_{\Omega} \sum_{i,j=1}^{3} \left(\frac{\partial v_i}{\partial x_j} \right)^2 dx < +\infty. \tag{62}$$

Allo scopo osserviamo che dalla (42) si ottiene

$$\frac{1}{2} \int_{-h}^{0} dt \int_{\omega(R)} v^2 dx + \int_{-h}^{0} dt \int_{-h}^{t} d\xi \int_{\omega(R)} \sum_{i,j=1}^{3} \left(\frac{\partial v_i}{\partial x_j} \right)^2 dx \leqslant$$

$$\leqslant C'_{18}\int_{-h}^{0} dt \int_{\sigma(R)} v^2 d\sigma + C'_{18}\int_{-h}^{0} dt \int_{-h}^{t} d\xi \int_{\omega(R)} \sum_{i,j=1}^{3} \left(\frac{\partial v_i}{\partial x_j}\right)^2 dx +$$

$$+C'_{18}R^{1/2}\left(\int_{-h}^{0} dt \int_{\sigma(R)} v^2 d\sigma\right)^{1/2} + \frac{h}{2}\int_{\omega(R)} v^2(-h)dx + C'_{18}h\int_{-h}^{0} dt \int_{\omega(R)} v^2 dx.$$

Da quest'ultima chiamando ancora con $N(R)$ la (43) con h al posto di τ_2 e tenendo presente la (56) e il teorema di Fubini si ottiene

$$N(R) \leqslant C'_{19}N'(R) + C'_{19}(RN(R))^{1/2} + C'_{19} \leqslant$$

$$\leqslant \frac{1}{20} RN'(R) + C'_{20} \qquad \forall R \geqslant R_{21}\,;$$

di qui, ragionando come prima, segue

$$N(R) \leqslant 2C'_{20} \qquad \forall R \geqslant R_0\,.$$

Quindi per il teorema di Fubini risulta che $\sum_{i,j=1}^{3}\left(\frac{\partial v_i}{\partial x_j}\right)^2$ è sommabile per quasi ogni $t \in [-h, 0]$; perciò ragionando come nel teorema precedente si dimostra che $v=0$, $p=0$ in $[-h, 0]$; ripetendo poi i ragionamenti già fatti si prova che $v=0$, $p=0$ in $[-2h, 0]$; ecc. Con ciò si dimostra il teorema.

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