Rendiconti
del
Seminario Matematico
della
Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 46 (1971), p. 191-198

<http://www.numdam.org/item?id=RSMUP_1971__46__191_0>

ON A CERTAIN APPROXIMATION PROPERTY FOR FIRST-ORDER ABSTRACT DIFFERENTIAL EQUATIONS

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Introduction.

This paper is closely related with some previous work of ours [1], [2]. There we insisted mainly on the so called « global existence theorems » and a main tool in the proof was a certain density result. Here we shall concentrate ourselves on density (or approximation) theorem; what we prove is essentially a generalization of Lemma 6-bis in [1]. The condition S in [1] is here replaced by condition S' below; the proofs are essentially those indicated in our Lecture Notes [2]; the condition S' is implied by S (Lemma 1 and Lemma 5 in [1]) and this indicates how our Theorem below is (slightly) more general than Lemma 6-bis in [1].

§ 1. Let H be a hilbert space; $(\, , \,)$ and $| \; |$ are the notations for scalar product and for the norm.

Consider in H a linear closed operator A with domain D_A and let A^* be the adjoint of H.

For any open interval $a<t<b$ of the real line, we define a class of test-functions (vector-valued); precisely

$$K_{A^*}(a, b) = \{\varphi(t),\ a<t<b \to D_{A^*}\ ;\ \varphi \in C^2(a, b;\ H);$$
$$A^*\varphi \in C(a, b;\ H);\ \operatorname{supp} \varphi \text{ is compact in } (a, b)\}.$$

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This research is supported by a grant of the N.R.C. of Canada.

Similarly is defined K_{A^*}, by taking (a, b)=whole real axis, and more generally the classes K_B, $K_B(a, b)$, for any linear closed operator B in H.

Let us give now a function $f(t)$, $a<t<b \to H$, belonging to $L^2(a, b; H)$, i.e. square Bochner integrable in (a, b), and a function $u(t)$ with the same property.

We say that $u(t)$ is weak solution on $a<t<b$ of the differential equation

$$u'(t)=Au(t)+f(t) \tag{1.1}$$

when the integral identity

$$\int_a^b (u(t), \varphi'(t)+(A^*\varphi)(t))dt=-\int_a^b (f(t), \varphi(t))dt \tag{1.2}$$

is verified, $\forall \varphi \in K_{A^*}(a, b)$.

Let us take now an arbitrary positive number $T>0$, and denote by V_T the set of weak solutions in $(-T, T)$ of the homogeneous differential equation

$$u'(t)=Au(t); \tag{1.3}$$

also call V_∞ the set of weak solutions of (1.3) on the whole real axis [1]). We give now some more definitions;

Definition 1. The abstract differential operator $\frac{d}{dt}-A$ has *the approximation property* if for any pair of positive numbers $T_1<T_2$, the set V_∞ is dense in V_{T_2} in the norm of $L^2(-T_1; T_1; H)$.

Definition 2. The abstract differential operator $\frac{d}{dt}+A^*$ has *the support property* if:

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[1]) Precisely, $u \in V_\infty$ if $u \in L^2_{\text{loc}}(-\infty, \infty; H)$ and the relation

$$\int_{-\infty}^{\infty} (u(t), \varphi'(t)+(A^*\varphi)(t))dt=0 \text{ holds, } \forall \varphi \in K_{A^*}.$$

i) $u \in L^2_{\text{loc}}(-\infty, \infty; H)$, supp u-compact

ii) $f \in L^2_{\text{loc}}(-\infty, \infty; H)$, supp $f \subset [a, b]$

iii) $u' + A^* u = f$ on $-\infty < t < \infty$ in weak sense, imply

iv) supp $u \subset [a, b]$ too.

Let us assume also that the following holds:

PROPERTY ($\mathfrak{F}$). For any finite interval $a < t < b$ there is a constant $C_{a,b}$ such that

$$\|\Psi\|_{L^2(a,b;H)} \leq C_{a,b} \|\Psi' + A^*\Psi\|_{L^2(a,b;H)}, \qquad \forall \Psi \in K_{A^*}(a, b) \tag{1.4}$$

is verified.

Now we finally define condition S'

DEFINITION 3. The linear closed operator A satisfies condition S' when

(1.5) $\frac{d}{dt} + A^*$ has *the support property*, and

(1.6) property ($\mathfrak{F}$) holds.

Then we have

THEOREM. *If A verifies condition S' then $\frac{d}{dt} - A$ has the approximation property.*

The proof is given below.

§ 2. We have

MAIN LEMMA. *If A verifies condition S', and if we take three positive numbers $T_1 < T_2 < T_3$, then V_{T_3} is dense in V_{T_2} in the norm of $L^2(-T_1, T_1; H)$.*

As well-known, it will be enough to prove that:

$$\int_{-T_1}^{T_1} (v(t), u(t))dt = 0 \qquad \forall u \in V_{T_3} \tag{2.1}$$

implies

$$\int_{-T_1}^{T_1} (v(t),\ w(t))dt=0 \qquad \forall w \in V_{T_2} \tag{2.2}$$

where $v(t)$ is an arbitrary function in $L^2(-T_1, T_1; H)$.

If we define outside $(-T_1, T_1)$, $\tilde{v}(t)=\theta$, $\tilde{v}(t)=v(t)$, $-T_1<t<T_1$, then (2.2) equals

$$\int_{-T_2}^{T_2} (\tilde{v}(t),\ w(t))dt=0, \qquad \forall w \in V_{T_2}. \tag{2.3}$$

If we take a sequence of mollifiers $(\alpha_n)_1^\infty \to \delta$-the Dirac distribution, and if we form convolutions

$$(\tilde{v} * \alpha_n)(t)=\int_{|t-\zeta|<\frac{1}{n}} \tilde{v}(\zeta)\alpha_n(t-\zeta)d\zeta \tag{2.4}$$

then $\tilde{v} * \alpha_n \in C^\infty(-\infty, \infty; H)$ and $\operatorname{supp}(\tilde{v} * \alpha_n) \subset (-T_2, T_2)$ for n large enough. Furthemore $\| \tilde{v} * \alpha_n - \tilde{v} \|_{L^2(-T_2, T_2; H)} \to 0$ as $n \to \infty$.

Let now $\mathfrak{M} \subset L^2(-T_3, T_3; H)$ be defined as the image of $K_{A^*}(a_3, b_3)$ through the operator $\frac{d}{dt}+A^*$, i.e.

$$\mathfrak{M}=\{\varphi'+A^*\varphi,\ \varphi \in K_{A^*}(-T_3, T_3)\}. \tag{2.5}$$

We have

PROPOSITION 1. $\tilde{v}(t) \in$ closure $\mathfrak{M}$ (in $L^2(-T_3; T_3; H)$).

As well known it is enough to show the following. For any $h(t) \in L^2(-T_3, T_3; H)$ such that $h(t) \perp \mathfrak{M}$, it follows $h(t) \perp \tilde{v}(t)$ (in $L^2(-T_3, T_3; H)$).

Thus, we assume that

$$\int_{-T_3}^{T_3} (h(t),\ \varphi'(t)+(A^*\varphi)(t))dt=0 \qquad \forall \varphi \in K_{A^*}(-T_3, T_3). \tag{2.6}$$

This means precisely that $h(t) \in V_{T_3}$; then

$$\int_{-T_3}^{T_3} (h(t), \tilde{v}(t))dt = \int_{-T_1}^{T_1} (h(t), v(t))dt = 0$$

in view of (2.1).

Now, the Proposition means that we have for a sequence

$$(\Psi_n)_1^\infty \in \mathfrak{M}, \lim_{n\to\infty} \| \tilde{v} - \Psi_n \|_{L^2(-T_3, T_3; H)} = 0;$$

here $\Psi_n = \varphi'_n + A^* \varphi_n$, where $\varphi_n \in K_{A^*}(-T_3, T_3)$, hence

$$\lim_{n\to\infty} \int_{-T_3}^{T_3} \| \tilde{v}(t) - (\varphi'_n + A^* \varphi_n) \|^2 dt = 0. \tag{2.7}$$

At this stage we shall apply property ($\mathfrak{F}$); as the sequence $\varphi'_n + A^* \varphi_n$ is convergent in $L^2(-T_3, T_3; H)$, the sequence $(\varphi_n)_1^\infty$ will be a Cauchy sequence in $L^2(-T_3, T_3; H)$; let $\Phi(t) = \lim \varphi_n(t)$ in $L^2(-T_3, T_3; H)$; so $\Phi(t)$ is well-defined on $-T_3 < t < T_3$; put also $\Phi(t) = 0$ outside this interval. We have

PROPOSITION 2. The $\operatorname{supp} \Phi(t) \subset [-T_1, T_1]$.

Consider in fact $\frac{d}{dt}\Phi + A^*\Phi$ on $-\infty < t < \infty$ in weak sense, i.e. the integral

$$\int_{-\infty}^{\infty} (\Phi(t), \zeta'(t) - (A\zeta)(t))dt, \qquad \forall \zeta \in K_A$$

(use $A^{**} = A$, for densely defined, linear closed operators in Hilbert spaces).

We have then

$$\int_{-\infty}^{\infty} (\Phi(t), \zeta'(t) - (A\zeta)(t))dt = \int_{-T_3}^{T_3} (\Phi(t), \zeta'(t) - (A\zeta)(t))dt = \tag{2.8}$$

$$= \lim_{n\to\infty} \int_{-T_3}^{T_3} (\varphi_n(t), \zeta'(t)-(A\zeta)(t))dt = -\lim_{n\to\infty} \int_{-T_3}^{T_3} (\varphi'_n(t)+(A^*\varphi_n)(t), \zeta(t))dt =$$

$$= -\int_{-T_3}^{T_3} (\tilde{v}(t), \zeta(t))dt = -\int_{-\infty}^{\infty} (\tilde{v}(t), \zeta(t))dt, \qquad \forall \zeta \in K_A$$

hence $\Phi' + A^*\Phi = \tilde{v}$, in weak sense.

We can apply here the support property; remark that $\operatorname{supp} \tilde{v} \subset \subset [-T_1, T_1]$ hence $\operatorname{supp} \Phi \subset [-T_1, T_1]$ too, as we assumed $\operatorname{supp} \Phi \subset \subset [-T_3, T_3]$-compact in R^1.

Now, in order to prove (2.3), it will suffice to show

$$\text{(2.9)} \qquad \lim_{n\to\infty} \int_{-T_2}^{T_2} ((\tilde{v} * \alpha_n)(t), w(t))dt = 0.$$

Let us use now (2.8) and Lemma 2.3 in [2], for $B = A^*$; we get that $(\Phi * \alpha_n)(t) \in D_{A^*}$, $\forall t$, and that

$$\text{(2.10)} \qquad (\Phi * \alpha_n)'(t) + A^*(\Phi * \alpha_n)(t) = (\tilde{v} * \alpha_n)(t), \qquad -\infty < t < \infty.$$

From (2.10) we obtain, for $n \geq n_0$

$$\text{(2.11)} \qquad \int_{-T_2}^{T_2} (\tilde{v} * \alpha_n, w)dt = \int_{-T_2}^{T_2} ((\Phi * \alpha_n)' + A^*(\Phi * \alpha_n), w)dt = 0,$$

if we remember that $w \in V_{T_2}$, and if remark that $\Phi * \alpha_n \in K_{A^*}(-T_2, T_2)$ for $n \geq n_0$ (because $\operatorname{supp} \Phi \subset [-T_1, T_1]$, hence $\sup (\Phi * \alpha_n) \subset (-T_2, T_2)$ for large n). Hence (2.9) follows.

This will prove Main Lemma.

We can pass now to the final step: take $0 < T_1 < T_2$, a function $u \in V_{T_2}$ and an $\varepsilon > 0$. We must find $u_\varepsilon > V_\infty$ such that

$$\| u_\varepsilon - u \|_{L^2(-T_1, T_1; H)} < \varepsilon.$$

Let us consider an increasing sequence $T_3 < T_4 < \ldots$, where $T_2 < T_3$, and $\lim_{n\to\infty} T_n = \infty$.

We shall use successively the Main Lemma to triplets (T_1, T_2, T_3), (T_2, T_3, T_4), ... etc. We get a function $u_1(t)\in V_{T_3}$ such that

$$\| u-u_1 \|_{L^2(-T_1, T_1; H)}<\frac{\varepsilon}{2}.$$

Then a function $u_2(t)\in V_{T_4}$ exists, such that

$$\| u_1-u_2 \|_{L^2(-T_2, T_2; H)}<\frac{\varepsilon}{2^2};$$

continuing this way we shall find $u_n(t)\in V_{T_{n-2}}$, such that

$$\| u_{n-1}-u_n \|_{L^2(-T_n, T_n; H)}<\frac{\varepsilon}{2^n}.$$

Now, on any finite interval $[a, b]\subset R^1$, the sequence $(u_n, u_{n+1}, ...,)$ is well-defined for $n\geq n_0$-depending on $[a, b]$ (when $-T_{n_0}<a<b<T_{n_0}$). Then, we shall have

$$\| u_{j-1}-u_j \|_{L^2(-T_{n_0}, T_{n_0}; H)}<\frac{\varepsilon}{2^j},$$

for $j>n_0$ and this implies that $\lim_{j\to\infty} u_{n_0+j}$ exists in $L^2(a, b; H)$.

Taking $[a, b]$ successively $=[-1, 1], [-2, 2], ...,$ we shall find functions $u^1(t), u^2(t), ...,$ so that $u^p(t)\in L^2(-p, p; H)$, and

$$u^p(t)=\lim_{\substack{n\to\infty \\ n\geq n_p}} u_n(t) \text{ in } L^2(-p; p; H).$$

Remark that $u^2(t)=u^1(t)$ a.e. on $-1\leq t\leq 1$; $u^3(t)=u^2(t)$, a.e. on $-2\leq t\leq 2$, ... etc. and we may put, $\forall t\in(-\infty, \infty)$

$$u_\varepsilon(t)=u^j(t) \text{ for } -j\leq t\leq j.$$

Remark that $u_\varepsilon(t)\in L^2[a, b; H]$ for any finite interval $[a, b]\subset R^1$, so $u_\varepsilon\in L^2_{\text{loc}}(-\infty, \infty; H)$. We see also that

$$\| u_\varepsilon-u \|_{L^2(-T_1, T_1; H)}=\lim_{n\to\infty} \| u_n-u \|_{L^2(-T_1, T_1; H)}.$$

But

$$\| u_n - u \|_{L^2(-T_1, T_1; H)} \leq \| u - u_1 \|_{L^2(-T_1, T_1; H)} + $$
$$+ \| u_1 - u_2 \|_{L^2(-T_1, T_1; H)} + \dots + \| u_{n-1} - u_n \|_{L^2(-T_1, T_1; H)} \leq$$
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \dots + \frac{\varepsilon}{2^n} = \frac{\frac{\varepsilon}{2} - \frac{\varepsilon}{2^{n+1}}}{1 - \frac{1}{2}} = \frac{\frac{\varepsilon}{2} - \frac{\varepsilon}{2^{n+1}}}{\frac{1}{2}} = \varepsilon - \frac{\varepsilon}{2^n}.$$

So

$$\| u_n - u \|_{L^2(-T_1, T_1; H)} \leq \varepsilon - \frac{\varepsilon}{2^n},$$

and

$$\lim_{n \to \infty} \| u_n - u \|_{L^2(-T_1, T_1; H)} \leq \varepsilon.$$

Finally, it remains to show that $u_\varepsilon \in V_\infty$, i.e. that $\frac{d}{dt} u_\varepsilon - A u_\varepsilon = 0$ in weak sense.

Take then an arbitrary $\varphi \in K_{A^*}$; for $i \geq i_0$ all $u_i(t)$ are defined on an interval $[a, b] \supset \operatorname{supp} \varphi$; moreover, because $u_i(t) \in V_{T_{i+2}}$, we get

$$\int_{-\infty}^{\infty} (u_i, \varphi' + A^* \varphi) dt = \int_a^b (u_i, \varphi' + A^* \varphi) dt = 0.$$

When $i \geq i_0$ tends to ∞, we get $\int_a^b (u_\varepsilon, \varphi' + A^* \varphi) dt = 0$, and this proves the theorem.

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Manoscritto pervenuto in redazione il 10 maggio 1971.