

RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

S. ZAIDMAN

On a certain approximation property for first-order abstract differential equations

Rendiconti del Seminario Matematico della Università di Padova,
tome 46 (1971), p. 191-198

http://www.numdam.org/item?id=RSMUP_1971__46__191_0

© Rendiconti del Seminario Matematico della Università di Padova, 1971, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

ON A CERTAIN APPROXIMATION PROPERTY FOR FIRST-ORDER ABSTRACT DIFFERENTIAL EQUATIONS

S. ZAIDMAN *)

Introduction.

This paper is closely related with some previous work of ours [1], [2]. There we insisted mainly on the so called « global existence theorems » and a main tool in the proof was a certain density result. Here we shall concentrate ourselves on density (or approximation) theorem; what we prove is essentially a generalization of Lemma 6-bis in [1]. The condition S in [1] is here replaced by condition S' below; the proofs are essentially those indicated in our Lecture Notes [2]; the condition S' is implied by S (Lemma 1 and Lemma 5 in [1]) and this indicates how our Theorem below is (slightly) more general than Lemma 6-bis in [1].

§ 1. Let H be a hilbert space; $(,)$ and $||$ are the notations for scalar product and for the norm.

Consider in H a linear closed operator A with domain D_A and let A^* be the adjoint of A .

For any open interval $a < t < b$ of the real line, we define a class of test-functions (vector-valued); precisely

$$K_{A^*}(a, b) = \{ \varphi(t), a < t < b \rightarrow D_{A^*}; \varphi \in C^2(a, b; H); \\ A^* \varphi \in C(a, b; H); \text{supp } \varphi \text{ is compact in } (a, b) \}.$$

*) Indirizzo dell'A.: Université de Montréal, Case Postale 6128, Montréal, 101 Canada.

This research is supported by a grant of the N.R.C. of Canada.

Similarly is defined K_{A^*} , by taking (a, b) = whole real axis, and more generally the classes K_B , $K_B(a, b)$, for any linear closed operator B in H .

Let us give now a function $f(t)$, $a < t < b \rightarrow H$, belonging to $L^2(a, b; H)$, i.e. square Bochner integrable in (a, b) , and a function $u(t)$ with the same property.

We say that $u(t)$ is weak solution on $a < t < b$ of the differential equation

$$(1.1) \quad u'(t) = Au(t) + f(t)$$

when the integral identity

$$(1.2) \quad \int_a^b (u(t), \varphi'(t) + (A^*\varphi)(t)) dt = - \int_a^b (f(t), \varphi(t)) dt$$

is verified, $\forall \varphi \in K_{A^*}(a, b)$.

Let us take now an arbitrary positive number $T > 0$, and denote by V_T the set of weak solutions in $(-T, T)$ of the homogeneous differential equation

$$(1.3) \quad u'(t) = Au(t);$$

also call V_∞ the set of weak solutions of (1.3) on the whole real axis¹⁾. We give now some more definitions;

DEFINITION 1. The abstract differential operator $\frac{d}{dt} - A$ has the *approximation property* if for any pair of positive numbers $T_1 < T_2$, the set V_∞ is dense in V_{T_2} , in the norm of $L^2(-T_1; T_1; H)$.

DEFINITION 2. The abstract differential operator $\frac{d}{dt} + A^*$ has the *support property* if:

¹⁾ Precisely, $u \in V_\infty$ if $u \in L^2_{loc}(-\infty, \infty; H)$ and the relation

$$\int_{-\infty}^{\infty} (u(t), \varphi'(t) + (A^*\varphi)(t)) dt = 0 \text{ holds, } \forall \varphi \in K_{A^*}.$$

- i) $u \in L^2_{loc}(-\infty, \infty; H)$, $\text{supp } u$ -compact
- ii) $f \in L^2_{loc}(-\infty, \infty; H)$, $\text{supp } f \subset [a, b]$
- iii) $u' + A^*u = f$ on $-\infty < t < \infty$ in weak sense, imply
- iv) $\text{supp } u \subset [a, b]$ too.

Let us assume also that the following holds:

PROPERTY (3). For any finite interval $a < t < b$ there is a constant $C_{a,b}$ such that

$$(1.4) \quad \|\Psi\|_{L^2(a,b;H)} \leq C_{a,b} \|\Psi' + A^*\Psi\|_{L^2(a,b;H)}, \quad \forall \Psi \in K_{A^*}(a, b)$$

is verified.

Now we finally define condition S'

DEFINITION 3. The linear closed operator A satisfies condition S' when

$$(1.5) \quad \frac{d}{dt} + A^* \text{ has the support property, and}$$

$$(1.6) \quad \text{property (3) holds.}$$

Then we have

THEOREM. If A verifies condition S' then $\frac{d}{dt} - A$ has the approximation property.

The proof is given below.

§ 2. We have

MAIN LEMMA. If A verifies condition S' , and if we take three positive numbers $T_1 < T_2 < T_3$, then V_{T_3} is dense in V_{T_2} in the norm of $L^2(-T_1, T_1; H)$.

As well-known, it will be enough to prove that:

$$(2.1) \quad \int_{-T_1}^{T_1} (v(t), u(t)) dt = 0 \quad \forall u \in V_{T_3}$$

implies

$$(2.2) \quad \int_{-T_1}^{T_1} (v(t), w(t)) dt = 0 \quad \forall w \in V_{T_2}$$

where $v(t)$ is an arbitrary function in $L^2(-T_1, T_1; H)$.

If we define outside $(-T_1, T_1)$, $\tilde{v}(t) = \theta$, $\tilde{v}(t) = v(t)$, $-T_1 < t < T_1$, then (2.2) equals

$$(2.3) \quad \int_{-T_2}^{T_2} (\tilde{v}(t), w(t)) dt = 0, \quad \forall w \in V_{T_2}.$$

If we take a sequence of mollifiers $(\alpha_n)_{n=1}^\infty \rightarrow \delta$ -the Dirac distribution, and if we form convolutions

$$(2.4) \quad (\tilde{v} * \alpha_n)(t) = \int_{|t-\zeta| < \frac{1}{n}} \tilde{v}(\zeta) \alpha_n(t-\zeta) d\zeta$$

then $\tilde{v} * \alpha_n \in C^\infty(-\infty, \infty; H)$ and $\text{supp}(\tilde{v} * \alpha_n) \subset (-T_2, T_2)$ for n large enough. Furthermore $\|\tilde{v} * \alpha_n - \tilde{v}\|_{L^2(-T_2, T_2; H)} \rightarrow 0$ as $n \rightarrow \infty$.

Let now $\mathfrak{N} \subset L^2(-T_3, T_3; H)$ be defined as the image of $K_{A^*}(a_3, b_3)$ through the operator $\frac{d}{dt} + A^*$, i.e.

$$(2.5) \quad \mathfrak{N} = \{\varphi' + A^*\varphi, \varphi \in K_{A^*}(-T_3, T_3)\}.$$

We have

PROPOSITION 1. $\tilde{v}(t) \in \text{closure } \mathfrak{N}$ (in $L^2(-T_3; T_3; H)$).

As well known it is enough to show the following. For any $h(t) \in L^2(-T_3, T_3; H)$ such that $h(t) \perp \mathfrak{N}$, it follows $h(t) \perp \tilde{v}(t)$ (in $L^2(-T_3, T_3; H)$).

Thus, we assume that

$$(2.6) \quad \int_{-T_3}^{T_3} (h(t), \varphi'(t) + (A^*\varphi)(t)) dt = 0 \quad \forall \varphi \in K_{A^*}(-T_3, T_3).$$

This means precisely that $h(t) \in V_{T_3}$; then

$$\int_{-T_3}^{T_3} (h(t), \tilde{v}(t)) dt = \int_{-T_1}^{T_1} (h(t), v(t)) dt = 0$$

in view of (2.1).

Now, the Proposition means that we have for a sequence

$$(\Psi_n)_{n \rightarrow \infty} \in \mathfrak{D}\mathcal{L}, \quad \lim_{n \rightarrow \infty} \|\tilde{v} - \Psi_n\|_{L^2(-T_3, T_3; H)} = 0;$$

here $\Psi_n = \varphi'_n + A^* \varphi_n$, where $\varphi_n \in K_{A^*}(-T_3, T_3)$, hence

$$(2.7) \quad \lim_{n \rightarrow \infty} \int_{-T_3}^{T_3} \|\tilde{v}(t) - (\varphi'_n + A^* \varphi_n)\|^2 dt = 0.$$

At this stage we shall apply property (\mathfrak{B}); as the sequence $\varphi'_n + A^* \varphi_n$ is convergent in $L^2(-T_3, T_3; H)$, the sequence $(\varphi_n)_{n \rightarrow \infty}$ will be a Cauchy sequence in $L^2(-T_3, T_3; H)$; let $\Phi(t) = \lim \varphi_n(t)$ in $L^2(-T_3, T_3; H)$; so $\Phi(t)$ is well-defined on $-T_3 < t < T_3$; put also $\Phi(t) = 0$ outside this interval. We have

PROPOSITION 2. The $\text{supp } \Phi(t) \subset [-T_1, T_1]$.

Consider in fact $\frac{d}{dt} \Phi + A^* \Phi$ on $-\infty < t < \infty$ in weak sense, i.e. the integral

$$\int_{-\infty}^{\infty} (\Phi(t), \zeta'(t) - (A\zeta)(t)) dt, \quad \forall \zeta \in K_A$$

(use $A^{**} = A$, for densely defined, linear closed operators in Hilbert spaces).

We have then

$$(2.8) \quad \int_{-\infty}^{\infty} (\Phi(t), \zeta'(t) - (A\zeta)(t)) dt = \int_{-T_3}^{T_3} (\Phi(t), \zeta'(t) - (A\zeta)(t)) dt =$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \int_{-T_3}^{T_3} (\varphi_n(t), \zeta'(t) - (A\zeta)(t)) dt = - \lim_{n \rightarrow \infty} \int_{-T_3}^{T_3} (\varphi'_n(t) + (A^*\varphi_n)(t), \zeta(t)) dt = \\
 &= - \int_{-T_3}^{T_3} (\tilde{v}(t), \zeta(t)) dt = - \int_{-\infty}^{\infty} (\tilde{v}(t), \zeta(t)) dt, \quad \forall \zeta \in K_A
 \end{aligned}$$

hence $\Phi' + A^*\Phi = \tilde{v}$, in weak sense.

We can apply here the support property; remark that $\text{supp } \tilde{v} \subset [-T_1, T_1]$ hence $\text{supp } \Phi \subset [-T_1, T_1]$ too, as we assumed $\text{supp } \Phi \subset [-T_3, T_3]$ -compact in R^1 .

Now, in order to prove (2.3), it will suffice to show

$$(2.9) \quad \lim_{n \rightarrow \infty} \int_{-T_2}^{T_2} ((\tilde{v} * \alpha_n)(t), w(t)) dt = 0.$$

Let us use now (2.8) and Lemma 2.3 in [2], for $B = A^*$; we get that $(\Phi * \alpha_n)(t) \in D_{A^*}$, $\forall t$, and that

$$(2.10) \quad (\Phi * \alpha_n)'(t) + A^*(\Phi * \alpha_n)(t) = (\tilde{v} * \alpha_n)(t), \quad -\infty < t < \infty.$$

From (2.10) we obtain, for $n \geq n_0$

$$(2.11) \quad \int_{-T_2}^{T_2} (\tilde{v} * \alpha_n, w) dt = \int_{-T_2}^{T_2} ((\Phi * \alpha_n)' + A^*(\Phi * \alpha_n), w) dt = 0,$$

if we remember that $w \in V_{T_2}$, and if remark that $\Phi * \alpha_n \in K_{A^*}(-T_2, T_2)$ for $n \geq n_0$ (because $\text{supp } \Phi \subset [-T_1, T_1]$, hence $\text{supp } (\Phi * \alpha_n) \subset (-T_2, T_2)$ for large n). Hence (2.9) follows.

This will prove Main Lemma.

We can pass now to the final step: take $0 < T_1 < T_2$, a function $u \in V_{T_2}$, and an $\varepsilon > 0$. We must find $u_\varepsilon \in V_\infty$ such that

$$\|u_\varepsilon - u\|_{L^2(-T_1, T_1; H)} < \varepsilon.$$

Let us consider an increasing sequence $T_3 < T_4 < \dots$, where $T_2 < T_3$, and $\lim_{n \rightarrow \infty} T_n = \infty$.

We shall use successively the Main Lemma to triplets (T_1, T_2, T_3) , (T_2, T_3, T_4) , ... etc. We get a function $u_1(t) \in V_{T_3}$ such that

$$\|u - u_1\|_{L^2(-T_1, T_1; H)} < \frac{\varepsilon}{2}.$$

Then a function $u_2(t) \in V_{T_4}$ exists, such that

$$\|u_1 - u_2\|_{L^2(-T_2, T_2; H)} < \frac{\varepsilon}{2^2};$$

continuing this way we shall find $u_n(t) \in V_{T_{n-2}}$, such that

$$\|u_{n-1} - u_n\|_{L^2(-T_n, T_n; H)} < \frac{\varepsilon}{2^n}.$$

Now, on any finite interval $[a, b] \subset R^1$, the sequence (u_n, u_{n+1}, \dots) is well-defined for $n \geq n_0$ depending on $[a, b]$ (when $-T_{n_0} < a < b < T_{n_0}$). Then, we shall have

$$\|u_{j-1} - u_j\|_{L^2(-T_{n_0}, T_{n_0}; H)} < \frac{\varepsilon}{2^j},$$

for $j > n_0$ and this implies that $\lim_{j \rightarrow \infty} u_{n_0+j}$ exists in $L^2(a, b; H)$.

Taking $[a, b]$ successively $=[-1, 1], [-2, 2], \dots$, we shall find functions $u^1(t), u^2(t), \dots$, so that $u^p(t) \in L^2(-p, p; H)$, and

$$u^p(t) = \lim_{\substack{n \rightarrow \infty \\ n \geq n_p}} u_n(t) \text{ in } L^2(-p; p; H).$$

Remark that $u^2(t) = u^1(t)$ a.e. on $-1 \leq t \leq 1$; $u^3(t) = u^2(t)$, a.e. on $-2 \leq t \leq 2$, ... etc. and we may put, $\forall t \in (-\infty, \infty)$

$$u_\varepsilon(t) = u^j(t) \text{ for } -j \leq t \leq j.$$

Remark that $u_\varepsilon(t) \in L^2[a, b; H]$ for any finite interval $[a, b] \subset R^1$, so $u_\varepsilon \in L^2_{loc}(-\infty, \infty; H)$. We see also that

$$\|u_\varepsilon - u\|_{L^2(-T_1, T_1; H)} = \lim_{n \rightarrow \infty} \|u_n - u\|_{L^2(-T_1, T_1; H)}.$$

But

$$\begin{aligned} & \|u_n - u\|_{L^2(-T_1, T_1; H)} \leq \|u - u_1\|_{L^2(-T_1, T_1; H)} + \\ & + \|u_1 - u_2\|_{L^2(-T_1, T_1; H)} + \dots + \|u_{n-1} - u_n\|_{L^2(-T_1, T_1; H)} \leq \\ & \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \dots + \frac{\varepsilon}{2^n} = \frac{\frac{\varepsilon}{2} - \frac{\varepsilon}{2^{n+1}}}{1 - \frac{1}{2}} = \frac{\frac{\varepsilon}{2} - \frac{\varepsilon}{2^{n+1}}}{\frac{1}{2}} = \varepsilon - \frac{\varepsilon}{2^n}. \end{aligned}$$

So

$$\|u_n - u\|_{L^2(-T_1, T_1; H)} \leq \varepsilon - \frac{\varepsilon}{2^n},$$

and

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{L^2(-T_1, T_1; H)} \leq \varepsilon.$$

Finally, it remains to show that $u_\varepsilon \in V_\infty$, i.e. that $\frac{d}{dt}u_\varepsilon - Au_\varepsilon = 0$ in weak sense.

Take then an arbitrary $\varphi \in K_{A^*}$; for $i \geq i_0$ all $u_i(t)$ are defined on an interval $[a, b] \supset \text{supp } \varphi$; moreover, because $u_i(t) \in V_{T_{i+2}}$, we get

$$\int_{-\infty}^{\infty} (u_i, \varphi' + A^*\varphi) dt = \int_a^b (u_i, \varphi' + A^*\varphi) dt = 0.$$

When $i \geq i_0$ tends to ∞ , we get $\int_a^b (u_\varepsilon, \varphi' + A^*\varphi) dt = 0$, and this proves the theorem.

REFERENCES

- [1] ZAIDMAN, S.: *A global existence theorem for some differential equations in Hilbert spaces*, Proc. Nat. Acad. Sci., June 1964.
- [2] ZAIDMAN, S.: *Equations différentielles abstraites*, Les Presses de l'Université de Montréal, Janvier 1966.