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DUAL-DEDEKIND SUBGROUPS IN FINITE GROUPS

Federico Menegazzo *)

If $G$ is a group and $H$ is a subgroup of $G$, $H$ is dual-Dedekind in $G$, or a $\mathfrak{D}$-subgroup of $G$ (written $H\mathfrak{D}G$) if the following conditions are fulfilled:

i) $X \supseteq Y \Rightarrow (Y \cup H) \cap X = Y \cup (H \cap X)$

ii) $H \supseteq Y \Rightarrow (Y \cup X) \cap H = Y \cup (X \cap H)$

for every pair $X, Y$ of subgroups of $G$ (for the dual notion, namely that of Dedekind subgroups, there called « modular subgroups », see [4]). In this paper we are particularly concerned with the properties of « minimum » $\mathfrak{D}$-subgroups (i.e. minimal in the set of non identity dual-Dedekind subgroups of a given group $G$); we establish some necessary conditions in order that a finite group $G$ has non-trivial (i.e. different from 1, $G$) $\mathfrak{D}$-subgroups. From these it will follow that a finite group having non-trivial $\mathfrak{D}$-subgroups cannot be simple (Theorem 3.3) — a similar result for Dedekind subgroups is proved in [2]; it is perhaps worth noting that the converse is false: $G$ non-simple is not a sufficient condition for $G$ to have a non-trivial $\mathfrak{D}$-subgroup. The proposition « if $N \triangleleft G$, then $N\mathfrak{D}G$ » for arbitrary $G$ is false; in the second half of the paper we determine all finite soluble groups where such a condition holds. The main result in this section is (Theorem 4.6): $G$ is soluble and every

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normal subgroup of $G$ is dual-Dedekind in $G$ iff $G = H_1 \times H_2 \times \ldots \times H_t$ with each $H_i$ a Hall subgroup of $G$ and either

i) $H_i$ is a modular $p$-group; or

ii) $H_i = (P_{i1} \times \ldots \times P_{is_i})Q_i$ with $P_{ij}, Q_i$ Sylow subgroups of $G$ for different primes, $P_{ij}$ abelian of odd order ($j = 1, \ldots, s_i$), $Q_i = \langle b_i \rangle$ with $b_i$ inducing a non-identity power automorphism on each $P_{ij}$.

1. Let $L$ be a lattice. An element $a \in L$ is a dual-Dedekind element of $L$ (a $\mathcal{D}$-element of $L$, $a \mathcal{D} L$) if

i) $x \geq y \Rightarrow (y \cup a) \cap x = y \cup (a \cap x)$;

ii) $a \geq y \Rightarrow (y \cup x) \cap a = y \cup (x \cap a)$

hold for every pair $(x, y)$ of elements in $L$. Notice that $a \mathcal{D} L$ if and only if $a$ is a Dedekind element in $L$ (the dual lattice of $L$); hence properties of $\mathcal{D}$-elements of $L$ are properties of Dedekind elements of $L$. We shall use mostly (see [4]):

I) $a \mathcal{D} L$ iff for every $b \in L$ the maps

$$
\varphi_b : x \mapsto x \cup b \quad \varphi_b : [a/a \cap b] \rightarrow [a \cup b/b]
$$

$$
\varphi_a : y \mapsto y \cap a \quad \varphi_a : [a \cup b/b] \rightarrow [a/a \cap b]
$$

are inverse lattice-isomorphisms.

II) if $a \mathcal{D} L$ and $b \in L$, then $a \cap b \mathcal{D} (b)$.

III) if $a_1 \mathcal{D} L$ and $a_2 \mathcal{D} L$, then $a_1 \cap a_2 \mathcal{D} L$.

IV) if $a_1 \mathcal{D} L$ and $a_2 \mathcal{D} (a_1)$, then $a_2 \mathcal{D} L$.

V) whenever $\varphi$ is a surjective lattice isomorphism of $L$ onto $L'$, we have $a \mathcal{D} L'$ iff $a \mathcal{D} L$.

A subgroup $H$ of a group $G$ is dual-Dedekind in $G$ ($H$ is a $\mathcal{D}$-subgroup of $G$, $H \mathcal{D} G$) if $H$ is a $\mathcal{D}$-element of the lattice $\mathcal{L}(G)$ of all subgroups of $G$. Normal subgroups are usually not $\mathcal{D}$-subgroups; the following are
dual-Dedekind subgroups in any group $G$:

- $G$ and the identity subgroup 1 of $G$ (they will be referred to as the « trivial » $\mathfrak{D}$-subgroups);
- the subgroups of the centre $Z(G)$;
- all normal cyclic subgroups;
- the subgroups of the so-called « kernel »

$$K(G) = \cap \{ \mathfrak{N}_c(X) \mid X \in \mathfrak{L}(G) \}.$$ 

All definitions and notations will be standard; throughout this paper « group » means « finite group ».

2. The existence of non-trivial $\mathfrak{D}$-elements in $\mathfrak{L}(G)$ rather severely restricts the structure of $G$. The following two lemmas provide examples where, in very simple cases, the structure of $G$ is completely determined.

**Lemma 2.1.** Assume $G = \langle a, b \rangle$, $|a| = |b| = p$, $p$ a prime, $G$ non-cyclic. Then $\langle a \rangle \mathfrak{D} G$ iff either $|G| = p^2$ or $G$ is a non-abelian group of order $pq$ ($q$ a prime greater than $p$).

Since in both cases $\mathfrak{L}(G)$ is a modular lattice, the sufficiency of the condition is obvious. The condition is also necessary: the intervals $[\langle a, b \rangle / \langle b \rangle]$ and $[(\langle a \rangle / \langle a \rangle \cap \langle b \rangle)]$ are isomorphic, hence $\langle b \rangle$ is a maximal subgroup of $G$ and, if $G$ is a $p$-group, then $|G| = p^2$. Assume that $G$ is not a $p$-group; then $\langle b \rangle$ is a cyclic $p$-Sylow subgroup, $\langle b \rangle$ is its own normalizer in $G$ and has a normal complement $N$. $\langle a \rangle$, as a conjugate of $\langle b \rangle$, is maximal in $G$ and $\langle a \rangle \cap N = 1$; for a $c \in N$ of prime order $q$ one has $N = (\langle a \rangle \cup \langle c \rangle) \cap N = \langle c \rangle \cup (\langle a \rangle \cap N) = \langle c \rangle$, and the conclusion follows.

**Lemma 2.2.** Assume $G = \langle a, b \rangle$, $|a| = p$, $|b| = q$, $p$, $q$ different prime numbers. Then $\langle a \rangle \mathfrak{D} G$ iff either $|G| = pq$, or $G$ contains an elementary abelian $p$-subgroup $N \triangleleft G$ such that $G = N \langle b \rangle$, and $\langle b \rangle$ operates irreducibly on $N$.

If $\langle a \rangle \mathfrak{D} G$, then $\langle b \rangle$ is a maximal subgroup of $G$ and from $\langle b \rangle \triangleleft G$ follows $|G| = pq$. If $\langle b \rangle$ is not normal, then $\langle b \rangle$ is a $q$-Sylow subgroup
which is its own normalizer in $G$, hence $\langle b \rangle$ has a normal complement $N$. Since any conjugate of $\langle b \rangle$ is maximal in $G$, no proper non-trivial subgroup of $N$ is normalized by $b$ nor by a conjugate of $b$. Let $S$ be a $p$-Sylow subgroup of $N$ containing $\langle a \rangle$; by the Frattini argument $G = N \triangleleft \mathcal{L}_G(S)$, hence $\mathcal{L}_G(S)$ contains a conjugate of $\langle b \rangle$, and $S = N$. The Frattini subgroup $\Phi(N) \neq N$, $\Phi(N) \triangleleft G$, so that $\Phi(N) = 1$ and the «only if» part is proved. Conversely, neglecting the case $|G| = pq$ where everything is obvious, we have to prove that if $1 \neq a \in N$ and $X \in \mathcal{L}(G)$ is arbitrary, $a \notin X$ implies that $X$ is maximal in $\langle a, X \rangle$. But $X \subseteq N$ implies that $\langle a, X \rangle$ is abelian, whereas if $X \not\subseteq N$ a conjugate of $\langle b \rangle$, say $\langle c \rangle$, lies in $X$, hence $X = \langle c \rangle$ is a maximal subgroup of $G = \langle a, c \rangle$.

3. Definition. Let $H$ be a subgroup of the group $G$. We shall say that $H$ is a minimum $\mathcal{D}$-subgroup of $G$ if $H$ is minimal in the set of all non-identity $\mathcal{D}$-subgroups of $G$.

**Theorem 3.1.** Let $H$ be a minimum $\mathcal{D}$-subgroup of $G$. If $|H|$ is not a prime number, then

i) $H$ is normal in $G$;

ii) for every prime number $p$ dividing $|H|$, all the elements of $G$ of order $p$ are in $1$-1; and

iii) $\mathcal{C}_G(H) = \{ x \in G \mid (|x|, |H|) = 1 \}$.

First of all, notice that the minimality of $H$ and III, V of section 1 imply that for every $g \in G$ either $g^{-1}Hg \cap H = 1$ or $g^{-1}Hg = H$; furthermore, if $1 \neq A \subseteq H$, then $\mathcal{L}_G(A) \subseteq \mathcal{L}_G(H)$: thus, if $g^{-1}Ag = A$, then $1 \neq A \subseteq H \cap g^{-1}Hg$, whence $g^{-1}Hg = H$. Choose now an element $a \in H$ of prime order $p$. For any $b \notin H$ such that $|b| = p$, $\langle a \rangle = \langle a, b \rangle \cap H \mathcal{D} \langle a, b \rangle$; by lemma 2.1 either $|\langle a, b \rangle| = p^2$, or $\langle a, b \rangle$ is a non-abelian group of order $pq$ ($q$ a prime greater than $p$). In the first case $[a, b] = 1$, hence $b \in \mathcal{L}_G(H)$; moreover, for every $x \in H$, $\langle x \rangle = \langle x, b \rangle \cap H \triangleleft \langle x, b \rangle$, i.e. $b$ is in the normalizer of every subgroup of $H$. Since the same conclusion holds for $ab$, it would follow that $a$ is in the kernel of $H$, so that $\langle a \rangle \mathcal{D} H$; by IV of section 1 this would imply $\langle a \rangle \mathcal{D} G$ and $H = \langle a \rangle$,
which contradicts our assumption on the order of $H$. In the second case $\langle b \rangle$ and $\langle a \rangle$ are conjugate in $\langle b \rangle \cup H$; $b$ lies in a conjugate $H_1$ of $H_{1 \neq H}$ and $\langle b \rangle = (\langle b \rangle \cup H) \cap H_1 \leq \langle b \rangle \cup H$, whence again $\langle a \rangle \lhd \langle b \rangle \cup H$, $\langle a \rangle \lhd H$ and $H = \langle a \rangle$, thus contradicting our hypothesis on $|H|$. We can now prove that $H \triangleleft G$: let $g \in G$ have order $q^n$ with $q$ a prime number; if $q \mid |H|$, by a previous remark $\langle g^{q^n-1} \rangle \subseteq H$ and $g \in \mathcal{N}_G(H)$; if $q \nmid |H|$, for $a \in H$ of prime order we have $\langle g^{-1}ag \rangle \subseteq \langle a, g \rangle \cap H = \langle a \rangle$, and by the same remark $g \in \mathcal{N}_G(H)$. Moreover, in the latter case, for every $x \in H$ we get $\langle x \rangle = \langle x, g \rangle \cap H \triangleleft \langle x, g \rangle$ and, assuming $|x|$ to be a prime number, from $[x, g] \neq 1$ it would follow that $x$ too normalizes every subgroup of $H$, which clearly cannot happen; i.e. $g$ centralizes $H$. On the other hand in our hypothesis $Z(H) = 1 = H \cap \mathcal{C}_G(H)$, and if $|g| \mid |H|$ then $\langle g \rangle \cap H \neq 1$; all this implies that $\mathcal{C}_G(H)$ is exactly the set of all the elements of $G$ whose order is prime to $|H|$.

The above theorem does not cover the minimum $\mathcal{S}$-subgroups of prime order; they will be dealt with in the following

**Theorem 3.2.** Let $a \in G$ have prime order $p$. If $\langle a \rangle \mathcal{S} G$, then either i) $\langle a \rangle^G$ is an elementary abelian $p$-group, or ii) $G = S(N \times K)$, where $K = \mathcal{C}_G(\langle a \rangle^G)$ is a Hall subgroup of $G$, $N$ is an elementary abelian $q$-group with $q$ a prime greater than $p$, $S$ is a $p$-Sylow subgroup of $G$ which is cyclic or generalized quaternion, and $\langle a \rangle^G = \langle a \rangle N$ is a $P$-group.

Let us first show that if we can find in $G$ an element $b$ of order $p$ such that $\langle a \rangle \cap \langle b \rangle = 1$ but $[a, b] = 1$, then $a$ permutes with every element of order $p$ in $G$; hence it will follow that, if this is the case, $\langle a \rangle^G$ is elementary abelian. Thus, choose if possible $c \in G$ such that $|c| = p$, $[a, c] \neq 1$; by lemma 2.1 $\langle a, c \rangle = \langle a, d \rangle$ where $\langle d \rangle \triangleleft (\langle a, c \rangle$ and $|d| = q$ (notice that $a$ and $c$ are conjugate). If $[b, c] = 1$, then $\langle a, b, c \rangle = \langle a, c \rangle \times \langle b \rangle$, $|db| = pq$, whereas $|adb| = p$ and lemma 2.1 imply that no elements of composite order are in $\langle a, adb \rangle$, so that we can assume $[b, c] \neq 1$. $\langle b \rangle$ is then conjugate to $\langle c \rangle$, whence $\langle b \rangle \mathcal{S} G$. If $b \in \mathcal{N}_G(\langle d \rangle)$ we get $\langle a, c \rangle \leq \langle a, b, c \rangle = \langle a, c \rangle \times \langle b' \rangle$, where $b'$ is a suitable element of $\langle a \rangle \times \langle b \rangle$, and the above technique leads to a
contradiction. Lemma 2.2 now implies \( \langle b, d \rangle = N(d) \) with \( N \) an elementary abelian normal \( p \)-subgroup of \( \langle b, d \rangle \) which in turn is normal in \( \langle a, b, c \rangle \); \( a \notin N \), and for every \( x \in N \) we have \( \langle x \rangle = \langle x, a \rangle \cap N \triangleleft \langle x, a \rangle \), i.e. \( a \in C_G(N) \). Hence \( \langle a \rangle = \langle a, d \rangle \cap C_{\langle a, b, c \rangle}(N) \triangleleft \langle a, d \rangle \), thus contradicting an earlier statement. So far we proved that, if \( \langle a \rangle^G \) is not an elementary abelian \( p \)-group, then \([a, b] \neq 1\) for every \( b \in G \) such that \( |b| = p \), \( \langle a \rangle \cap \langle b \rangle = 1 \); as a consequence, all \( p \)-Sylow subgroups of \( G \) are either cyclic or generalized quaternion. We now proceed to show that for any pair \( x, y \) of elements of \( G \) such that \( |x| = |y| = p \), \( \langle x \rangle \cap \langle y \rangle = 1 \), the subgroup \( \langle x, y \rangle \) is non abelian and \( \langle x, y \rangle | = pq \), \( q \) being independent from the choice of \( x, y \); since there is in \( G \) just one class of conjugate subgroups of order \( p \), it is enough if we prove that \( |b| = |c| = p \), \( \langle a \rangle \cap \langle b \rangle = \langle a \rangle \cap \langle c \rangle = 1 \) implies \( |\langle a, b \rangle| = |\langle a, c \rangle| \). Let \( u \in G \) be such that \( \langle a, b \rangle = \langle a, u \rangle \), \( |u| = q \), \( \langle u \rangle \triangleleft \langle a, b \rangle \); \( c \in C_G(u) \) (were this not the case, by lemma 2.2 two independent conjugates of \( a \) would permute), hence \( \langle u \rangle \triangleleft \langle a, b, c \rangle = \langle u \rangle \langle a, c \rangle \). Looking at \( \langle a, c \rangle \), which by lemma 2.1 is also non abelian of order, say, \( pr \), we see that \( \langle a, c \rangle = \langle a, v \rangle \) where \( |v| = r \), \( \langle v \rangle \triangleleft \langle a, c \rangle \) and \( \langle v \rangle = \langle a, c \rangle \cap C_{\langle a, b, c \rangle}(u) \), so that \( \langle a, b, c \rangle = (\langle u \rangle \times \langle v \rangle) \langle a \rangle \). The subgroups \( \langle au \rangle \), \( \langle av \rangle \), being conjugate to \( \langle a \rangle \), are dual-Dedekind in \( G \); by lemma 2.1 no element of composite order lies in \( \langle au, av \rangle \), hence \( v^{-1}u \in \langle au, av \rangle \) has prime order: but then \( q = r \) (notice that we have also proved that every element of order \( p \) normalizes every subgroup of order \( q \)). By an easy induction argument one can now prove that any set of elements of order \( p \) generates a \( P \)-group of order \( pq^n \) for a suitable \( n \), so that \( \langle a \rangle^G \), which is generated by all such elements of \( G \), is a \( P \)-group: \( \langle a \rangle^G = \langle a \rangle N \), with \( N \) an elementary abelian \( q \)-subgroup on which \( a \) induces a non identity power automorphism. Our next step is to prove that for every pair \( x, y \) of elements of \( G \) such that \( |x| = q^m \), \( |y| = p \), one has \( \langle x \rangle \triangleleft \langle x, y \rangle \); by an earlier remark we can assume \( m > 1 \) and use induction. \( \langle x^q \rangle \) is then normal in \( \langle y, x^q \rangle \); \( \langle y, x^q \rangle / \langle x^q \rangle \triangleleft \langle y, x \rangle / \langle x^q \rangle \): if \( \langle y, x \rangle / \langle x^q \rangle | = pq \) we are through. If this is not the case, then \( \langle y, x \rangle / \langle x^q \rangle = \langle (x^q) \rangle \cap (N / \langle x^q \rangle) \) with \( N / \langle x^q \rangle \) an elementary abelian normal \( p \)-subgroup of \( \langle y, x \rangle / \langle x^q \rangle \) (lemma 2.2); \( \langle x^q \rangle \) is a \( q \)-Sylow subgroup of \( N \), whence \( N = M \langle x^q \rangle \) for a suitable elementary abelian \( p \)-subgroup \( M \) containing \( \langle y \rangle \). But then \( M = \langle y \rangle \) and again \( \langle y, x \rangle / \langle x^q \rangle | = pq \). For a conjugate \( b \) of \( a \) such that \( \langle b \rangle \cap \langle a \rangle = 1 \) one has either \( \langle x \rangle \cap \langle a, b \rangle = 1 \), which
implies \([a, x] \neq 1\); or \(\langle x \rangle \cap \langle a, b \rangle = 1\), and \(\langle x, a, b \rangle = \langle x \rangle \times \langle u \rangle (a)\), where \(\langle u \rangle\) is the \(q\)-Sylow subgroup of \(\langle a, b \rangle\), and \(a\) induces a non identity power automorphism on \(\langle x \rangle \times \langle u \rangle\), whence again \([a, x] \neq 1\); but then \(\langle x \rangle = \langle [a, x] \rangle \leq \langle a \rangle^G\), i.e. \(N\) is the (unique) \(q\)-Sylow subgroup of \(G\). Now put \(K = \mathcal{O}_c(\langle a \rangle^G)\); \(K \cap \langle a \rangle^G = 1\) and, since a \(p\)-Sylow subgroup is either cyclic or generalized quaternion and its subgroup of order \(p\) lies in \(\langle a \rangle^G\), \(K \subseteq \{ g \in G \mid (|g|, pq) = 1 \}\). On the other hand, if \((|g|, pq) = 1\), for every \(y \in G\) with \(|y| = p\) one has \(\langle y \rangle = \langle y, g \rangle \cap \langle a \rangle^G \leq \langle y, g \rangle\); therefore \(g\) is in the normalizer of every subgroup of order \(pq\) in \(\langle a \rangle^G\): but this implies \([g, \langle a \rangle^G] = 1\), which concludes the proof of the theorem.

The following result is a trivial corollary to theorems 3.1, 3.2:

**Theorem 3.3.** Let \(G\) be a finite group. If \(G\) has non-trivial dual-Dedekind subgroups, then \(G\) is not simple.

**Remark.** Finite non simple groups with no non-trivial \(\mathcal{D}\)-subgroups do exist: e.g. the symmetric group \(S_n\) is such whenever \(n > 3\) (it is a simple matter to verify that no normal subgroup of \(S_n\) satisfies the theorems 3.1, 3.2); the case \(n = 4\) provides an example of a soluble group which has no non-trivial \(\mathcal{D}\)-subgroups.

4. We have already pointed out that, generally speaking, normal subgroups need not be \(\mathcal{D}\)-subgroups; in order to evaluate, in a sense, the gap between these two classes we proceed to study the groups where every normal subgroup is also a \(\mathcal{D}\)-subgroup (in the main result of this section we restrict ourselves to soluble groups).

**Proposition 4.1.** Assume that every normal subgroup of the group \(G\) is a \(\mathcal{D}\)-subgroup of \(G\). If \(N \triangleleft G\), then every normal subgroup of \(G/N\) is a \(\mathcal{D}\)-subgroup of \(G/N\).

Thus, \(K/N \triangleleft G/N\) implies \(K \triangleleft G\), \(K \mathcal{D} G\), hence \(K \mathcal{D}[G/N]\) and obviously \(K/N \mathcal{D} G/N\).

**Proposition 4.2.** Let \(N\) be a minimum normal subgroup of \(G\). If every normal subgroup of \(G\) is also a \(\mathcal{D}\)-subgroup, then \(N\) is simple.

Assume first that \(N\) is abelian; then \(|N| = p^\alpha\) with \(p\) a prime and \(\alpha \geq 1\); the number \(k\) of its subgroups of order \(p\) is congruent to 1
(mod $p$). The normal subgroup $P = \cap \{ \triangledown_C(H) \mid H \triangleleft N, \mid H \mid = p \}$ contains every element of $G$ whose order is prime to $p$: thus, if $(\mid x \mid, p) = 1$, then $\langle x \rangle \cap N = 1$ and, for any such an $H$, $H = (H \cup \langle x \rangle) \cap N \triangleleft H \cup \langle x \rangle$. So $G$ acts as a $p$-group of permutations on the set of the $k$ subgroups of order $p$ in $N$, hence it has at least a fixed point, i.e. $\mid N \mid = p$.

Assume now that $N$ is abelian; let $N_1$ be a simple direct factor of $N$. If $N_1 \neq N$ and $x \in G$ is such that $x^{-1}N_1x \neq N_1$, then $N_1 \times x^{-1}N_1x \subseteq \langle N_1 \cup \langle x \rangle \rangle \cap N = N_1(\langle x \rangle \cap N)$ and $x^{-1}N_1x$ would be isomorphic to a subgroup of $\langle x \rangle$, which is clearly not the case.

**COROLLARY 4.3.** Let $G$ be a soluble group. If every normal subgroup of $G$ is a $\tilde{D}$-subgroup of $G$, then $G$ is supersoluble.

**PROPOSITION 4.4.** Let $G$ be a nilpotent group. If $H \tilde{D} G$, then $H$ is quasi-normal in $G$.

This is a trivial consequence of a result of Napolitani, [1].

**PROPOSITION 4.5.** Let $G$ be a $p$-group ($p$ a prime). If every normal subgroup of $G$ is a $\tilde{D}$-subgroup, then $G$ is modular.

For $u \in Z(G)$, with $\mid u \mid = p$, $G/\langle u \rangle$ is by induction a modular $p$-group. Assume that $G/\langle u \rangle$ is either abelian or Hamiltonian: for arbitrary $x \in G$, $\langle x, u \rangle$ is abelian, hence $\langle x \rangle \tilde{D} \langle x, u \rangle$; moreover $\langle x, u \rangle \triangleleft G$ implies $\langle x, u \rangle \tilde{D} G$ and $\langle x \rangle \tilde{D} G$; by proposition 4.4 $\langle x \rangle$ is a quasi-normal subgroup of $G$, i.e. $G$ is modular. We may then assume that $G/\langle u \rangle$ is neither abelian nor Hamiltonian, so that $G = \langle t, A \rangle$ with $u \in A$, $A/\langle u \rangle$ abelian, $t^{-1}at = a^{1 + p^su^{s(a)}}$ for every $a \in A$ and suitable $s(a)$, $s \geq 2$ if $p = 2$ ([3], p. 13). Just as before one sees that every subgroup of $A$ is dual-Dedekind, whence quasi-normal, in $G$; it follows that $A$ is a modular group. Moreover $A^p = \{ a^p \mid a \in A \}$ is a subgroup of $A$, any of whose subgroups is normalized by $t$; $t^p$ normalizes every subgroup of $A$, inducing on every cyclic subgroup a power automorphism which is congruent to 1 (mod. $p$), and congruent to 1 (mod. 4) if $p = 2$. $A$ cannot be a Hamiltonian group: thus, if $A = Q \times B$ with $Q$ a quaternion group of order 8 and $B^2 = 1$, from $u \in Q$ it would follow that $G/\langle u \rangle$ is abelian, whereas, if $u \notin Q$, $G/\langle u \rangle$ would be a modular 2-group containing a quaternion group, and $G/\langle u \rangle$ would be a Hamiltonian group. There are two cases left:
i) $A$ is abelian. By a previous remark, $\langle t^p, A \rangle$ is modular and all its subgroups are quasi-normal in $G$. Let $y \in G$ be such that $y \notin \langle t^p, A \rangle$, so that $G = A \langle y \rangle$. If $\langle y \rangle \cap A \neq 1$, since $\langle y \rangle \cap A \triangleleft G$, then by induction $G/\langle y \rangle \cap A$ is modular, hence $\langle y \rangle$ is quasi-normal in $G$. Assume now $\langle y \rangle \cap A = 1$: for every $a \in A$ we get $\langle a \rangle = \langle a, y \rangle \cap A \triangleleft \langle a, y \rangle$, i.e. $y$ induces a power automorphism on the abelian group $A$, which is congruent to $1$ (mod. $p$). If $p \neq 2$ there is nothing more to prove; if $p = 2$ we remark that, if we had $A^4 = 1$, $G/\langle u \rangle$ would be abelian; hence $A^4 \neq 1$, $G/A^4$ is by induction a modular group, and the power induced by $y$ is congruent to $1$ (mod. $4$), which implies that $G$ is modular.

ii) $A$ is neither abelian nor Hamiltonian. We have $A = \langle v, B \rangle$, $B$ abelian, $v^{-1}xv = x^n$ with $n = 1$ (mod. $p$) for every $x \in B$ and $n$ independent from the choice of $x$ ($n = 1$ (mod. $4$) if $p = 2$; we remark here that $B^4 \neq 1$, otherwise $A$ would be abelian). $A^p \subseteq Z(A)$, hence every subgroup of $A^p$ is normal in $G$; both of $A/A^p$ and $A/B$ are abelian, so that $\langle u \rangle = A' \subseteq A^p \cap B$; moreover, we can write $B$ as $B = \langle b \rangle \times B_1$ where $u \in \langle b \rangle$, $\exp B_1 < |b|$ and $|b| \geq 8$ if $p = 2$. We will show that $\langle g_1, g_2 \rangle = \langle g_1 \rangle \langle g_2 \rangle$ for every pair $g_1, g_2$ of elements of $G$ (without loss of generality, we can assume $g_i \notin A$, since every subgroup of $A$ is quasi-normal in $G$). Write $\langle g_1 \rangle = \langle a_1 t^{p^k} \rangle$, $\langle g_2 \rangle = \langle a_2 t^{p^k} \rangle$; assuming $0 \leq h \leq k$ we get $g_2 \in A\langle g_1 \rangle$, $\langle g_1, g_2 \rangle = \langle g_1, a \rangle$ for suitable $a_1, a_2, a \in A$. Should $\langle g_1 \rangle$ contain a non-identity normal subgroup $K$ of $G$, since $G/K$ would be a modular group by the induction hypothesis, then $\langle g_1 \rangle$ would be quasi-normal in $G$; hence we can assume $\langle g_1 \rangle \cap A^p = 1$, which implies $u \notin \langle g_1 \rangle$. Suppose $\langle g_1 \rangle \cap A = 1$; then $\langle a \rangle = \langle a, g_1 \rangle \cap A \triangleleft \langle a, g_1 \rangle$, and, if $p \neq 2$, $\langle a, g_1 \rangle$ is modular, whence $\langle g_1, g_2 \rangle = \langle g_1 \rangle \langle g_2 \rangle$. Under the same assumptions, but with $p = 2$, $g_1$ induces a power automorphism on the abelian group $B$; $G/B^4$ being modular, this power is congruent to $1$ (mod. $4$), so that if $a \in B$ then $\langle a, g_1 \rangle$ is modular. Let now $a \notin B$; $u \in \langle a, g_1 \rangle$ if and only if $u \notin \langle a \rangle$, hence if either $u \notin \langle a \rangle$ or $u \in \langle a^p \rangle$ we again conclude that $\langle a, g_1 \rangle$ is modular; we are left with one more possibility: $u = a^2 = b^{2l}$; but $[g_1, b^{2l-1}] = 1$ (for $g_1, b$ is modular), $[g_1, ab^{2l-1}] = 1$ since $|ab^{2l-1}| = 2$, so that $\langle g_1, a \rangle$ is abelian. Assume now $1 = \langle g_1 \rangle \cap A^p \subseteq \langle c \rangle \cap A = \langle c \rangle$ with $|c| = p$, $uc \in \langle a, g_1 \rangle \cap A = \langle a, c \rangle$; if $|a| \neq p$ then $\langle g_1 \rangle \triangleleft \langle a, g_1 \rangle = \langle g_1 \rangle \langle g_2 \rangle$; if $|a| > p$ but $u \notin \langle a \rangle$ we should have $c \in \langle a \rangle \times \langle u \rangle$, whence $c \in \langle a^p \rangle \times \langle u \rangle \subseteq A^p$, contradicting an earlier
hypothesis. We have then $|a| \geq p, u \in \langle a \rangle$: so $\langle a \rangle \triangleleft G$ and, if either $p \neq 2$ or $p = 2$, $u \in \langle a^4 \rangle, \langle g_1, a \rangle$ is modular. It follows that we are left with one last case: $p = 2, u = a = b^2$. Since $\langle b \rangle \triangleleft G, u \in \langle b^4 \rangle$ and $G/\langle u \rangle$ is modular, we see that $\langle g_1, b \rangle$ is also modular, whence $[g_1, b^{2-1}] = 1$; if $a \in \langle b \rangle$, $\langle g_1, a \rangle$ is abelian, whereas, if $a \notin \langle b \rangle$, $b_{2-1} \in Z(G)$ and finally $\langle g_1 \rangle \triangleleft \langle g_1, g_2 \rangle \leq \langle g_1, b^{2-1}, a^{-1}b^{2-1} \rangle$, which disposes of the case and ends the proof.

**Theorem 4.6.** The group $G$ is soluble and every normal subgroup of $G$ is dual-Dedekind in $G$ if and only if $G = H_1 \times H_2 \times \ldots \times H_t$ with $H_i$ a Hall subgroup of $G$ $(i = 1, \ldots, t)$ and either

1) $H_i$ is a modular $p$-group; or

2) $H_i = (P_{i1} \times \cdots \times P_{is_i})Q_i$ with $P_{ij}$, $Q_i$ Sylow subgroups of $G$ for different primes, $P_{ij}$ abelian of odd order $(j = 1, \ldots, s_i), Q_i = \langle b_i \rangle$, and $b_i$ inducing a non-identity power automorphism on each $P_{ij}$.

**Proof of necessity.** Assume $S$, a $p$-Sylow subgroup of $G$ for some prime $p$, is normal in $G$; then, unless $S$ is a direct factor of $G$, $S \subseteq \Gamma_\omega(G)$ where $\Gamma_\omega(G)$ denotes the intersection of all normal subgroups of $G$ whose factor group is nilpotent. Thus $S \triangleleft G$ and for $a \in S, x \in G$ such that $(|x|, p) = 1$ we have $\langle a \rangle = \langle a \rangle \cup (\langle x \rangle \cap S) = \langle a, x \rangle \cap S \triangleleft \langle a, x \rangle$; if $S$ is not a direct factor of $G$, we can choose $a, x$ such that $[a, x] \neq 1$, but then $([a, x]) = \langle a \rangle$ and $a$ also induces a power automorphism on $S$. Let now $b$ be arbitrary in $S$; if $[b, x] \neq 1$ the above argument shows that $b$ operates on $S$ as a power automorphism, whereas if $[b, x] = 1$ we have $[ab, x] \neq 1$ and the same conclusion holds for $ab$, hence for $b$. It follows that $S$ is abelian of odd order, $x^{-1}yx = y^r$ with $r \equiv 1 \pmod{p}$, $r$ independent from the choice of $y \in S, [G, S] = S$ and $S \subseteq \Gamma_\omega(G)$. Choosing for $p$ the maximum prime divisor of $|G|$, by the supersolubility of $G$ the $p$-Sylow subgroup is certainly normal, so that an easy induction proves that $\Gamma_\omega(G)$ is a Hall subgroup of $G$. Moreover $G$ has a normal 2-complement whose quotient group is clearly nilpotent, so that $|\Gamma_\omega(G)|$ is odd; again, by the supersolubility of $G$, $\Gamma_\omega(G)$ is nilpotent, hence it is a direct product of normal Sylow subgroups of $G$ which are all abelian by the preceding remark, and every element of $G$ operates by conjugation on $\Gamma_\omega(G)$ as a power automorphism. $G/\Gamma_\omega(G)$ is a direct
product of modular $p$-groups for different primes; notice that every Sylow subgroup of $G$ which is a direct factor has trivial intersection with $\Gamma_\omega(G)$, and is modular; therefore, we can factor out all such subgroups, and write $G = T \times G_1$ with $T$ a modular, nilpotent, Hall subgroup of $G$ and $G_1$ also satisfying all our assumptions; from now on we shall assume $G = G_1$. Let $P$ be a normal Sylow subgroup of $G$; we have already seen that $P \subseteq \Gamma_\omega(G)$ and that every element of $G$ operates on $P$ as a power automorphism; we claim that $G/C_\omega(P)$ is a (cyclic) group of prime power order. Deny: then there are a $q$-Sylow subgroup $Q$ and an $r$-Sylow subgroup $R$ of $G$ such that $[Q, R] = 1$, $Q \cap \Gamma_\omega(G) = R \cap \Gamma_\omega(G) = 1$, $[Q, P] = [R, P] = P$; choose $a \in Q$, $b \in R$, $a \in P$ such that $[a, P] \neq 1$, $[b, P] \neq 1$, $|u| = p (p \mid |P|)$. The Hall subgroup $Q \Gamma_\omega(G)$ is normal, hence dual-Dedekind, in $G$, which is a contradiction to $\langle au \rangle = \langle au \rangle \cup \langle b \rangle \cap Q \Gamma_\omega(G) = \langle \langle au \rangle \cup \langle b \rangle \rangle \cap Q \Gamma_\omega(G)$ (this owing to the fact that the former group has $q$-power order, whereas the latter contains $\langle u \rangle = \langle [au, b] \rangle$ which has order $p$). Therefore we get $G = QC_\omega(P)$ for a suitable $q$-Sylow subgroup $Q$ of $G$; we shall now prove, by induction on $q^b = |Q|$, that $Q$ is cyclic. Without loss of generality we can assume $P = \Gamma_\omega(G)$ (were this not the case, we would work on $G/C$ with $C$ the complement of $P$ in $\Gamma_\omega(G)$). If $Q \cap C_\omega(P) = 1$, since $G/C_\omega(P)$ is cyclic, then $Q$ is also cyclic. Assume then $Q \cap C_\omega(P) \neq 1$; $C_\omega(P) \cap Z(Q)$ is a non-trivial normal subgroup of $G$ and by the inductive hypothesis $Q/C_\omega(P) \cap Z(Q)$ is cyclic; therefore $Q$ is abelian and all subgroups of $Q \cap P$ containing $P$ are normal, hence dual-Dedekind subgroups of $G$. If now $Q$ were not cyclic we could pick $a$ and $b$ in $Q$ in such a way that $a \in C_\omega(P)$, $a^q \in C_\omega(P)$, $b \in Q$, $|b| = q$, $[b, P] = 1$, $\langle a \rangle \cap \langle b \rangle = 1$; for $u \in P$ with $|u| = p$ we would have $\langle au \rangle = \langle au \rangle \cup \langle ab \rangle \cap \langle a \rangle P = \langle \langle au \rangle \cup \langle ab \rangle \rangle \cap \langle a \rangle P \supseteq \langle [au, ab] \rangle = \langle [u, a] \rangle = \langle u \rangle$ i.e. $[u, a] = 1$ contrary to our choice of $a$. Now let $Q_1$ be a non normal Sylow subgroup of $G$, and let $P_{11}, P_{12}, \ldots, P_{1s_1}$ be those Sylow subgroups of $\Gamma_\omega(G)$ which are not centralized by $Q_1$; $H_1 = (P_{11} \times \ldots \times P_{1s_1})Q_1$ is a direct factor of $G$, and if $G = H_1$ the theorem is proved. Assume $G \neq H_1$; let $Q_2$ be a normal Sylow subgroup of $G$, not contained in $H_1$, and let $P_{21}, \ldots, P_{2s_2}$ be those Sylow subgroups of $\Gamma_\omega(G)$ which are not centralized by $Q_2$: $H_2 = (P_{21} \times \ldots \times P_{2s_2})Q_2$ is also a direct factor of $G$, and $H_1 \cap H_2 = 1$; in this way we clearly get a decomposition of $G$ as a direct product of factors of the prescribed type.
PROOF OF SUFFICIENCY. Since such a decomposition as is described in the theorem is both group- and lattice-theoretical, it will be enough if we prove the theorem for each one of the factors (nothing is to be proved for the modular ones). Without loss of generality, we can assume $G = (P_1 \times \ldots \times P_s)Q$ where the $P_i$'s and $Q$ are Sylow subgroups of $G$, $Q$ is cyclic, $P_i$ is abelian of odd order ($i = 1, \ldots, s$) and $Q$ operates on $P_1 \times \ldots \times P_s$ as a group of power automorphisms, with $\mathbb{C}_O(P_i) \neq Q$. Let $H \trianglelefteq G$; we renumber the $P_i$'s so that $[H, P_i] = P_i$ for $i = 1, \ldots, r$ and $[H, P_i] = 1$ for $i = r + 1, \ldots, s$. We shall prove that $\varphi^K : X \to X \cup K$ ($\varphi^K : [H/H \cap K] \to [HK/K]$) and $\varphi_H : Y \to Y \cap H$ ($\varphi_H : [HK/K] \to [H/H \cap K]$) are inverse lattice isomorphisms, whenever $K$ is a subgroup of $G$; since $H \trianglelefteq G$, we have only to prove that $X \varphi^K \varphi_H = X$ for every $X \in [H/H \cap K]$. Assume first that

$$K \subseteq (P_1 \times \ldots \times P_s)H = (H \cap Q)(P_1 \times \ldots \times P_r) \times (P_{r+1} \times \ldots \times P_s);$$

we have

$$K = (K \cap (H \cap Q)(P_1 \times \ldots \times P_r)) \times (K \cap (P_{r+1} \times \ldots \times P_s)) = (H \cap K)L$$

with $L = K \cap (P_{r+1} \times \ldots \times P_s) \trianglelefteq G$. We have $H \cup K = H \cup (H \cap K) \cup L = H \cup L$ and, for every $X \in [H/H \cap K]$,

$$X \varphi^K \varphi_H = (X \cup K) \cap H = (X \cup (H \cap K) \cup L) \cap H =$$

$$= (X \cup L) \cap H = X \cup (L \cap H) = X.$$ 

Assume now that $K \subseteq (P_1 \times \ldots \times P_s)H$; there exists a $q$-Sylow subgroup $T$ of $G$ with $T \cap K$ $q$-Sylow in $K$; we have $T \cap H \subseteq T \cap K$. If we call $M = H \cap (P_1 \times \ldots \times P_s)$, then $H = M(T \cap H) = M(H \cap K)$; notice that, since every subgroup of $M$ is normal in $G$, $M \trianglelefteq G$. Now for every $X \in [H/H \cap K]$ we get $X = (X \cap M) \cup (H \cap K)$ and

$$(X \cup K) \cap H = X \varphi^K \varphi_H = ((X \cap M) \cup (H \cap K) \cup K) \cap H =$$

$$=((X \cap M) \cup K) \cap H = (X \cap M) \cup (H \cap K) = X,$$

thus ending our proof.
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