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DUAL-DEDEKIND SUBGROUPS IN FINITE GROUPS

Federico Menegazzo *)

If $G$ is a group and $H$ is a subgroup of $G$, $H$ is dual-Dedekind in $G$, or a $\mathfrak{D}$-subgroup of $G$ (written $H\mathfrak{D}G$) if the following conditions are fulfilled:

i) $X \supseteq Y \Rightarrow (Y \cup H) \cap X = Y \cup (H \cap X)$

ii) $H \supseteq Y \Rightarrow (Y \cup X) \cap H = Y \cup (X \cap H)$

for every pair $X, Y$ of subgroups of $G$ (for the dual notion, namely that of Dedekind subgroups, there called « modular subgroups », see [4]).

In this paper we are particularly concerned with the properties of « minimum » $\mathfrak{D}$-subgroups (i.e. minimal in the set of non identity dual-Dedekind subgroups of a given group $G$); we establish some necessary conditions in order that a finite group $G$ has non-trivial (i.e. different from 1, $G$) $\mathfrak{D}$-subgroups. From these it will follow that a finite group having non-trivial $\mathfrak{D}$-subgroups cannot be simple (Theorem 3.3) — a similar result for Dedekind subgroups is proved in [2]; it is perhaps worth noting that the converse is false: $G$ non-simple is not a sufficient condition for $G$ to have a non-trivial $\mathfrak{D}$-subgroup. The proposition « if $N \trianglelefteq G$, then $N\mathfrak{D}G$ » for arbitrary $G$ is false; in the second half of the paper we determine all finite soluble groups where such a condition holds.

The main result in this section is (Theorem 4.6): $G$ is soluble and every

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normal subgroup of \( G \) is dual-Dedekind in \( G \) iff \( G = H_1 \times H_2 \times \ldots \times H_t \) with each \( H_i \) a Hall subgroup of \( G \) and either

i) \( H_i \) is a modular \( p \)-group; or

ii) \( H_i = (P_{i1} \times \ldots \times P_{i_s})Q_i \) with \( P_{ij}, Q_i \) Sylow subgroups of \( G \) for different primes, \( P_{ij} \) abelian of odd order \((j = 1, \ldots, s_i)\), \( Q_i = \langle b_i \rangle \) with \( b_i \) inducing a non-identity power automorphism on each \( P_{ij} \).

1. Let \( L \) be a lattice. An element \( a \in L \) is a dual-Dedekind element of \( L \) (a \( \bar{D} \)-element of \( L \), \( a\bar{D}L \)) if

i) \( x \geq y \Rightarrow (y \cup a) \cap x = y \cup (a \cap x) \);

ii) \( a \geq y \Rightarrow (y \cup x) \cap a = y \cup (x \cap a) \)

hold for every pair \((x, y)\) of elements in \( L \). Notice that \( a\bar{D}L \) if and only if \( a \) is a Dedekind element in \( \bar{L} \) (the dual lattice of \( L \)); hence properties of \( \bar{D} \)-elements of \( L \) are properties of Dedekind elements of \( \bar{L} \). We shall use mostly (see [4]):

I) \( a\bar{D}L \) iff for every \( b \in L \) the maps

\[
\varphi_b : x \rightarrow x \cup b \quad \varphi^b : [a/a \cap b] \rightarrow [a \cup b/b]
\]

\[
\varphi_a : y \rightarrow y \cap a \quad \varphi_a : [a \cup b/b] \rightarrow [a/a \cap b]
\]

are inverse lattice-isomorphisms.

II) if \( a\bar{D}L \) and \( b \in L \), then \( a \cap b\bar{D}(b) \).

III) if \( a\bar{D}L \) and \( a_2\bar{D}L \), then \( a_1 \cap a_2\bar{D}L \).

IV) if \( a\bar{D}L \) and \( a_2\bar{D}(a_1) \), then \( a_2\bar{D}L \).

V) whenever \( \varphi \) is a surjective lattice isomorphism of \( L \) onto \( L' \), we have \( a\varphi\bar{D}L' \) iff \( a\bar{D}L \).

A subgroup \( H \) of a group \( G \) is dual-Dedekind in \( G \) (\( H \) is a \( \bar{D} \)-subgroup of \( G \), \( H\bar{D}G \)) if \( H \) is a \( \bar{D} \)-element of the lattice \( \mathcal{L}(G) \) of all subgroups of \( G \). Normal subgroups are usually not \( \bar{D} \)-subgroups; the following are
dual-Dedekind subgroups in any group $G$:

- $a$) $G$ and the identity subgroup 1 of $G$ (they will be referred to as the « trivial » $\mathfrak{D}$-subgroups);
- $b$) the subgroups of the centre $Z(G)$;
- $c$) all normal cyclic subgroups;
- $d$) the subgroups of the so-called « kernel »

$$K(G) = \cap \{ \mathfrak{C}_c(X) \mid X \in \mathfrak{L}(G) \}.$$ 

All definitions and notations will be standard; throughout this paper « group » means « finite group ».

2. The existence of non-trivial $\mathfrak{D}$-elements in $\mathfrak{L}(G)$ rather severely restricts the structure of $G$. The following two lemmas provide examples where, in very simple cases, the structure of $G$ is completely determined.

**Lemma 2.1.** Assume $G = \langle a, b \rangle$, $|a| = |b| = p$, $p$ a prime, $G$ non-cyclic. Then $\langle a \rangle \mathfrak{D} G$ iff either $|G| = p^2$ or $G$ is a non-abelian group of order $pq$ ($q$ a prime greater than $p$).

Since in both cases $\mathfrak{L}(G)$ is a modular lattice, the sufficiency of the condition is obvious. The condition is also necessary: the intervals $[(a, b)/\langle b \rangle]$ and $[(a)/\langle a \rangle \cap \langle b \rangle]$ are isomorphic, hence $\langle b \rangle$ is a maximal subgroup of $G$ and, if $G$ is a $p$-group, then $|G| = p^2$. Assume that $G$ is not a $p$-group; then $\langle b \rangle$ is a cyclic $p$-Sylow subgroup, $\langle b \rangle$ is its own normalizer in $G$ and has a normal complement $N$. $\langle a \rangle$, as a conjugate of $\langle b \rangle$, is maximal in $G$ and $\langle a \rangle \cap N = 1$; for a $c \in N$ of prime order $q$ one has $N = (\langle a \rangle \cup \langle c \rangle) \cap N = \langle c \rangle \cup (\langle a \rangle \cap N) = \langle c \rangle$, and the conclusion follows.

**Lemma 2.2.** Assume $G = \langle a, b \rangle$, $|a| = p$, $|b| = q$, $p, q$ different prime numbers. Then $\langle a \rangle \mathfrak{D} G$ iff either $|G| = pq$, or $G$ contains an elementary abelian $p$-subgroup $N < G$ such that $G = N \langle b \rangle$, and $\langle b \rangle$ operates irreducibly on $N$.

If $\langle a \rangle \mathfrak{D} G$, then $\langle b \rangle$ is a maximal subgroup of $G$ and from $\langle b \rangle < G$ follows $|G| = pq$. If $\langle b \rangle$ is not normal, then $\langle b \rangle$ is a $q$-Sylow subgroup.
which is its own normalizer in $G$, hence $\langle b \rangle$ has a normal complement $N$. Since any conjugate of $\langle b \rangle$ is maximal in $G$, no proper non-trivial subgroup of $N$ is normalized by $b$ nor by a conjugate of $b$. Let $S$ be a $p$-Sylow subgroup of $N$ containing $\langle a \rangle$; by the Frattini argument $G=NG_{G}(S)$, hence $G_{G}(S)$ contains a conjugate of $\langle b \rangle$, and $S=N$. The Frattini subgroup $\Phi(N) \neq N$, $\Phi(N) \triangleleft G$, so that $\Phi(N)=1$ and the «only if» part is proved. Conversely, neglecting the case $|G|=pq$ where everything is obvious, we have to prove that if $1 \neq a \in N$ and $X \in \mathcal{L}(G)$ is arbitrary, $a \notin X$ implies that $X$ is maximal in $\langle a, X \rangle$. But $X \subseteq N$ implies that $\langle a, X \rangle$ is abelian, whereas if $X \nsubseteq N$ a conjugate of $\langle b \rangle$, say $\langle c \rangle$, lies in $X$, hence $X=\langle c \rangle$ is a maximal subgroup of $G=\langle a, c \rangle$.

3. Definition. Let $H$ be a subgroup of the group $G$. We shall say that $H$ is a minimum $\mathfrak{D}$-subgroup of $G$ if $H$ is minimal in the set of all non-identity $\mathfrak{D}$-subgroups of $G$.

**Theorem 3.1.** Let $H$ be a minimum $\mathfrak{D}$-subgroup of $G$. If $|H|$ is not a prime number, then

i) $H$ is normal in $G$;

ii) for every prime number $p$ dividing $|H|$, all the elements of $G$ of order $p$ are in $1$-1; and

iii) $\mathfrak{C}_{G}(H) = \{ x \in G \mid (|x|, |H|) = 1 \}$.

First of all, notice that the minimality of $H$ and III, V of section 1 imply that for every $g \in G$ either $g^{-1}Hg \cap H = 1$ or $g^{-1}Hg = H$; furthermore, if $1 \neq A \subseteq H$, then $\mathfrak{N}_{G}(A) \subseteq \mathfrak{N}_{G}(H)$: thus, if $g^{-1}A g = A$, then $1 \neq A \subseteq H \cap g^{-1}Hg$, whence $g^{-1}Hg = H$. Choose now an element $a \in H$ of prime order $p$. For any $b \notin H$ such that $|b| = p$, $\langle a, b \rangle \cap H \triangleleft \langle a, b \rangle$; by lemma 2.1 either $|\langle a, b \rangle| = p^2$, or $\langle a, b \rangle$ is a non-abelian group of order $pq$ ($q$ a prime greater than $p$). In the first case $[a, b] = 1$, hence $b \in \mathfrak{N}_{G}(H)$; moreover, for every $x \in H$, $\langle x \rangle = \langle x, b \rangle \cap H \triangleleft \langle x, b \rangle$, i.e. $b$ is in the normalizer of every subgroup of $H$. Since the same conclusion holds for $ab$, it would follow that $a$ is in the kernel of $H$, so that $\langle a \rangle \triangleleft H$; by IV of section 1 this would imply $\langle a \rangle \triangleleft G$ and $H = \langle a \rangle$,
which contradicts our assumption on the order of \( H \). In the second case \( \langle b \rangle \) and \( \langle a \rangle \) are conjugate in \( \langle b \rangle \cup H \); \( b \) lies in a conjugate \( H_1 \) of \( H \) \((H_1 \neq H)\) and \( \langle b \rangle = \langle (b) \cup H \rangle \cap H_1 \bar{\Delta} \langle b \rangle \cup H \), whence again \( \langle a \rangle \bar{\Delta} \langle b \rangle \cup H \), \( \langle a \rangle \bar{\Delta} H \) and \( H = \langle a \rangle \), thus contradicting our hypothesis on \( |H| \). We can now prove that \( H \triangleleft G \): let \( g \in G \) have order \( q^n \) with \( q \) a prime number; if \( q \mid |H| \), by a previous remark \( \langle g^{q^{n-1}} \rangle \subseteq H \) and \( g \in \mathfrak{S}_d(H) \); if \( q \nmid |H| \), for \( a \in H \) of prime order we have \( \langle g^{-1}ag \rangle \subseteq \langle a, g \rangle \cap H = \langle a \rangle \), and by the same remark \( g \in \mathfrak{S}_d(H) \). Moreover, in the latter case, for every \( x \in H \) we get \( \langle x \rangle = \langle x, g \rangle \cap H \triangleleft \langle x, g \rangle \) and, assuming \( |x| \) to be a prime number, from \( [x, g] \neq 1 \) it would follow that \( x \) too normalizes every subgroup of \( H \), which clearly cannot happen; i.e. \( g \) centralizes \( H \). On the other hand in our hypothesis \( Z(H) = 1 = H \cap \mathfrak{C}_d(H) \), and if \( |g|, |H| \neq 1 \) then \( \langle g \rangle \cap H \neq 1 \); all this implies that \( \mathfrak{C}_d(H) \) is exactly the set of all the elements of \( G \) whose order is prime to \( |H| \).

The above theorem does not cover the minimum \( \bar{\Delta} \)-subgroups of prime order; they will be dealt with in the following

**Theorem 3.2.** Let \( a \in G \) have prime order \( p \). If \( \langle a \rangle \bar{\Delta} G \), then either i) \( \langle a \rangle^G \) is an elementary abelian \( p \)-group, or ii) \( G = \Phi(N \times K) \), where \( K = \mathfrak{C}_d(\langle a \rangle^G) \) is a Hall subgroup of \( G \), \( N \) is an elementary abelian \( q \)-group with \( q \) a prime greater than \( p \), \( S \) is a \( p \)-Sylow subgroup of \( G \) which is cyclic or generalized quaternion, and \( \langle a \rangle^G = \langle a \rangle N \) is a \( P \)-group.

Let us first show that if we can find in \( G \) an element \( b \) of order \( p \) such that \( \langle b \rangle \cap \langle a \rangle = 1 \) but \([a, b] = 1\), then \( a \) permutes with every element of order \( p \) in \( G \); hence it will follow that, if this is the case, \( \langle a \rangle^G \) is elementary abelian. Thus, choose if possible \( c \in G \) such that \( |c| = p \), \([a, c] \neq 1\); by lemma 2.1 \( \langle a, c \rangle = \langle a, d \rangle \) where \( \langle d \rangle \triangleleft \langle a, c \rangle \) and \( |d| = q \) (notice that \( a \) and \( c \) are conjugate). If \([b, c] = 1\), then \( \langle a, b, c \rangle = \langle a, c \rangle \times \langle b \rangle \), \( |db| = pq \), whereas \( |adb| = p \) and lemma 2.1 imply that no elements of composite order are in \( \langle a, adb \rangle \), so that we can assume \([b, c] \neq 1\). \( \langle b \rangle \) is then conjugate to \( \langle c \rangle \), whence \( \langle b \rangle \bar{\Delta} G \). If \( b \in \mathfrak{S}_d(\langle d \rangle) \) we get \( \langle a, c \rangle \triangleleft \langle a, b, c \rangle = \langle a, c \rangle \times \langle b' \rangle \), where \( b' \) is a suitable element of \( \langle a \rangle \times \langle b \rangle \), and the above technique leads to a
contradiction. Lemma 2.2 now implies $\langle b, d \rangle = N\langle d \rangle$ with $N$ an elementary abelian normal $p$-subgroup of $\langle b, d \rangle$ which in turn is normal in $\langle a, b, c \rangle$; $a \notin N$, and for every $x \in N$ we have $\langle x \rangle = \langle x, a \rangle \cap N \triangleleft \langle x, a \rangle$, i.e. $a \in \mathcal{C}(N)$. Hence $\langle a \rangle = \langle a, d \rangle \cap \mathcal{C}_{\langle a, b, c \rangle}(N) \triangleleft \langle a, d \rangle$, thus contradicting an earlier statement. So far we proved that, if $\langle a \rangle^G$ is not an elementary abelian $p$-group, then $[a, b] \neq 1$ for every $b \in G$ such that $|b| = p$, $\langle a \rangle \cap \langle b \rangle = 1$; as a consequence, all $p$-Sylow subgroups of $G$ are either cyclic or generalized quaternion. We now proceed to show that for any pair $x, y$ of elements of $G$ such that $|x| = |y| = p$, $\langle x \rangle \cap \langle y \rangle = 1$, the subgroup $\langle x, y \rangle$ is non abelian and $|\langle x, y \rangle| = pq$, $q$ being independent from the choice of $x, y$; since there is in $G$ just one class of conjugate subgroups of order $p$, it is enough if we prove that $\langle a \rangle \cap \langle b \rangle = \langle a \rangle \cap \langle c \rangle = 1$ implies $|\langle a, b \rangle| = |\langle a, c \rangle|$. Let $u \in G$ be such that $\langle a, b \rangle = \langle a, u \rangle$, $|u| = q$, $\langle a, b \rangle = \langle a, b \rangle$; $c \in \mathcal{C}_G(\langle u \rangle)$ (were this not the case, by lemma 2.2 two independent conjugates of $a$ would permute), hence $\langle u \rangle \triangleleft \langle a, b, c \rangle = \langle u \rangle \langle a, c \rangle$. Looking at $\langle a, c \rangle$, which by lemma 2.1 is also non abelian of order, say, $pr$, we see that $\langle a, c \rangle = \langle a, v \rangle$ where $|v| = r$, $\langle v \rangle \triangleleft \langle a, c \rangle$ and $\langle v \rangle = \langle a, c \rangle \cap \mathcal{C}_{\langle a, b, c \rangle}(u)$, so that $\langle a, b, c \rangle = (\langle u \rangle \times \langle v \rangle)\langle a \rangle$. The subgroups $\langle au \rangle$, $\langle av \rangle$, being conjugate to $\langle a \rangle$, are dual-Dedekind in $G$; by lemma 2.1 no element of composite order lies in $\langle au, av \rangle$, hence $v^{-1}u \in \langle au, av \rangle$ has prime order: but then $q = r$ (notice that we have also proved that every element of order $p$ normalizes every subgroup of order $q$). By an easy induction argument one can now prove that any set of elements of order $p$ generates a $P$-group of order $pq^n$ for a suitable $n$, so that $\langle a \rangle^G$, which is generated by all such elements of $G$, is a $P$-group: $\langle a \rangle^G = \langle a \rangle N$, with $N$ an elementary abelian $q$-subgroup on which $a$ induces a non identity power automorphism. Our next step is to prove that for every pair $x, y$ of elements of $G$ such that $|x| = q^m$, $|y| = p$, one has $\langle x \rangle \triangleleft \langle x, y \rangle$; by an earlier remark we can assume $m > 1$ and use induction. $\langle x^p \rangle$ is then normal in $\langle y, x^p \rangle$, $\langle y, x^p \rangle / \langle x^p \rangle \triangleleft \langle y, x \rangle / \langle x^p \rangle$: if $|\langle y, x \rangle / \langle x^p \rangle| = pq$ we are through. If this is not the case, then $\langle y, x \rangle / \langle x^p \rangle = (\langle x \rangle / \langle x^p \rangle)N / \langle x^p \rangle$ with $N / \langle x^p \rangle$ an elementary abelian normal $p$-subgroup of $\langle y, x \rangle / \langle x^p \rangle$ (lemma 2.2); $\langle x^p \rangle$ is a $q$-Sylow subgroup of $N$, whence $N = M \langle x^q \rangle$ for a suitable elementary abelian $p$-subgroup $M$ containing $\langle y \rangle$. But then $M = \langle y \rangle$ and again $|\langle y, x \rangle / \langle x^p \rangle| = pq$. For a conjugate $b$ of $a$ such that $\langle b \rangle \cap \langle a \rangle = 1$ one has either $\langle x \rangle \cap \langle a, b \rangle = 1$, which
implies \([a, x] \neq 1\); or \(\langle x \rangle \cap \langle a, b \rangle = 1\), and \(\langle x, a, b \rangle = \langle (x) \times \langle u \rangle \rangle \langle a \rangle\), where \(\langle u \rangle\) is the \(q\)-Sylow subgroup of \(\langle a, b \rangle\), and \(a\) induces a non identity power automorphism on \(\langle x \rangle \times \langle u \rangle\), whence again \([a, x] \neq 1\); but then
\[\langle x \rangle = \langle [a, x] \rangle \leq \langle a \rangle^q,\]
ineq i.e. \(N\) is the (unique) \(q\)-Sylow subgroup of \(G\). Now put \(K = \langle a \rangle^q\); \(K \cap \langle a \rangle^q = 1\) and, since a \(p\)-Sylow subgroup is either cyclic or generalized quaternion and its subgroup of order \(p\) lies in \(\langle a \rangle^q\), \(K \leq \{g \in G \mid (|g|, pq) = 1\}\). On the other hand, if \((|g|, pq) = 1\), for every \(y \in G\) with \(|y| = p\) one has \(\langle y \rangle = \langle y, g \rangle \cap \langle a \rangle^q \leq \langle y, g \rangle\); therefore \(g\) is in the normalizer of every subgroup of order \(pq\) in \(\langle a \rangle^q\): but this implies \([g, \langle a \rangle^q] = 1\), which concludes the proof of the theorem.

The following result is a trivial corollary to theorems 3.1, 3.2:

**Theorem 3.3.** Let \(G\) be a finite group. If \(G\) has non-trivial dual-Dedekind subgroups, then \(G\) is not simple.

**Remark.** Finite non simple groups with no non-trivial \(\mathfrak{D}\)-subgroups do exist: e.g. the symmetric group \(S_n\) is such whenever \(n > 3\) (it is a simple matter to verify that no normal subgroup of \(S_n\) satisfies the theorems 3.1, 3.2); the case \(n = 4\) provides an example of a soluble group which has no non-trivial \(\mathfrak{D}\)-subgroups.

**4.** We have already pointed out that, generally speaking, normal subgroups need not be \(\mathfrak{D}\)-subgroups; in order to evaluate, in a sense, the gap between these two classes we proceed to study the groups where every normal subgroup is also a \(\mathfrak{D}\)-subgroup (in the main result of this section we restrict ourselves to soluble groups).

**Proposition 4.1.** Assume that every normal subgroup of the group \(G\) is a \(\mathfrak{D}\)-subgroup of \(G\). If \(N \trianglelefteq G\), then every normal subgroup of \(G/N\) is a \(\mathfrak{D}\)-subgroup of \(G/N\).

Thus, \(K/N \trianglelefteq G/N\) implies \(K \trianglelefteq G\), \(K \mathfrak{D} G\), hence \(K \mathfrak{D}[G/N]\) and obviously \(K/N \mathfrak{D} G/N\).

**Proposition 4.2.** Let \(N\) be a minimum normal subgroup of \(G\). If every normal subgroup of \(G\) is also a \(\mathfrak{D}\)-subgroup, then \(N\) is simple.

Assume first that \(N\) is abelian; then \(|N| = p^\alpha\) with \(p\) a prime and \(\alpha \geq 1\); the number \(k\) of its subgroups of order \(p\) is congruent to 1
(mod $p$). The normal subgroup $P = \cap \{ \mathfrak{N}_G(H) \mid H \subseteq N, \ |H| = p \}$ contains every element of $G$ whose order is prime to $p$: thus, if $|x|, p \equiv 1$, then $\langle x \rangle \cap N = 1$ and, for any such an $H$, $H = (H \cup \langle x \rangle) \cap N \triangleleft H \cup \langle x \rangle$. So $G$ acts as a $p$-group of permutations on the set of the $k$ subgroups of order $p$ in $N$, hence it has at least a fixed point, i.e. $|N| = p$.

Assume now that $N$ is abelian; let $N_1$ be a simple direct factor of $N$. If $N_1 \neq N$ and $x \in G$ is such that $x^{-1}N_1x \neq N_1$, then $N_1 \times x^{-1}N_1x \subseteq (N_1 \cup \langle x \rangle) \cap N = N_1(\langle x \rangle \cap N)$ and $x^{-1}N_1x$ would be isomorphic to a subgroup of $\langle x \rangle$, which is clearly not the case.

**Corollary 4.3.** Let $G$ be a soluble group. If every normal subgroup of $G$ is a $\mathfrak{S}$-subgroup of $G$, then $G$ is supersoluble.

**Proposition 4.4.** Let $G$ be a nilpotent group. If $H \mathfrak{S} G$, then $H$ is quasi-normal in $G$.

This is a trivial consequence of a result of Napolitani, [1].

**Proposition 4.5.** Let $G$ be a $p$-group ($p$ a prime). If every normal subgroup of $G$ is a $\mathfrak{D}$-subgroup, then $G$ is modular.

For $u \in Z(G)$, with $|u| = p$, $G/\langle u \rangle$ is by induction a modular $p$-group. Assume that $G/\langle u \rangle$ is either abelian or Hamiltonian: for arbitrary $x \in G$, $\langle x, u \rangle$ is abelian, hence $\langle x \rangle \mathfrak{S} \langle x, u \rangle$; moreover $\langle x, u \rangle \triangleleft G$ implies $\langle x, u \rangle \mathfrak{S} G$ and $\langle x \rangle \mathfrak{S} G$; by proposition 4.4 $\langle x \rangle$ is a quasi-normal subgroup of $G$, i.e. $G$ is modular. We may then assume that $G/\langle u \rangle$ is not abelian nor Hamiltonian, so that $G = \langle t, A \rangle$ with $u \in A$, $A/\langle u \rangle$ abelian, $t^{-1}at = a^{1+p^s}u^{\alpha(a)}$ for every $a \in A$ and suitable $\alpha(a)$, $s \geq 2$ if $p = 2$ ([3], p. 13). Just as before one sees that every subgroup of $A$ is dual-Dedekind, whence quasi-normal, in $G$; it follows that $A$ is a modular group. Moreover $A^p = \{ a^p \mid a \in A \}$ is a subgroup of $A$, any of whose subgroups is normalized by $t$; $t^p$ normalizes every subgroup of $A$, inducing on every cyclic subgroup a power automorphism which is congruent to 1 (mod. $p$), and congruent to 1 (mod. 4) if $p = 2$. $A$ cannot be a Hamiltonian group: thus, if $A = Q \times B$ with $Q$ a quaternion group of order 8 and $B^2 = 1$, from $u \in Q$ it would follow that $G/\langle u \rangle$ is abelian, whereas, if $u \not\in Q$, $G/\langle u \rangle$ would be a modular 2-group containing a quaternion group, and $G/\langle u \rangle$ would be a Hamiltonian group. There are two cases left:
i) $A$ is abelian. By a previous remark, $\langle t^p, A \rangle$ is modular and all its subgroups are quasi-normal in $G$. Let $y \in G$ be such that $y \not\in \langle t^p, A \rangle$, so that $G = A \langle y \rangle$. If $\langle y \rangle \cap A \neq 1$, since $\langle y \rangle \cap A \leq G$, then by induction $G/\langle y \rangle \cap A$ is modular, hence $\langle y \rangle$ is quasi-normal in $G$. Assume now $\langle y \rangle \cap A = 1$: for every $a \in A$ we get $\langle a \rangle = \langle a, y \rangle \cap A \triangleleft \langle a, y \rangle$, i.e. $y$ induces a power automorphism on the abelian group $A$, which is congruent to 1 (mod. $p$). If $p \neq 2$ there is nothing more to prove; if $p = 2$ we remark that, if we had $A^4 = 1$, $G/\langle u \rangle$ would be abelian; hence $A^4 \neq 1$, $G/A^4$ is by induction a modular group, and the power induced by $y$ is congruent to 1 (mod. 4), which implies that $G$ is modular.

ii) $A$ is neither abelian nor Hamiltonian. We have $A = \langle v, B \rangle$, $B$ abelian, $v^{-1}xv = x^n$ with $n = 1$ (mod. $p$) for every $x \in B$ and $n$ independent from the choice of $x$ ($n = 1$ (mod. 4) if $p = 2$; we remark here that $B^4 \neq 1$, otherwise $A$ would be abelian). $A^p \subseteq Z(A)$, hence every subgroup of $A^p$ is normal in $G$; both of $A/A^p$ and $A/B$ are abelian, so that $\langle u \rangle = A' \subseteq A^p \cap B$; moreover, we can write $B$ as $B = \langle b \rangle \times B_1$ where $u \in \langle b \rangle$, $\exp B_1 < |b|$ and $|b| \geq 8$ if $p = 2$. We will show that $\langle g_1, g_2 \rangle = \langle g_1 \rangle \langle g_2 \rangle$ for every pair $g_1, g_2$ of elements of $G$ (without loss of generality, we can assume $g_i \notin A$, since every subgroup of $A$ is quasi-normal in $G$). Write $\langle g_1 \rangle = \langle a_1 t^{p^h} \rangle$, $\langle g_2 \rangle = \langle a_2 t^{p^k} \rangle$; assuming $0 \leq h \leq k$ we get $g_2 \in A(g_1)$, $\langle g_1, g_2 \rangle = \langle g_1, a \rangle$ for suitable $a_1, a_2, a \in A$. Should $\langle g_1 \rangle$ contain a non-identity normal subgroup $K$ of $G$, since $G/K$ would be a modular group by the induction hypothesis, then $\langle g_1 \rangle$ would be quasi-normal in $G$; hence we can assume $\langle g_1 \rangle \cap A^p = 1$, which implies $u \notin \langle g_1 \rangle$. Suppose $\langle g_1 \rangle \cap A = 1$; then $\langle a \rangle = \langle a, g_1 \rangle \cap A \triangleleft \langle a, g_1 \rangle$, and, if $p \neq 2$, $\langle a, g_1 \rangle$ is modular, whence $\langle g_1, g_2 \rangle = \langle g_1 \rangle \langle g_2 \rangle$. Under the same assumptions, but with $p = 2$, $g_1$ induces a power automorphism on the abelian group $B$; $G/B^4$ being modular, this power is congruent to 1 (mod. 4), so that if $a \in B$ then $\langle a, g_1 \rangle$ is modular. Let now $a \notin B$; $u \notin \langle a, g_1 \rangle$ if and only if $u \notin \langle a \rangle$, hence if either $u \notin \langle a \rangle$ or $u \in \langle a \rangle^4$ we again conclude that $\langle a, g_1 \rangle$ is modular; we are left with one more possibility: $u = a^2 = b^{2l}$; but $[g_1, b^{2l-1}] = 1$ (for $[g_1, b]$ is modular), $[g_1, ab^{2l-1}] = 1$ since $|ab^{2l-1}| = 2$, so that $\langle g_1, a \rangle$ is abelian. Assume now $1 = \langle g_1 \rangle \cap A^p \subseteq \langle g_1 \rangle \cap A = \langle c \rangle$ with $|c| = p$, $u \in \langle a, g_1 \rangle \cap A = \langle a, c \rangle$; if $|a| = p$ then $\langle g_1 \rangle \triangleleft \langle a, g_1 \rangle = \langle g_1 \rangle \langle g_2 \rangle$; if $|a| > p$ but $u \notin \langle a \rangle$ we should have $c \in \langle a \rangle \times \langle u \rangle$, whence $c \in \langle a^p \rangle \times \langle u \rangle \subseteq A^p$, contradicting an earlier
hypothesis. We have then \( |a| > p, u \in \langle a \rangle \): so \( \langle a \rangle \trianglelefteq G \) and, if either \( p \neq 2 \) or \( p = 2, u \in \langle a^4 \rangle, \langle g_1, a \rangle \) is modular. It follows that we are left with one last case: \( p = 2, a = d = b^2 \). Since \( \langle b \rangle \trianglelefteq G, u \in \langle b^4 \rangle \) and \( G/\langle u \rangle \) is modular, we see that \( \langle g_1, b \rangle \) is also modular, whence \( [g_1, b^{2^l-1}] = 1 \); if \( a \in \langle b \rangle, \langle g_1, a \rangle \) is abelian, whereas, if \( a \notin \langle b \rangle, |a^{-1}b^{2^l-1}| = 2, a^{-1}b^{2^l-1} \in \mathcal{O}_L(\langle g_1 \rangle) \) and finally \( \langle g_1 \rangle \trianglelefteq \langle g_1, g_2 \rangle \subseteq \langle g_1, b^{2^l-1}, a^{-1}b^{2^l-1} \rangle \), which disposes of the case and ends the proof.

**Theorem 4.6.** The group \( G \) is soluble and every normal subgroup of \( G \) is dual-Dedekind in \( G \) if and only if \( G = H_1 \times H_2 \times \ldots \times H_t \) with \( H_i \) a Hall subgroup of \( G \) (\( i = 1, \ldots, t \)) and either

1) \( H_i \) is a modular \( p \)-group; or

2) \( H_i = (P_{i1} \times \ldots \times P_{is_i})Q_i \) with \( P_{ij}, Q_i \) Sylow subgroups of \( G \) for different primes, \( P_{ij} \) abelian of odd order (\( j = 1, \ldots, s_i \)), \( Q_i = \langle b_i \rangle \), and \( b_i \) inducing a non identity power automorphism on each \( P_{ij} \).

**Proof of necessity.** Assume \( S \), a \( p \)-Sylow subgroup of \( G \) for some prime \( p \), is normal in \( G \); then, unless \( S \) is a direct factor of \( G \), \( S \trianglelefteq \Gamma_\infty(G) \) where \( \Gamma_\infty(G) \) denotes the intersection of all normal subgroups of \( G \) whose factor group is nilpotent. Thus \( S \trianglelefteq G \) and for \( a \in S, x \in G \) such that \( (|x|, p) = 1 \) we have \( \langle a \rangle = \langle a \rangle \cup (\langle x \rangle \cap S) = \langle a, x \rangle \cap S \trianglelefteq \langle a, x \rangle \); if \( S \) is not a direct factor of \( G \), we can choose \( a, x \) such that \( [a, x] \neq 1 \), but then \( ([a, x]) = \langle a \rangle \) and \( a \) also induces a power automorphism on \( S \). Let now \( b \) be arbitrary in \( S \); if \( [b, x] \neq 1 \) the above argument shows that \( b \) operates on \( S \) as a power automorphism, whereas if \( [b, x] = 1 \) we have \( [ab, x] \neq 1 \) and the same conclusion holds for \( ab \), hence for \( b \).

It follows that \( S \) is abelian of odd order, \( x^{-ry}_r = y^r \) with \( r \neq 1 \) (mod. \( p \)), \( r \) independent from the choice of \( y \in S, [G, S] = S \) and \( S \trianglelefteq \Gamma_\infty(G) \). Choosing for \( p \) the maximum prime divisor of \( |G| \), by the supersolubility of \( G \) the \( p \)-Sylow subgroup is certainly normal, so that an easy induction proves that \( \Gamma_\infty(G) \) is a Hall subgroup of \( G \). Moreover \( G \) has a normal 2-complement whose quotient group is clearly nilpotent, so that \( |\Gamma_\infty(G)| \) is odd; again, by the supersolubility of \( G \), \( \Gamma_\infty(G) \) is nilpotent, hence it is a direct product of normal Sylow subgroups of \( G \) which are all abelian by the preceding remark, and every element of \( G \) operates by conjugation on \( \Gamma_\infty(G) \) as a power automorphism. \( G/\Gamma_\infty(G) \) is a direct
product of modular $p$-groups for different primes; notice that every Sylow subgroup of $G$ which is a direct factor has trivial intersection with $\Gamma_\omega(G)$, and is modular; therefore, we can factor out all such subgroups, and write $G = T \times G_1$ with $T$ a modular, nilpotent, Hall subgroup of $G$ and $G_1$ also satisfying all our assumptions; from now on we shall assume $G = G_1$. Let $P$ be a normal Sylow subgroup of $G$; we have already seen that $P \subseteq \Gamma_\omega(G)$ and that every element of $G$ operates on $P$ as a power automorphism; we claim that $G/\mathcal{C}_G(P)$ is a (cyclic) group of prime power order. Deny: then there are a $q$-Sylow subgroup $Q$ and an $r$-Sylow subgroup $R$ of $G$ such that $[Q, R] = 1$, $Q \cap \Gamma_\omega(G) = R \cap \Gamma_\omega(G) = 1$, $[Q, P] = [R, P] = 1$; choose $a \in Q$, $b \in R$, $u \in P$ such that $[a, P] \neq 1$, $[b, P] \neq 1$, $|u| = p$ ($p \mid |P|$). The Hall subgroup $Q\Gamma_\omega(G)$ is normal, hence dual-Dedekind, in $G$, which is a contradiction to $\langle au \rangle = \langle au \rangle \cup (\langle b \rangle \cap Q\Gamma_\omega(G)) = \langle au \rangle \cup \langle b \rangle \cap Q\Gamma_\omega(G)$ (this owing to the fact that the former group has $q$-power order, whereas the latter contains $\langle u \rangle = \langle [au, b] \rangle$ which has order $p$). Therefore we get $G = Q\mathcal{C}_G(P)$ for a suitable $q$-Sylow subgroup $Q$ of $G$; we shall now prove, by induction on $q^b = |Q|$, that $Q$ is cyclic. Without loss of generality we can assume $P = \Gamma_\omega(G)$ (were this not the case, we would work on $G/C$ with $C$ the complement of $P$ in $\Gamma_\omega(G)$). If $Q \cap \mathcal{C}_G(P) = 1$, since $G/\mathcal{C}_G(P)$ is cyclic, then $Q$ is also cyclic. Assume then $Q \cap \mathcal{C}_G(P) \neq 1$; $\mathcal{C}_G(P) \cap Z(Q)$ is a non-trivial normal subgroup of $G$ and by the inductive hypothesis $Q/\mathcal{C}_G(P) \cap Z(Q)$ is cyclic; therefore $Q$ is abelian and all subgroups of $QP$ containing $P$ are normal, hence dual-Dedekind subgroups of $G$. If now $Q$ were not cyclic we could pick $a$ and $b$ in $Q$ in such a way that $a \notin \mathcal{C}_G(P)$, $a^d \in \mathcal{C}_G(P)$, $b \in Q$, $|b| = q$, $[b, P] = 1$, $\langle a \rangle \cap \langle b \rangle = 1$; for $u \in P$ with $|u| = p$ we would have $\langle au \rangle = \langle au \rangle \cup (\langle ab \rangle \cap \langle a \rangle P) = \langle [au, ab] \rangle \cap \langle a \rangle P \supseteq \langle [u, a] \rangle = \langle u \rangle$ i.e. $[u, a] = 1$ contrary to our choice of $a$. Now let $Q_1$ be a non normal Sylow subgroup of $G$, and let $P_{11}, P_{12}, \ldots, P_{1s_1}$ be those Sylow subgroups of $\Gamma_\omega(G)$ which are not centralized by $Q_1$; $H_1 = (P_{11} \times \ldots \times P_{1s_1})Q_1$ is a direct factor of $G$, and if $G = H_1$ the theorem is proved. Assume $G \neq H_1$; let $Q_2$ be a normal Sylow subgroup of $G$, not contained in $H_1$, and let $P_{21}, \ldots, P_{2s_2}$ be those Sylow subgroups of $\Gamma_\omega(G)$ which are not centralized by $Q_2$: $H_2 = (P_{21} \times \ldots \times P_{2s_2})Q_2$ is also a direct factor of $G$, and $H_1 \cap H_2 = 1$; in this way we clearly get a decomposition of $G$ as a direct product of factors of the prescribed type.
PROOF OF SUFFICIENCY. Since such a decomposition as is described in the theorem is both group- and lattice-theoretical, it will be enough if we prove the theorem for each one of the factors (nothing is to be proved for the modular ones). Without loss of generality, we can assume \( G=(P_1 \times \cdots \times P_s)Q \) where the \( P_i \)'s and \( Q \) are Sylow subgroups of \( G \), \( Q \) is cyclic, \( P_i \) is abelian of odd order (\( i=1, \ldots, s \)) and \( Q \) operates on \( P_1 \times \cdots \times P_s \) as a group of power automorphisms, with \( \mathcal{C}_Q(P_i)\neq Q \). Let \( H \trianglelefteq G \); we renumber the \( P_i \)'s so that \([H, P_i]=P_i\) for \( i=1, \ldots, r \) and \([H, P_i]=1\) for \( i=r+1, \ldots, s \). We shall prove that \( \varphi^K : X \rightarrow X \cup K \) \( (\varphi^K : [H/H \cap K] \rightarrow [HK/K]) \) and \( \varphi_H : Y \rightarrow Y \cap H \) \( (\varphi_H : [HK/K] \rightarrow [H/H \cap K]) \) are inverse lattice isomorphisms, whenever \( K \) is a subgroup of \( G \); since \( H \trianglelefteq G \), we have only to prove that \( X\varphi^K\varphi_H=X \) for every \( X \in [H/H \cap K] \). Assume first that

\[
K \subseteq (P_1 \times \cdots \times P_s)H = (H \cap Q)(P_1 \times \cdots \times P_s) \times (P_{r+1} \times \cdots \times P_s);
\]

we have

\[
K=(K \cap (H \cap Q)(P_1 \times \cdots \times P_s)) \times (K \cap (P_{r+1} \times \cdots \times P_s)) = (H \cap K)L
\]

with \( L=K \cap (P_{r+1} \times \cdots \times P_s) \trianglelefteq G \). We have \( H \cup K = H \cup (H \cap K) \cup L = L \cup L \) and, for every \( X \in [H/H \cap K] \),

\[
X\varphi^K\varphi_H=(X \cup K) \cap H=(X \cup (H \cap K) \cup L) \cap H=
\]

\[
=(X \cup L) \cap H=X \cup (L \cap H)=X.
\]

Assume now that \( K \subseteq (P_1 \times \cdots \times P_s)H \); there exists a \( q \)-Sylow subgroup \( T \) of \( G \) with \( T \cap K \) \( q \)-Sylow in \( K \); we have \( T \cap H \subseteq T \cap K \). If we call \( M=H \cap (P_1 \times \cdots \times P_s) \), then \( H=M(T \cap H)=M(H \cap K) \); notice that, since every subgroup of \( M \) is normal in \( G \), \( M \trianglelefteq G \). Now for every \( X \in [H/H \cap K] \) we get \( X=(X \cap M) \cup (H \cap K) \) and

\[
(X \cup K) \cap H=X\varphi^K\varphi_H=((X \cap M) \cup (H \cap K) \cup K) \cap H=
\]

\[
=((X \cap M) \cup K) \cap H=(X \cap M) \cup (H \cap K)=X,
\]

thus ending our proof.
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