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Normal neighborhood spaces

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NORMAL NEIGHBORHOOD SPACES

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The object of this paper is to extend to neighborhood spaces the well-known Urysohn's lemma for topological spaces.

DEFINITION 1. Let $X$ be a set and $k$ a set-valued set-function mapping the power set, of $X$, to itself. Then $(X, k)$ is said to be a neighborhood space iff,

1. $k\emptyset = \emptyset$
2. $A \subset kA$ for every $A \subset X$ and
3. $kA \subset kB$ if $A \subset B \subset X$.

The neighborhood space $(X, k)$ is said to be directed iff $k(A \cup B) = kA \cup kB$ for all $A, B \subset X$.

For a subset $A$ of $X$, write $cA = X - A$.

DEFINITION 2. Let $(X, k)$ be a neighborhood space. Take $i = ckc$. Then a set $A$ is said be a neighborhood of a set $B$ iff $B \subset iA$.

DEFINITION 3. A neighborhood space $(X, k)$ is said to be normal iff $A, B \subset X$ and $kA, kB$ are disjoint imply $kA, kB$ have disjoint neighborhoods.

It is obvious that a neighborhood space $(X, k)$ is normal iff $A, B \subset X$ and $kA \subset iB$ imply there is $C \subset X$ such that $kA \subset iC$ and $kC \subset iB$.

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**Definition 4.** Let \((X, k), (Y, m)\) be two neighborhood spaces and \(f\) a function from \(X\) to \(Y\). Then \(f\) is said to be continuous at the point \(x\) of \(X\) iff the inverse, under \(f\), of every neighborhood of \(f(x)\) is a neighborhood of \(x\). We will say \(f\) is continuous iff \(f\) is continuous at each point of \(X\).

It is easily seen that \(f\) is continuous iff \(f k \subset m f\).

Let \(R\) denote the reals and \(n\) the closure function of the usual topology for \(R\). Denote by \(I\) the closed unit interval \([0, 1]\) and let \(p\) be the restriction of \(n\) to \(I\).

**Lemma 1.** Let \((X, k)\) be a directed neighborhood space and \(D\) a dense subset of the positive reals. For each \(t\) in \(D\) let \(S(t)\) be a subset of \(X\) such that

1. \(\bigcup \{S(t) : t \in D\} = X\) and
2. \(k S(t) \subset i S(u)\) if \(t < u\).

Take \(f(x) = \inf \{t : x \in S(t)\}\). Then \(f\) is a continuous function from \((X, k)\) to \((R, n)\).

**Proof.** Let \(x \in X\). To prove \(f\) is continuous it is enough to show that \(f(x) < v\) implies \(E = \{y : f(y) < v, y \in X\}\) is a neighborhood of \(x\) and that \(u < f(x)\) implies \(F = \{y : f(y) > u, y \in X\}\) is a neighborhood of \(x\).

Now \(f(x) < v\) implies there are \(w, z\) in \(D\) such that \(f(x) < w < z < v\). Hence \(x \in S(w)\). Also \(S(z) \subset E\) since \(y \in S(z)\) implies \(f(y) \leq z < v\). Therefore \(x \in i S(z)\) and so \(E\) is a neighborhood of \(x\).

Also \(u < f(x)\) implies there are \(r, s\) in \(D\) such that \(u < r < s < f(x)\). Then \(x \in c S(s)\) since \(x \in S(s)\) implies \(f(x) \leq s\). Next, \(y \in c S(r)\) implies \(f(y) \geq r > u\) and so \(y \in F\); hence \(c S(r) \subset F\). Now \(k S(r) \subset i S(s)\) and so \(k c S(s) \subset i c S(r)\). Hence \(F\) is a neighborhood of \(x\).

The next lemma can be proved in the same way as the corresponding part of Urysohn's lemma; for instance we can use the method of proof of Lemma 4 on page 115 of Kelley [1].

**Lemma 2.** Let \((X, k)\) be a normal directed neighborhood space and \(A, B \subset X\) such that \(kA, kB\) are disjoint. Then there is a continuous function \(f\) from \((X, k)\) to \((I, p)\) such that \(f\) is 0 on \(kA\) and 1 on \(kB\).

**Definition 5.** A directed neighborhood space \((X, k)\) is said to be completely normal iff \(A, B \subset X\) and \(kA, kB\) are disjoint imply there
is a continuous function \( f \) from \((X, k)\) to \((I, p)\) such that \( f \) is 0 on \( kA \) and 1 on \( kB \).

The next result now easily follows.

**THEOREM 1.** A directed neighborhood space is normal iff it is completely normal.

Define a neighborhood space for the reals \( R \) as follows. For a real number \( x \) let \( \mathcal{N}(x) \) be the family of all subsets \( N \) of \( R \) such that \( \{ y : y < v \} \subseteq N \) for some \( v > x \) or \( \{ y : u < y \} \subseteq N \) for some \( u < x \). For a subset \( A \) of the reals, let \( hA \) be the set of all points \( x \) such that each \( N \) in \( \mathcal{N}(x) \) intersects \( A \). Then \((R, h)\) is a neighborhood space. Let \( g \) be the restriction of \( h \) to \( I \).

**DEFINITION 6.** A neighborhood space \((X, k)\) is said to be completely normal iff \( A, B \subseteq X \) and \( kA, kB \) are disjoint imply there is a continuous function \( f \) from \((X, k)\) to \((I, g)\) such that \( f \) is 0 on \( kA \) and 1 on \( kB \).

We then have the following result.

**THEOREM 2.** A neighborhood space is normal iff it is completely normal.

**REFERENCES**
