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ON THE ASYMPTOTIC BEHAVIOR OF THE ONE-SIDED GREEN'S FUNCTION FOR A DIFFERENTIAL OPERATOR NEAR A SINGULARITY

STEVEN BANK *)

1. Introduction.

In this paper we consider n_{th} order linear differential operators Ω, whose coefficients are complex functions defined and analytic in unbounded sectorial regions, and have asymptotic expansions, as the complex variable $x \to \infty$ in such regions, in terms of real (but not necessarily integral) powers of x and/or functions which are of smaller rate of growth ($<$) than all powers of x as $x \to \infty$. (We are using here the concept of asymptotic equivalence ($\sim$) as $x \to \infty$, and the order relation « $<$ » introduced in [8; § 13]. (A summary of the necessary definitions from [8] appears in § 2 below.) However, it should be noted (see [8; § 128 (g)] that the class of operators treated here includes, as a special case, those operators whose coefficients are analytic and possess asymptotic expansions (in the customary sense) of the form $\Sigma c_j^{-\lambda_j}$ with λ_j real and $\lambda_j \to +\infty$ as $j \to \infty$). More specifically, we are concerned here with the asymptotic behavior of the one-sided Green's function $H(x, \zeta)$ for the operator Ω (see [7; p. 33] or § 3 below), near the singular point at ∞. This function plays a major role in determining the asymptotic behavior near ∞ of solutions of the non-homogeneous equation $\Omega(y) = f$ (for functions f analytic in a sectoral region D), since

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the function $y(x)=\int_{x_0}^{x} H(x, \zeta)f(\zeta)d\zeta$ is a solution of $\Omega(y)=f$ satisfying zero initial conditions at the point x_0 in D. (The proof of this fact for the real domain given in [7; p. 34] is easily seen to be valid for the complexsimply-connected region D, where of course, the contour of integration is any rectifiable path in D from x_0 to x).

If $\{\psi_1, \ldots, \psi_n\}$ is a fundamental set of solutions for $\Omega(y)=0$, then the Green's function $H(x, \zeta)$ is a function of the form $\sum_{j=1}^{n} \psi_j(x)w_j(\zeta)$. In this paper, we determine the asymptotic behavior of $H(x, \zeta)$ by determining the asymptotic behavior near ∞ of the functions $w_j(\zeta)$, when $\{\psi_1, \ldots, \psi_n\}$ is a particular fundamental set whose existence was proved in [1, 2] and whose asymptotic behavior in subsectorial regions is known. The asymptotic behavior of $\{\psi_1, \ldots, \psi_n\}$ is as follows: Associated with Ω is a polynomial $P(\alpha)$ of degree $p\leq n$ ([2; § 3(*e*)]). If $\alpha_0, \ldots, \alpha_r$ are the distinct roots of $P(\alpha)$ with α_j of multiplicity m_j, then $\psi_1, \ldots, \psi_p$ are solutions of $\Omega(y)=0$ where each ψ_j is $\sim$ to a constant multiple of a distinct function of the form $x^{\alpha_i}(\log x)^{m-1}$, where $1\leq m\leq m_i$. For the remaining solutions $\psi_{p+1}, \ldots, \psi_n$, each ψ_k is $\sim$ to a function of the form $\exp\int V_k$ where each V_k is $\sim$ to a function of the form $c_k x^{-1+d_k}$ for $d_k>0$ and complex non-zero c_k. (The functions $c_k x^{-1+d_k}$ involved can be determined in advance by an algorithm. For a complete discussion, see § 4 below).

If the above fundamental set $\{\psi_1, \ldots, \psi_n\}$ is used to calculate the Green's function, $H(x, \zeta)=\sum_{j=1}^{n} \psi_j(x)w_j(\zeta)$, directly from the definition of $H(x, \zeta)$ (see § 3 below), the asymptotic behavior of the functions $w_j(\zeta)$ is difficult to determine since each w_j depends on the quotient of the Wronskian of $\{\psi_1, \ldots, \psi_n\}-\{\psi_j\}$ by the Wronskian of $\{\psi_1, \ldots, \psi_n\}$. However, in this paper we do succeeed in determining the asymptotic behavior of the functions $w_j(\zeta)$ by taking advantage of a factorization result proved in [1]. It was shown in [1; § 7] that under a simple change of dependent variable and multiplication by a suitable function, the operator Ω is transformed into an operator Φ which possesses an exact factorization into first order operators $\dot{f}_j$ of the form $\dot{f}_j(y)=y-(y'/f_j)$, where the asymptotic behavior of the functions $f_1, \ldots, f_n$ involved is known precisely. Since the Green's functions $K(x, \zeta)$ for a factored operator $\Phi=\Phi_1\Phi_2$ is related to the Green's functions K_1 and K_2 for Φ_1 and Φ_2 respectively, by $K(x, \zeta)=\int_\zeta^x K_2(x, s)K_1(s, \zeta)ds$ (see [7; p. 41] for the proof in the real domain and § 8 B below for the proof in the complex domain), we are in a position to use an inductive proof to determine the behavior of the Green's function for Φ (see § 6 below), and this easily leads to a result for Ω. In this connection, we make use of results in [3, 4] in determining the asymptotic behavior of the integrals which arise.

Our main result (§ 5) states that if Ω has been suitably normalized by dividing through by a known function of the form cx^β, and if the distinct roots $\alpha_0, \ldots, \alpha_r$ of $P(\alpha)$ also have distinct real parts, then there exists a fundamental set of solutions $\{\psi_1, \ldots, \psi_n\}$ for $\Omega(y)=0$ having the asymptotic behavior which was previously described such that the asymptotic behavior of each function $w_j(\zeta)$, in the Green's function $H(x, \zeta)=\sum_{j=1}^{n}\psi_j(x)w_j(\zeta)$ for Ω, is related to the asymptotic behavior of the corresponding function $\psi_j(x)$ as follows: If $1\leq j\leq p$, we know $\psi_j(x)$ is $\sim$ to a function of the form $a_jx^{\alpha_i}(\log x)^{m-1}$ where $1\leq m\leq m_i$ and a_j is a constant. We prove that $w_j(\zeta)$ is $\sim$ to a constant multiple of $\zeta^{-1-\alpha_i}(\log\zeta)^{m_i-m}$. For $p+1\leq k\leq n$, we know $\psi_k(x)$ is $\sim$ to a function of the form $\exp\int^x V_k$. We prove that $w_k(\zeta)$ is $\sim$ to a function of the form $\exp\int^\zeta U_k$, where $U_k\sim -V_k$, and in fact, we obtain more detailed information on U_k. (The condition concerning distinctness of the real parts of the α_j is needed in the proof since it guarantees that any two of the functions $\psi_1, \ldots, \psi_p$ are comparable with respect to the order relation « $<$ » (see § 2 (b))). Since the functions $w_1, \ldots, w_n$ comprise a fundamental set of solutions of the equation $\Omega^*(y)=0$ where Ω^* is the adjoint of Ω (see [7; p. 38]), we have therefore succeeded in also determining the asymptotic behavior of a fundamental set of solutions of the adjoint equation $\Omega^*(y)=0$.

In § 8, we prove certain results which are needed in the proof of the main theorem.

2. Concepts from [5] and [8].

(a) [8; § 94]. Let $-\pi \leq a < b \leq \pi$. For each non-negative real valued function g on $(0, (b-a)/2)$, let $E(g)$ be the union (over $\delta \in (0, (b-a)/2)$) of all sectors, $a+\delta < \arg(x-h(\delta)) < b-\delta$ where $h(\delta) = g(\delta)\exp(i(a+b)/2)$. The set of all $E(g)$ (for all choices of g) is denoted $F(a, b)$ and is a filter base which converges to ∞. Each $E(g)$ is simply-connected by [8; § 93]. If $V(x)$ is analytic in $E(g)$ then the symbol $\int^x V$ will stand for a primitive of $V(x)$ in $E(g)$. A statement is said to hold *except in finitely many directions* (briefly *e.f.d.*) in $F(a, b)$ if there are finitely many points $r_1 < r_2 < \dots < r_q$ in (a, b) such that the statement holds in each of $F(a, r_1)$, $F(r_1, r_2)$, ..., $F(r_q, b)$ separately.

(b) [8; § 13]. If f is analytic in some $E(g)$, then $f \to 0$ in $F(a, b)$ means that for any $\varepsilon > 0$, there is a g_1 such that $|f(x)| < \varepsilon$ for all $x \in E(g_1)$. $f < 1$ in $F(a, b)$ means that in addition to $f \to 0$, all functions $\theta_j^k f \to 0$ where θ_j is the operator $\theta_j f = (x \log x \dots \log_{j-1} x) f'$. Then $f_1 < f_2$, $f_1 \sim f_2$, $f_1 \approx f_2$, $f_1 \lesssim f_2$ mean respectively, $f_1/f_2 < 1$, $f_1 - f_2 < f_2$, $f_1 \sim c f_2$ for some constant $c \neq 0$, and finally either $f_1 < f_2$ or $f_1 \approx f_2$. If $f \sim c$, we write $f(\infty) = c$, while if $f < 1$, we write $f(\infty) = 0$. The relation «$<$» has the property ([8; § 28]) that if $f < 1$ then $\theta_j f < 1$ for all j. If $f \sim K x^{\alpha_0} (\log x)^{\alpha_1}$ for complex α_0 and K and real α_1, then $\delta_0(f)$ will denote α_0. It is easily verified that for every $\varepsilon > 0$, $x^{Re(\alpha_0)-\varepsilon} < x^{\alpha_0} < x^{Re(\alpha_0)+\varepsilon}$, from which it easily follows that if $Re(\delta_0(f)) < Re(\delta_0(h))$ then $f < h$. If $f \sim c x^{-1+d}$ where c is a non-zeroconstant and $d \geq 0$ then, the *indicial function* of f is the function on $(-\pi, \pi)$ defined by $IF(f)(\varphi) = \cos(d\varphi + \arg c)$. Finally, a function h is called *trivial* if $h < x^\alpha$ for all real α.

(c) [8; § 49] (and [10; § 53]). A logarithmic domain of rank zero (briefly, an LD_0) over $F(a, b)$ is a complex vector space L of functions (each analytic in some $E(g)$), which contains the constants, and such that any finite linear combination of elements of L, with coefficients which are functions of the form cx^α (for real α), is either $\sim$ to a function of this latter form or is trivial.

(*d*) [5; § 3]. If $G(z)=\sum_{j=0}^{n} b_j(x)z^j$, where the b_j belong to an LD_0, then a function $N(x)$ of the form cx^α (for real α) is called a *critical monomial* of G, if there is a function $h \sim N$ such that $G(h)$ is not $\sim G(N)$. (An algorithm for finding all critical monomials can be found in [5; § 26]). The critical monomial N of G is called *simple* if N is not a critical monomial of $\partial G/\partial z$.

3. The Green's function.

If $\Omega(y)=\sum_{j=0}^{n} a_j(x)y^{(j)}$ where the coefficients $a_j(x)$ are analytic in a simply-connected region D, and $a_n(x)$ has no zero in D, then the one-sided Green's function for Ω is the function $H(x, \zeta)$ on $D\times D$ defined as follows: If $B=\{\psi_1, \dots, \psi_n\}$ is a fundamental set of solutions in D for $\Omega(y)=0$, and if W is the Wronskian of B while W_j is the Wronskian of $B-\{\psi_j\}$, then $H(x, \zeta)=\sum_{j=1}^{n} \psi_j(x)v_j(\zeta)$ where

$$v_j(\zeta)=(-1)^{n+j}W_j(\zeta)/(a_n(\zeta)W(\zeta)).$$

(Remark: It follows from he uniqueness theorem for solutions of linear differential equations that the Green's function is indpendent of which fundamental set is used, since it is easily verified (as in [7; p. 33]) that no matter which fundamental set is used, the corresponding $H(x, \zeta)$ is a solution of $\Omega(y)=0$ for each $\zeta\in D$, satisfying the following initial conditions at $x=\zeta$: $\partial^k H(x, \zeta)/\partial^k=0$ for $0\le k\le n-2$; $\partial^{n-1}H(x, \zeta)/\partial x^{n-1}=$ $=(1/a_n(\zeta)))$.

4. Results from [1] and [2].

Let $\Omega(y)$ be an n^{th} order linear differential polynomial, coefficients in an LD_0 over $F(a, b)$. If θ is the operator $\theta y=xy'$, $\Omega(y)$ may be written $\Omega(y)=\sum_{j=0}^{n} B_j(x)\theta^j y$ where the functions B_j belong to an LD_0. We assume B_n is non-trivial. By dividing through by the highest power of x which is $\sim$ to a coefficient B_j, we may assume that for each j, $B_j \lesssim 1$ and there is an integer $p \geq 0$ such that $B_p \approx 1$ while for $j > p$, $B_j < 1$. Let $q = \min\{j : B_j \approx 1\}$. By dividing through by $B_q(\infty)$, we may assume $B_q \sim 1$. Let $P(\alpha) = \sum_{j=0}^{n} B_j(\infty)\alpha^j$ and let $\alpha_1, \ldots, \alpha_r$ be the distinct non-zero roots of $P(\alpha)$ with α_j of multiplicity m_j. (Thus $q + \sum_{j=1}^{r} m_j = p$). Define $M_1, \ldots, M_p$ as follows: $M_j = (\log x)^{j-1}$ if $1 \leq j \leq q$; $M_{q+j} = x^{\alpha_1}(\log x)^{j-1}$ if $1 \leq j \leq m_1$, and in general, $M_{q+m_1+\ldots+m_k+j} = x^{\alpha_{k+1}}(\log x)^{j-1}$ for $1 \leq k < r$ and $1 \leq j \leq m_{k+1}$. Define a sequence of integers $p = t(0) < t(1) < \ldots < t(\sigma) = n$ as follows: $t(0) = p$ and if $t(j)$ has been defined and is less than n, let $t(j+1)$ be the largest k such that $t(j) < k \leq n$ and such that $B_i \lesssim B_k$ for all i, $t(j) < i \leq n$. Let $G(z) = \sum_{j=0}^{\sigma} x^{t(j)} B_{t(j)} z^{t(j)-p}$, and assume that the critical monomials $N_1, \ldots, N_{n-p}$ of G are each simple (§ 2 (d)), and are arranged so that $N_j \lesssim N_{j+1}$ for each j. Then e.f.d. in $F(a, b)$, the following conclusions hold:

(a) Each N_j is of the form $c_j x^{-1+d_j}$ where c_j is a non-zero constant and $d_j > 0$.

(b) The equation $\Omega(y) = 0$ possesses a linearly independent set of solutions $\{g_1, \ldots, g_p\}$ where $g_j \sim M_j$ for $1 \leq j \leq p$.

(c) If we set $h_j = (\log x)^{-q} g_j$ for $1 \leq j \leq p$ and define functions $f_1, \ldots, f_p, \Psi_0, \ldots, \Psi_{p-1}$ recursively by the formulas, $\Psi_0 = h_1$ and $f_{j+1} = \Psi'_j/\Psi_j$ where $\Psi_j = (\dot{f}_j \ldots \dot{f}_1)(h_{j+1})$ (recalling that $\dot{f}_j(y) = y - (y'/f_j)$), then there exist functions $f_{p+1}, \ldots, f_n$ with $f_k \sim N_{k-p}$ such that,

(i) The equation $\Omega(y) = 0$ possesses solutions $g_{p+1}, \ldots, g_n$ such that g_k is of the form $g_k = R_k \exp \int^x f_k$ where $R_k \sim (\log x)^q \prod_{j=1}^{k-1} (f_j/(f_j - f_k))$ for $p+1 \leq k \leq n$.

(ii) The solutions $g_1, \ldots, g_n$ form a fundamental set of solutions for $\Omega(y) = 0$.

(iii) If $\Phi_0(z) = (1/q!)\Omega((\log x)^q z)$, then for some function $E \sim 1$, the operator Φ_0 possesses the exact factorization $\Phi_0 = E\dot{f}_n \ldots \dot{f}_1$ where $\dot{f}_j(y) = y - (y'/f_j)$.

(iv) If $h_k=(\log x)^{-q}g_k$ for $1\leq k\leq n$, then $f_k \ldots f_1(h_k)=0$ for each $k\in\{1, \ldots, n\}$.

(v) The functions $f_1, \ldots, f_p$ have the following asymptotic behavior: $f_j\sim -(q-j+1)x^{-1}(\log x)^{-1}$ if $1\leq j\leq q$; $f_{q+j}\sim \alpha_1 x^{-1}$ if $1\leq j\leq m_1$, and in general, $f_{q+m_1+\ldots+m_k+j}\sim \alpha_{k+1}x^{-1}$ for $1\leq k<r$ and $1\leq j\leq m_{k+1}$.

(REMARK. (*a*) is proved in [1; § 5]; (*b*) is proved in [2; §§ 5, 7, 10]; For (*c*), (i) is proved in [1; § 9] in light of [1; § 8]; (ii) is proved in [1; § 9]; (iii) and (v) are proved in [1; § 7]; (iv) for $1\leq k\leq p$ follows from the definition of f_j, while for $p+1\leq k\leq n$, it is proved in [1; § 9]).

In view of the above results, and with the above notation, we can make the following definition:

DEFINITION. A fundamental system of solutions $(\psi_1, \ldots, \psi_n)$ of $\Omega(y)=0$ is called *asymptotically canonical* if $\psi_j\approx M_j$ for $1\leq j\leq p$ while for $p+1\leq k\leq n$, ψ_k is $\approx$ to a function of the form $R_k \exp\int^x f_k$.

5. The Main Theorem.

Let $\Omega(y)$ be an n^{th} order linear differential polynomial with coefficient in an LD_0 over $F(a, b)$. By dividing through by a convenient function of form cx^{β} (as in § 4), we may assume $\Omega(y)=\sum_{j=0}^{n} B_j(x)\theta^j y$, where θ is the operator $\theta y=xy'$, and where the coefficients B_j belong to an LD_0 over $F(a, b)$ and have the following asymptotic properties: $B_j\lesssim 1$ for each j; For some integers $0\leq q\leq p$, $B_p\approx 1$, $B_q\sim 1$ and $B_j<1$ if $j>p$ or $j<q$. Let B_n be non-trivial in $F(a, b)$. Let $P(\alpha)=\sum_{j=0}^{n} B_j(\infty)\alpha^j$ and let P have the property that if α and β are roots of P with $\alpha\neq\beta$, then α and β have distinct real parts. Let $\alpha_1, \ldots, \alpha_r$ be the distinct non-zero roots of P, with α_j of multiplicity m_j, and let $M_1, \ldots, M_p$ be as in § 4. Let $G(z)$ be the polynomial constructed as in § 4, and assume, as in § 4, that the critical monomials $N_1, \ldots, N_{n-p}$ of $G(z)$ are each simple

and are arranged so that $N_j \lesssim N_{j+1}$ for each j. Define functions $u(x_1)$, ..., $u_n(x)$ e.f.d. in $F(a, b)$ as follows: $u_j(x) = x^{-1}(\log x)^{q-j}$ if $1 \leq j \leq q$; $u_{q+j}(x) = x^{-1-\alpha_1}(\log x)^{m_1-j}$ for $1 \leq j \leq m_1$, and in general $u_{q+m_1+\dots+m_k+j}(x) = x^{-1-\alpha_{k+1}}(\log x)^{m_{k+1}-j}$ for $1 \leq k < r$ and $1 \leq j \leq m_{k+1}$; For $p+1 \leq k \leq n$, let $u_k(x)$ be a function of the form $u_k(x) = E_k(x) \exp\left(-\int^x f_k\right)$ where $E_k = f_k \prod_{j=k+1}^{n} (f_j/(f_j - f_k))$, the f_j being in § 4. Then e.f.d. $F(a, b)$, the the following conclusions hold:

(1) The equation $\Omega(y) = 0$ possesses an asymptotically canonical fundamental system of solutions $(\psi_1, \dots, \psi_n)$ in the sense of § 4 (i.e. $\psi_j \approx M_j$ for $1 \leq j \leq p$, while $\psi_k \approx R_k \exp \int^x f_k$ for $p+1 \leq k \leq n$) such that such that the one-sided Green's function for Ω is of the form $H(x, \zeta) = \sum_{j=1}^{n} \psi_j(x) w_j(\zeta)$ where $w_j \approx u_j$ for each $j = 1, \dots, n$.

(2) The equation $\Omega^*(y) = 0$, where Ω^* is the adjoint of Ω, possesses a fundamental set of solutions $\{\psi^*_1, \dots, \psi^*_n\}$ where $\psi^*_j \sim u_j$ for each $j = 1, \dots, n$.

REMARK. It suffices to prove Part (1), since (2) will follow from (1) (see [7; p. 38]). In view of § 4 (c) (iii) we first prove a lemma concerning the Green's function for a factored operator, $\Phi = \dot{f}_n \dots \dot{f}_1$. The proof will make use of results proved in § 8, and the proof of the main theorem will be concluded in § 7.

6. LEMMA. Let $0 \leq q \leq p \leq n$, and let $m_1, \dots, m_r$ be positive integers such that $q + \sum_{j=1}^{n} m_j = p$. Let $\alpha_1, \dots, \alpha_r$ be distinct non-zero complex numbers such that $Re(\alpha_j) < Re(\alpha_{j+1})$ for each j. If $q > 0$, assume also that $Re(\alpha_j) \neq 0$ for each j. Let $M_1, \dots, M_p$ be as defined in § 4. Let I be an open subinterval of $(-\pi, \pi)$ and let $h_1, \dots, h_p$ be functions such that $h_j \sim (\log x)^{-q} M_j$ in $F(I)$ for $1 \leq j \leq p$. Let $f_1, \dots, f_p$, $\Psi_0, \dots, \Psi_{p-1}$ be defined as in § 4 (*c*) and let f_1, f_p have the asymptotic behavior described in § 4 (*c*) (v). Let $N_1, \dots, N_{n-p}$ be distinct functions, each of the form $c_j x^{-1+d_j}$ for complex $c_j \neq 0$ and $d_j > 0$, arranged so that $N_j \lessapprox N_{j+1}$ for each j. For $p+1 \leq k \leq n$, let f_k be a function $\sim N_{k-p}$ in $F(I)$ and let h_k be a function of the form $h_k = A_k \exp \int^x f_k$ where $A_k \sim \prod_{j=1}^{k-1} (f_j/f_j - f_k))$ in $F(I)$. Assume that $h_1, \dots, h_n$ are linearly independent and that for each $j \in \{1, \dots, n\}$, $\dot{f}_j \dots \dot{f}_1(h_j) = 0$ (where $\dot{f}_j(y) = y - (y'/f_j)$). Let $\Phi = \dot{f}_n \dots \dot{f}_1$ and let $u_1, \dots, u_n$ be as in § 5. Then, e.f.d. in $F(I)$, there exists a fundamental set of solutions $\{\varphi_1 \dots, \varphi_n\}$ of $\Phi(y) = 0$, such that $\varphi_j \approx h_j$ for $j = 1, \dots, n$ and such that the one-sided Green's function for Φ is of the form $H_0(x, \zeta) = \sum_{j=1}^{n} \varphi_j(x) v_j(\zeta)$ where $v_j \approx u_j$ for $j = 1, \dots, n$.

PROOF. The proof will be by induction on n. We consider first the case $n = 1$. Here $\Phi = \dot{f}_1$, and since $\dot{f}_1(h_1) = 0$, we have by § 8 *A* that the Green's function for Φ is,

(1) $$H_0(x, \zeta) = h_1(x) v_1(\zeta) \text{ where } v_1(\zeta) \approx f_1(\zeta)/h_1(\zeta).$$

We distinguish the two cases $p < n$ and $p = n$. If $p < n$ then $p = 0$ (since $n = 1$). Thus by § 5, $u_1(\zeta) = E_1(\zeta) \exp(-\int^\zeta f_1)$ where $E_1 = f_1$. But since $\dot{f}_1(h_1) = 0$, clearly $h_1(\zeta) = \exp \int^\zeta f_1$ and hence by (1), $v_1 \approx u_1$ so the result holds if $p < n$. If $p = n = 1$, we distinguish the two subcases $q < p$ and $q = p$. If $q < p$, then $q = 0$. Hence $h_1 \sim x^{\alpha_1}$ and $f_1 \sim \alpha_1 x^{-1}$. Thus by (1), $v_1(\zeta) \approx \zeta^{-1-\alpha_1}$, so $v_1 \approx u_1$. If $q = p$, then $h_1 \sim (\log x)^{-1}$ and $f_1 \sim -x^{-1} (\log x)^{-1}$. Hence by (1), $v_1(\zeta) \approx \zeta^{-1}$, so again $v_1 \approx u_1$. Thus the lemma holds for $n = 1$.

Now let $n > 1$, and assume that the lemma holds for $n - 1$. Let $h_1 \dots, h_n$ and $\Phi = \dot{f}_n \dots \dot{f}_1$ be given as in the statement of the lemma. (We show that the conclusion of the lemma holds for Φ). It follows from the hypothesis, that $h_1, \dots, h_{n-1}$ are solutions of $\Phi_1(y) = 0$ where

(2) $$\Phi_1 = \dot{f}_{n-1} \dots \dot{f}_1 .$$

We distinguish the two cases, $p=n$ and $p<n$.

CASE I. $p=n$. In this case, we will distinguish three subcases.

SUBCASE A. $q<p$ and $m_r=1$. Then $h_n \sim x^{\alpha_r}(\log x)^{-q}$. It is easily verified that using the given solutions $h_1, \ldots, h_{n-1}$ of $\Phi_1(y)=0$, the operator Φ_1 satisfies the induction hypothesis, where the corresponding functions u_j are precisely $u_1, \ldots, u_{n-1}$ as defined in the statement of the lemma (see § 5). Hence by the inductive assumption, there exists e.f.d. in $F(I)$, a fundamental set of solutions $\{\varphi_1, \ldots, \varphi_{n-1}\}$ of $\Phi_1(y)=0$ such that $\varphi_j \approx h_j$ for each j and such that the Green's function for Φ_1 is of the form $H_1(x, \zeta)=\sum_{j=1}^{n-1} \varphi_j(x)w_j(\zeta)$ where $w_j \approx u_j$ for each j. Now by definition of $\dot{f}_n$, we have $\dot{f}_n(\Psi_{n-1})=0$, where $\Psi_{n-1}=\dot{f}_{n-1} \ldots \dot{f}_1(h_n)$. In view of the asymptotic relations for the $\dot{f}_j$ given in § 4 (c) (v), it is easily verified using [1; § 6 (B), (D)] that

$$\Psi_{n-1} \approx x^{\alpha_r}. \tag{3}$$

Since $\dot{f}_n(\Psi_{n-1})=0$ and $f_n \sim \alpha_r x^{-1}$, it follows from § 8 A that the Green's function for the operator $\dot{f}_n$ is $H_2(x, \zeta)=\Psi_{n-1}(x)w(\zeta)$ where (using (3)), $w(\zeta) \approx \zeta^{-1-\alpha_r}$. Since $\Phi=\dot{f}_n\Phi_1$ (by (2)), we have by § 8 B that the Green's function for Φ is $H_0(x, \zeta)=\int_\zeta^x H_1(x, s)H_2(s, \zeta)ds$. Hence,

$$H_0(x, \zeta)=\sum_{j=1}^{n-1} \varphi_j(x)w(\zeta)\int_\zeta^x w_j(s)\Psi_{n-1}(s)ds. \tag{4}$$

Now $w_j \approx u_j$, so in view of (3), $w_j(s)\Psi_{n-1}(s) \approx s^{\alpha_r} u_j(s)$. Hence by the asymptotic relations for the u_j (see § 5), clearly for $1 \leq j \leq n-1$, $(\delta_0(w_j\Psi_{n-1})$ is either α_r-1 or $\alpha_r-1-\alpha_k$ for some $k<r$. Since $\alpha_r \neq 0$ and $\alpha_k \neq \alpha_r$ for $k<r$, we have that $\delta_0(w_j\Psi_{n-1}) \neq -1$ for each j. Thus by § 8 D (a), for each $j=1, \ldots, n-1$, there exists e.f.d. in $F(I)$, a function $Q_j(s) \approx s^{\alpha_r+1}u_j(s)$ such that $Q'_j=w_j\Psi_{n-1}$ Hence the right side of (4) is

$\sum_{j=1}^{n-1} \varphi_j(x)w(\zeta)(Q_j(x)-Q_j(\zeta))$, so (4) may be written,

$$H_0(x, \zeta)=\sum_{j=1}^{n-1} \varphi_j(x)v_j(\zeta)+V(x)w(\zeta) \tag{5}$$

where $v_j(\zeta)=-w(\zeta)Q_j(\zeta)$ and $V(x)=\sum_{j=1}^{n-1} \varphi_j(x)Q_j(x)$. Since $w(\zeta)\approx\zeta^{-\alpha-1}r$ and $Q_j(\zeta)\approx\zeta^{\alpha_r+1}u_j(\zeta)$, clearly $v_j\approx u_j$ for $1<j<n-1$. Furthemore, since $u_n(\zeta)\approx\zeta^{-1-\alpha_r}$, we have $w\approx u_n$. Hence in view of (5), the conclusion of the lemma will hold for Φ, if it can be shown that $\{\varphi_1, \dots, \varphi_{n-1}, V\}$ is a fundamental set for $\Phi(y)=0$ and that

$$V(x)\approx h_n(x). \tag{6}$$

To prove (6), we note first that $\varphi_1, \dots, \varphi_{n-1}$ are independent solutions of $\Phi(y)=0$, since they form a fundamental set for $\Phi_1(y)=0$. Hence in view of (5), we have by § 8 (C) that $\varphi_1, \dots, \varphi_{n-1}, V$ form a fundamental $\Phi(y)=0$. Since $h_1, \dots, h_n$ also form a fundamental set, there exist constant β_k and γ_k such that,

$$V=\sum_{k=1}^{n} \beta_k h_k, \tag{7}$$

and

$$h_n=\sum_{k=1}^{n-1} \gamma_k\varphi_k+\gamma_n V. \tag{8}$$

Now by hypothesis, for $q+1\leq j\leq n-1$, we have $\mathrm{Re}\,(\delta_0(h_j))<\mathrm{Re}\,(\alpha_r)$. Thus $h_j<h_n$ (see § 2 (b)), and since $\varphi_j\approx h_j$, we have $\varphi_j<h_n$ also. Hence $U=\sum_{j=q+1}^{n-1} \beta_j h_j$ is $<h_n$ and $W=\sum_{j=q+1}^{n-1} \gamma_j\varphi_j<h_n$, and so (7) and (8) may be written,

$$V=\beta_n h_n+\sum_{i=1}^{q} \beta_i h_i+U, \text{ where } U<h_n, \text{ and} \tag{9}$$

$$h_n=\gamma_n V+\sum_{i=1}^{q} \gamma_i\varphi_i+W \text{ where } W<h_n. \tag{10}$$

Now if $q=0$, then (6) will follow from (9) if $\beta_n \neq 0$. But this is clear, for if $\beta_n=0$, then by (9), $V<h_n$, and hence from (10) we would obtain $h_n<h_n$ (since $q=0$) which is a contradiction. Now consider the case $q>0$. Then by assumption, either $\mathrm{Re}(\alpha_r)>0$ or $\mathrm{Re}(\alpha_r)<0$. If $\mathrm{Re}(\alpha_r)>0$, then for $1 \leq i \leq q$, $h_i<h_n$ (and hence $\varphi_i<h_n$) since $\delta_0(h_i)= =0<\mathrm{Re}(\alpha_r)$. Thus again, (6) will follow from (9) if $\beta_n \neq 0$. But if $\beta_n=0$, then from (9), $V<h_n$ and so from (10) we woud obtain $h_n<h_n$ which is impossible. If $\mathrm{Re}(\alpha_r)<0$, we consider each term $\varphi_j Q_j$ in V. Since $\varphi_j \approx h_j$, we have for $1 \leq j \leq n-1$, $\varphi_j Q_j \approx x^{\alpha_r+1}(\log x)^{-q} M_j u_j$. By the asympotitic relations for M_j and u_j, clearly $\delta_0(M_j u_j)=-1$, and hence,

$$\delta_0(\varphi_j Q_j)=\alpha_r \text{ for } 1 \leq j \leq n-1. \tag{11}$$

Since $\mathrm{Re}(\alpha_r)<0$ and $\delta_0(h_k)=0$ for $1 \leq k \leq q$, we thus obtain $V<h_k$ and $h_n<h_k$ for $1 \leq k \leq q$. Thus from (9), $\sum_{i=1}^{q} \beta_i h_i<h_k$ for each $k \leq q$. Since $h_1<h_2< \dots <h_q$, this implies $\beta_i=0$ for $1 \leq i \leq q$, for in the contrary case, setting $j_0=\max\{i: 1 \leq i \leq q,\ \beta_i \neq 0\}$, we would obtain the contradiction, $h_{j_0} \approx \sum_{i=1}^{q} \beta_i h_i<h_{j_0}$. Thus from (9), $V=\beta_n h_n+U$, so (6) will hold if $\beta_n \neq 0$. But if $\beta_n=0$, then $V<h_n$, so since $\mathrm{Re}(\alpha_r)<0$ and $\varphi_k \approx h_k$, it would follow by (10), that $\sum_{i=1}^{q} \gamma_i \varphi_i<\varphi_k$ for $1 \leq k \leq q$. This would imply, as above that $\gamma_i=0$ for $1 \leq i \leq q$, so from (10) (and $V<h_n$) we would again obtain the contradiction $h_n<h_n$. Thus $\beta_n \neq 0$ so (6) holds. Thus in this subcase, the conclusion of the lemma holds for Φ.

SUBCASE B. $q<p$ and $m_r>1$. Since $p=n$, we have $h_n \sim x^{\alpha_r}(\log x)^{-q+m_r-1}$. For convenience, let $\sigma(j)=q+m_1+ \dots +m_{r-1}+j$ for $0 \leq j \leq m_r$. As in Subcase A, $h_1, \dots, h_{n-1}$ form a fundamental set for $\Phi_1(y)=0$, and we want to calculate the corresponding functions u_k for $h_1, \dots, h_{n-1}$. Now the α_j and m_j involved in $h_1, \dots, h_{\sigma(0)}$ are the same as in the statement for the lemma, and so the corresponding functions u_k for $k \leq \sigma(0)$, are precisely $u_1, \dots, u_{\sigma(0)}$ as defined in the statement of the lemma. The remaining solutions in $\{h_1, \dots, h_{n-1}\}$ are $h_{\sigma(j)}$ for $1 \leq j \leq m_r-1$. Thus the corresponding functions u_k for these solutions are obtained by using $m'_r=m_r-1$ in place of m_r in the definition of $u_{\sigma(j)}$ given in § 5. Since $u_{\sigma(j)}=x^{-1-\alpha_r}(\log x)^{m_r-j}$, using m'_r in place of m_r clearly results in $(\log x)^{-1}u_{\sigma(j)}$ as the corresponding u for $h_{\sigma(j)}$. Hence, by applying the inductive assumption to Φ_1, there exists e.f.d. in $F(I)$, a fundamental set $\{\varphi_1, \dots, \varphi_{n-1}\}$ for $\Phi_1(y)=0$ such that $\varphi_j \approx h_j$ for each j, and such that the Green's function for Φ_1 is of the form $H_1(x, \zeta)=$ $=\sum_{k=1}^{n-1}\varphi_j(x)w_j(\zeta)$ where $w_k \approx u_k$ for $1 \le k \le \sigma(0)$ while $w_{\sigma(j)} \approx (\log x)^{-1}u_{\sigma(j)}$ for $1 \le k \le m_r-1$. Now $f_n \sim \alpha_r x^{-1}$, and by using [1; § 6], it is easily verified that $\Psi_{n-1} \approx x^{\alpha_r}$. Hence as in Subcase A. the Green's function for the operator f_n is $H_2(x, \zeta)=\Psi_{n-1}(x)w(\zeta)$ where $w(\zeta) \approx \zeta^{-1-\alpha_r}$. Since $\Phi=f_n\Phi_1$, we have using § 8 (B) that the Green's function for Φ is,

$$H_0(x, \zeta)=\sum_{k=1}^{n-1}\varphi_k(x)w(\zeta)\int_\zeta^x w_k(s)\Psi_{n-1}(s)ds. \tag{12}$$

Now for $1 \le k \le \sigma(0)$, $w_k \approx u_k$ and hence $w_k(s)\Psi_{n-1}(s) \approx s^{\alpha_r}u_k(s)$. Hence as in Subcase A, $\delta_0(w_k\Psi_{n-1}) \ne -1$, and thus by § 8 D (*a*), for $1 \le k \le \sigma(0)$, there exists e.f.d. in $F(I)$, a function $Q_k(s) \approx s^{\alpha_r+1}u_k(s)$ such that $Q'_k=w_k\Psi_{n-1}$. Now for $\sigma(1) \le k \le n-1$, say $k=\sigma(j)$ where $1 \le j \le m_r-1$, we have $w_k \approx (\log x)^{-1}u_k$. Since $u_k \approx x^{-1-\alpha}(\log x)^{m_r-j}$, and also that $m_r-j-1>-1$ (since $j<m_r$), and so by § 8 D (*b*), for $k=\sigma(j)$ there exists e.f.d. in $F(I)$, a function $Q_k(s) \approx (\log s)^{m_r-j}$ such that $Q'_k=w_k\Psi_{n-1}$. Hence the right side of (12) is $\sum_{k=1}^{n-1}\varphi_k(x)w(\zeta)(Q_k(x)-Q_k(\zeta))$,.

$$H_0(x, \zeta)=\sum_{k=1}^{n-1}\varphi_k(x)v_k(\zeta)+V(x)w(\zeta), \tag{13}$$

where $v_k(\zeta)=-w(\zeta)Q_k(\zeta)$ and $V(x)=\sum_{k=1}^{n-1}\varphi_k(x)Q_k(x)$. Now for $1 \le k \le$ $\le \sigma(0)$, $Q_k(\zeta) \approx \zeta^{\alpha_r+1}u_k(\zeta)$ and so $v_k \approx u_k$ since $w(\zeta) \approx \zeta^{-1-\alpha_r}$. For $\sigma(1) \le$ $\le k \le n-1$, say $k=\sigma(j)$, we have $Q_k(\zeta) \approx (\log \zeta)^{m_r-j}$. Thus $v_k(\zeta) \approx$ $\approx \zeta^{-1-\alpha_r}(\log \zeta)^{m_r-j}$ and so again $v_k \approx u_k$. Furthemore $w \approx u_n$, and so in view of (13), the conclusion of the lemma will hold for Φ, if it can be shown that $\{\varphi_1, \dots, \varphi_{n-1}, V\}$ is a fundamental set for $\Phi(y)=0$ and that

$$V \approx h_n. \tag{14}$$

The proof of (14) is very similar to the proof of (6) in Subcase A. As in Subcase A, there exist constants β_k and γ_k such that (7) and (8) hold. By hypothesis for, $q+1 \leq k \leq \sigma(0)$, $\operatorname{Re}(\delta_0(h_k)) < \operatorname{Re}(\alpha_r)$ so $h_k < h_n$. For $\sigma(1) \leq k \leq n-1$, say $k=\sigma(j)$ (where $1 \leq j \leq m_r - 1$), we have $h_k \sim$ $\sim x^{\alpha} (\log x)^{-q+j-1}$ so $h_k < h_n$ since $j < m_r$. Thus setting $U = \sum_{j=q+1}^{n-1} \beta_j h_j$ and $W = \sum_{j=q+1}^{n-1} \gamma_j \varphi_j$, we have $U < h_n$ and $W < h_n$, and so we obtain (9) and (10). The proof now proceeds exactly as in Subcase A to establish (14). (We remark that the relation (11) which is needed in the proof is easy to verify, as in Subcase A, by using the definition of Q_j.)

Subcase C. $q=p$. Thus $q=n$ by this case. As before, $h_1, \ldots, h_{n-1}$ form a fundamental set for $\Phi_1(y)=0$ given by (2). Now $h_j \sim (\log x)^{-q+j-1}$ for $1 \leq j \leq n-1$, and this does not fit the induction hypothesis for Φ_1. (i.e. Since $\Phi_1 = \dot{f}_{q-1} \ldots \dot{f}_1$, the corresponding q for Φ_1 is $q-1$, and hence in order to apply the inductive assumption to Φ_1, the j^{th} solution must be $\sim (\log x)^{-(q-1)M_j}$ which is clearly not the case for h_j.) To remedy this, we set $\Lambda(z) = \Phi_1((\log x)^{-1} z)$. Then for $1 \leq j \leq n-1$, the functions $h_j^{\#} = (\log x) h_j$ solve $\Lambda(z)=0$. Clearly, $h_j^{\#} \sim (\log x)^{-q+j}$, so,

$$h_j^{\#} \sim (\log x)^{-(q-1)M_j} \text{ for } 1 \leq j \leq n-1. \tag{15}$$

Define functions $U_1, \ldots, U_{n-1}, \psi_0, \ldots, \psi_{n-2}$ recursively by $\psi_0 = h_1^{\#}$ and $U_{j+1} = \psi'_j / \psi_j$ where $\psi_j = \dot{U}_j \ldots \dot{U}_1 (h_{j+1}^{\#})$. Then clearly,

$$(\dot{U}_j \ldots \dot{U}_1)(h_j^{\#}) = 0 \text{ for } 1 \leq j \leq n-1. \tag{16}$$

In view of (15), it follows easily using [1; § 6 (A), (D)] that for $1 \leq j \leq n-1$,

$$U_j \sim -(q-j) x^{-1} (\log x)^{-1} \text{ and } \psi_{j-1} \approx h_j^{\#}. \tag{17}$$

Let $\Lambda_1 = \dot{U}_{n-1} \ldots \dot{U}_1$. In view of (15), (16), (17), it is clear that Λ_1, with the solutions $h_1^{\#}, \ldots, h_{n-1}^{\#}$, satisfies the inductive assumption using $q-1$ for q. The corresponding functions u_j are clearly obtained by using $q-1$ for q in the defintion of u_j given in § 5. Since

$u_j = x^{-1}(\log x)^{q-j}$, using $q-1$ for q clearly results in $(\log x)^{-1}u_j$ as the corresponding u for $h_j^{\#}$. Hence by the inductive assumption, there exists e.f.d. in $F(I)$, a fundamental set $\{\varphi_1^{\#}, \ldots, \varphi_{n-1}^{\#}\}$ for $\Lambda_1(y)=0$ such that $\varphi_j^{\#} \approx h_j^{\#}$ for each j, and such that the Green's function for Λ_1 is of the form

$$K(x, \zeta) = \sum_{j=1}^{n-1} \varphi_j^{\#}(x) w_j(\zeta), \tag{18}$$

where $w_j \approx (\log x)^{-1}u_j$ for $1 \leq j \leq n-1$. We now prove,

$$\Lambda = a(x)\Lambda_1, \text{ where } a(x) = \Phi_1((\log x)^{-1}). \tag{19}$$

To prove (19), we apply the division algorithm for linear differential operators ([9; § 2]), and divide Λ by $\dot{U}_1$. Since $\dot{U}_1$ is of order one, there exist an operator Γ_1 and a function $b_1(x)$ such that $\Lambda = \Gamma_1\dot{U}_1 + b_1$. Since Φ_1 is of order $n-1$, clearly Λ is of order $n-1$ and hence Γ_1 is of order $n-2$ by [9; § 5 (a)]. Since $\Lambda(h_1^{\#})=0$ and $\dot{U}_1(h_1^{\#})=0$ (by (16)), we have $b_1 h_1^{\#}=0$. Since $h_1^{\#} \not\equiv 0$ by (15), $b_1 \equiv 0$ so $\Lambda = \Gamma_1\dot{U}_1$. Dividing Γ_1 by $\dot{U}_2$, there exists an operator Γ_2 of order $n-3$ and a function b_2 such that $\Gamma_1 = \Gamma_2\dot{U}_2 + b_2$. Since $\Lambda(h_2^{\#})=0$ and $\dot{U}_2\dot{U}_1(h_2^{\#})=0$ (by (16)), we have $b_2\dot{U}_1(h_2^{\#})=0$. Since $\dot{U}_1(h_2^{\#})=\psi_1$ and $\psi_1 \not\equiv 0$ by (17) we obtain $b_2 \equiv 0$, so $\Lambda = \Gamma_2\dot{U}_2\dot{U}_1$. Continuing this way, we clearly obtain $\Lambda = \Gamma_{n-1}\Lambda_1$ where Γ_{n-1} is an operator of order zero. Thus for some function $a(x)$, $\Lambda(z) = a(x)\Lambda_1(z)$. Evaluating at $z=1$ (and noting that $\dot{U}_j(1)=1$), we obtain (19).

From (19) and the defintion of Λ, we have, $\Lambda_1(z) = (1/a(x))\,\Phi_1((\log x)^{-1}z)$. Thus by § 8 (A), the Green's function $H_1(x, \zeta)$ for Φ_1 is related to the Green's function $K(x, \zeta)$ for Λ_1 by $K(x, \zeta) = a(\zeta)(\log x)H_1(x, \zeta)$. Thus from (18), we obtain,

$$H_1(x, \zeta) = \sum_{j=1}^{n-1} \varphi_j(x)(w_j(\zeta)/a(\zeta)), \tag{20}$$

where $\varphi_j(x) = (\log x)^{-1}\varphi_j^{\#}(x)$. Since $\varphi_j^{\#} \approx h_j^{\#}$, clearly,

$$\varphi_j \approx h_j \text{ for } 1 \leq j \leq n-1. \tag{21}$$

Now by (19), $a(x)=\dot{f}_{n-1} \ldots \dot{f}_1((\log x)^{-1})$, and by definition, $\Psi_{n-1}=$ $=\dot{f}_{n-1} \ldots \dot{f}_1(h_n)$, where by assumption, $h_n \sim (\log x)^{-1}$. Since $q=n$, we have $f_j \sim -(q-j+1)x^{-1}(\log x)^{-1}$, and so it easily follows using [1; § 6 (D)], that

$$(22) \qquad a(x) \approx (\log x)^{-1} \text{ and } \Psi_{n-1} \approx (\log x)^{-1}.$$

Since $\dot{f}_n(\Psi_{n-1})=0$ and $f_n \sim -x^{-1}(\log x)^{-1}$, it follows from § 8 A that the Green's function for f_n is $H_2(x, \zeta)=\Psi_{n-1}(x)w(\zeta)$ where (using (22)), $w(\zeta) \approx \zeta^{-1}$. Since $\Phi=\dot{f}_n\Phi_1$, we have by § 8 B and (20) that the Green's function for Φ is,

$$(23) \qquad H_0(x, \zeta)=\sum_{j=1}^{n-1} \varphi_j(x)w(\zeta) \int_\zeta^x (w_j(s)\Psi_{n-1}(s)/a(s))ds.$$

Now $w_j(s) \approx (\log s)^{-1}u_j(s)$ and $\Psi_{n-1}(s)/a(s) \approx 1$ by (22). Hence since $u_j(s) \approx s^{-1}(\log s)^{q-j}$, we have $w_j(s)\Psi_{n-1}(s)/a(s) \approx s^{-1}(\log s)^{q-j-1}$ for $1 \leq j \leq$ $\leq n-1$. Since $q=n$ and $j<n$, $q-j-1>-1$. Thus by § 8 D (*b*), for each $j=1, \ldots, n-1$, there exists e.f.d. in $F(I)$, a function $Q(s) \approx (\log s)^{q-j}$ such that $Q'_j=w_j\Psi_{n-1}/a$. Hence the right side of (23) is $\sum_{j=1}^{n-1} \varphi_j(x)w(\zeta)$ $(Q_j(x)-Q_j(\zeta))$ and so (23) my be written,

$$(24) \qquad H_0(x, \zeta)=\sum_{j=1}^{n-1} \varphi_j(x)v_j(\zeta)+V(x)w(\zeta),$$

where $v_j(\zeta)=-w(\zeta)Q_j(\zeta)$ and $V(x)=\sum_{j=1}^{n-1} \varphi_j(x)Q_j(x)$. Since $w(\zeta) \approx \zeta^{-1}$, $v_j(\zeta) \approx \zeta^{-1}(\log \zeta)^{q-j}$ so $v_j \approx u_j$ for $1 \leq j \leq n-1$. Furthemore $w \approx u_n$, so in view of (21) and (24), the conclusion of the lemma will hold for Φ, if it can be shown that $\{\varphi_1, \ldots, \varphi_{n-1}, V\}$ is a fundamental set of solutions for $\Phi(y)=0$ and that

$$(25) \qquad V \approx h_n.$$

To prove (25), we note first that since $\{\varphi_1^\#, \ldots, \varphi_{n-1}^\#\}$ is a fundamental set for $\Lambda_1(y)=0$, clearly $\{\varphi_1, \ldots, \varphi_{n-1}\}$ is a fundamental set for

$\Phi_1(y)=0$. Since $\Phi=\dot{f}_n\Phi_1$, $\{\varphi_1, \ldots, \varphi_{n-1}\}$ is therefore an independent set of solutions of $\Phi(y)=0$, and hence in view of (24), it follows from § 8 c that $\varphi_1, \ldots, \varphi_{n-1}, V$ form a fundamental set for $\Phi(y)=0$. Since h_n is a solution $\Phi(y)=0$ by hypothesis, there exist constants γ_j such that,

$$h_n=\sum_{j=1}^{n-1}\gamma_j\varphi_j+\gamma_n V. \tag{26}$$

Since $n=q$, $h_j\sim(\log x)^{-q+j-1}$ and so $h_j<h_n$ for $j<n$. Since $\varphi_j\approx h_j$ by (21), $\varphi_j<h_n$ for $j<n$. Thus $\gamma_n\neq 0$, for othervise by (26), we would obtain the contradiction $h_n<h_n$. Hence $\gamma_n\neq 0$, and so $h_n\approx V$ by (26). This proves (25), and so the conclusion of the lemma holds for Φ in Subcase C, which completes Case I.

CASE II. $p<n$. Then $h_n=A_n\exp\int^x f_n$. Now $h_1, \ldots, h_{n-1}$ form a fundamental set for $\Phi_1(y)=0$ (see (2)), and we want to calculate the corresponding functions u_j for $h_1, \ldots, h_{n-1}$. Since $p\leq n-1$, the α_j and m_j involved in $h_1, \ldots, h_p$ are the same as in the statement of the lemma, and so the corresponding functions u_j are precisely $u_1, \ldots, u_p$ as defined in the statement of the lemma. For the remaining solutions $h_{p+1}, \ldots, h_{n-1}$, the corresponding functions u_k are clearly obtained by using $n-1$ in place of n in the definitions $u_{p+1}, \ldots, u_{n-1}$ given in the statement of the lemma (i.e. § 5). Since for $p+1\leq k\leq n-1$, u_k is defined as $E_k(x)\exp(-\int^x f_k)$ where $E_k=f_k\prod_{j=k+1}^{n}(f_j/(f_j-f_k))$, using $n-1$ for n clearly results in $E_k^{\#}\exp(-\int f_k)$, where $E_k\ =f_k\prod_{j=k+1}^{n-1}(f_j/(f_j-f_k))$, as the corresponding u for h_k. Hence by applying the inductive assumption to Φ_1, there exists e.f.d. in $F(I)$, a fundamental set $\{\varphi_1, \ldots, \varphi_{n-1}\}$ for $\Phi_1(y)=0$ such that $\varphi_j\approx h_j$ for each j, and such the Green's function for Φ_1 is of the form $H_1(x, \zeta)=\sum_{j=1}^{n-1}\varphi_j(x)w_j(\zeta)$ where $w_j\approx u_j$ for $1\leq j\leq p$, while $w_k(\zeta)\approx E_k^{\#}(\zeta)\exp(-\int^\zeta f_k)$ for $p+1\leq k\leq n-1$. Let $z_0(x)$ be a function of the form $\exp\int^x f_n$. Since $\dot{f}_n(z_0)=0$, it follows from § 8 A that the

Green's function for f_n is $H_2(x, \zeta)=z_0(x)w(\zeta)$, where

$$w(\zeta)\approx f_n(\zeta)/z_0(\zeta). \tag{27}$$

Since $\Phi=\dot{f}_n\Phi_1$, we have by § 8 (B) that the Green's function for Φ is,

$$H_0(x, \zeta)=\sum_{j=1}^{n-1}\varphi_j(x)w(\zeta)\int_{\zeta}^{x} w_j(s)z_0(s)ds. \tag{28}$$

Now for $1\leq j\leq p$, $w_j\approx u_j$ so w_j is $\approx$ to a function of the form $x^{\lambda}(\log x)^{\sigma}$. Since $p<n$, f_n is $\sim$ to a function of the form cx^{-1+d} where $d>0$. Thus clearly (see § 2 (b)), $IF(f_n)$ has only finitely many zeros on $(-\pi, \pi)$. Since $z_0(s)=\exp\int^{s} f_n$, it follows from [3; § 10 (b)] that for $1\leq j\leq p$, there exists e.f.d. in $F(I)$, a function of the form $Q_j(s)=$ $=a_j(s)z_0(s)$ where $a_j\sim w_j/f_n$, such that $Q'_j=w_jz_0$. For $p+1\leq k\leq n-1$, $w_k(s)z_0(s)$ is $\approx$ to a function of the form $E_k{}^{\#}(s)\exp\int^{s}(f_n-f_k)$. Now for $p+1\leq k<j\leq n$, $f_j-f_k\approx f_j$ (since $N_{k-p}\lesseqgtr N_{j-p}$ and $N_{k-p}\neq N_{j-p}$), and so it easily follows that $E_k{}^{\#}$ is $\approx$ to a function of the form $x^{\lambda}(\log x)^{\sigma}$. Since $f_n-f_k\approx f_n$, $IF(f_n-f_k)$ has only finitely many zeros. Thus it follows from [3; § 10 (b)] that for $p+1\leq k\leq n$, there exists e.f.d. in $F(I)$, a function of the form $Q_k(s)=T_k(s)\exp\int^{s}(f_n-f_k)$, where $T_k\approx E_k{}^{\#}/(f_n-f_k)$ such that $Q'_k=w_kz_0$. Hence the right side of (28) is $\sum_{j=1}^{n-1}\varphi_j(x)w(\zeta)(Q_j(x)-$ $=Q_j(\zeta))$, so (28) can be written,

$$H_0(x, \zeta)=\sum_{j=1}^{n-1}\varphi_j(x)v_j(\zeta)+V(x)w(\zeta), \tag{29}$$

where $v_j(\zeta)=-w(\zeta)Q_j(\zeta)$ and $V(x)=\sum_{j=1}^{n-1}\varphi_j(x)Q_j(x)$. Now in view of (27), for $1\leq j\leq p$, $v_j\approx f_na_j$. Since $a_j\sim w_j/f_n$, and $w_j\approx u_j$, we have $v_j\approx u_j$.

By (27), $w(\zeta)\approx f_n(\zeta)\exp(-\int^{\zeta} f_n)$. Thus for $p+1\leq k\leq n$, clearly $v_k(\zeta)\approx$ $\approx(f_n E_k^{\#}/(f_n-f_k))\exp(-\int^{\zeta} f_k)$ and hence $v_k(\zeta)\approx E_k(\zeta)\exp(-\int^{\zeta} f_k)$. Thus $v_k\approx u_k$. Furthemore by (27), $w(\zeta)\approx u_n(\zeta)$, so in view of (29), the conclusion of the lemma will hold for Φ if it can be shown that $\{\varphi_1, \ldots, \varphi_{n-1}, V\}$ is a fundamental set for $\Phi(y)=0$ and that

$$V\approx h_n. \tag{30}$$

To prove (30), we note that since $\{\varphi_1, \ldots, \varphi_{n-1}\}$ is a fundamental set for $\Phi_1(y)=0$, and since $\Phi=f_n\Phi_1$, it follows from (29) and § 8 D that $\{\varphi_1, \ldots, \varphi_{n-1}, V\}$ is a fundamental set for $\Phi(y)=0$ Since $\Phi(h_n)=0$, there exist constants β_j such that

$$h_n=\sum_{j=1}^{n-1}\beta_j\varphi_j+\beta_n V, \tag{31}$$

whence

$$(h_n-\beta_n V)/h_n=\sum_{j=1}^{n-1}\beta_j(\varphi_j/h_n). \tag{32}$$

We now calculate each term $\varphi_j Q_j$ in V. For $1\leq j\leq p$, $\varphi_j Q_j=\varphi_j a_j z_0$. For $p+1\leq k\leq n-1$, we have $\varphi_k\approx h_k$, $h_k=A_k\exp\int^x f_k$ and $Q_k=$ $=T_k\exp\int^x(f_n-f_k)$. Since $z_0=\exp\int^x f_n$, it follows easily that $\varphi_k Q_k=$ $=\Delta_k A_k T_k z_0$ where $\Delta_k\approx 1$. Thus clearly,

$$V=Uz_0, \tag{33}$$

where $U=\sum_{j=1}^{p}\varphi_j a_j+\sum_{k=p+1}^{n-1}\Delta_k A_k T_k$. Now $A_k\sim\prod_{j=1}^{k-1}(f_j/(f_j-f_k))$. Since f_j-f_k is $\approx f_k$ if $j<k$ and $k\geq p+1$, it follows easily that

(34) $\qquad A_k$ is $\approx$ to a function of the form $x^{\lambda}(\log x)^{\sigma}$.

In particular, A_k is $<$ some power of x. Since $\Delta_k T_k \approx E_k^{\#}/(f_n - f_k)$, it follows similarly $\Delta_k T_k$ is $<$ some power of x for $p+1 \leq k \leq n-1$. Since $\varphi_j \approx h_j$ and $a_j \approx u_j/f_n$ for $j \leq p$, it follows easily that φ_j and a_j are each $<$ some power of x. Thus each term in U is $<$ some power of x, so clearly,

(35) $$U(x) < x^{\sigma} \text{ for some real number } \sigma.$$

Since $h_n = A_n \exp \int^x f_n$, clearly $h_n = cA_n z_0$ for some $c \neq 0$. Hence in view of (33), the left side of (32) is $(cA_n - \beta_n U)/(cA_n)$, which by (34) and (35) is clearly $<$ some power of x. Thus by (32),

(36) $$\sum_{j=1}^{n-1} \beta_j(\varphi_j/h_n) < x^{\lambda}$$

for some real number λ.

Consider $\sum_{j=1}^{p} \beta_j \varphi_j$. Now by hypothesis, $\mathrm{Re}(\alpha_i) < \mathrm{Re}(\alpha_j)$ if $i<j$, and if $q>0$, $\mathrm{Re}\,\alpha_j \neq 0$. It easily follows (since $\varphi_j \approx h_j$) that for $1 \leq i < j \leq p$, either $\varphi_i < \varphi_j$ or $\varphi_j < \varphi_i$ (see § 2 (*b*)). Hence clearly, if not all of $\beta_1, .., \beta_p$ are zero, then there exists an index $j_0 \in \{1, ..., p\}$ such that $\beta_{j_0} \neq 0$ and $\varphi_i < \varphi_{j_0}$ if $i \leq p$ and $i \neq j_0$. Thus

(37) $$\sum_{j=1}^{p} \beta_j \varphi_j = \varphi_{j_0}(\beta_{j_0} + b(x)) \text{ where } b < 1.$$

(If all of $\beta_1, ..., \beta_p$ are zero, set β_{j_0} and b equal zero so (37) still holds.) For $p+1 \leq k \leq n-1$, set $D_k = \varphi_k/h_n$. Then we may write,

(38) $$\sum_{j=1}^{n-1} \beta_j(\varphi_j/h_n) = (1/h_n) \sum_{j=1}^{p} \varphi_j \beta_j + \sum_{k=p+1}^{n-1} \beta_k D_k.$$

Now for $p+1 \leq k \leq n$, clearly $IF(f_k)$ has only finitely many zeros (see § 2 (*b*)). For $p+1 \leq j < k \leq n$, $f_j - f_k \approx f_k$ so $IF(f_j - f_k)$ also has only finitely many zeros. Thus if we let Γ be the union of all zeros in I of all the above functions $IF(f_k)$ and $IF(f_j - f_k)$, then Γ is a finite set, say $\varepsilon_1 < ... < \varepsilon_m$. If $I = (\varepsilon_0, \varepsilon_{m+1})$, then letting J be any subinterval of any of the intervals $(\varepsilon_j, \varepsilon_{j+1})$ such that $\{\varphi_1, \ldots, \varphi_{n-1}, V\}$ exist on $F(J)$, we have that (36) is valid on $F(J)$ and all $IF(f_k)$ and $IF(f_j-f_k)$ as above, are nowhere zero on J. Now clearly, since $\varphi_k \approx h_k$, we have $D_k \approx$ $\approx (A_k/A_n) \exp \int^x (f_k-f_n)$. In view of (34) and the fact that $IF(f_k-f_n)$ is nowhere zero on J, it follows from [3; § 10 (a)], that for each $k \in \{p+1, \ldots, n-1\}$,

(39) Either D_k is trivial in $F(J)$ (i.e. $D_k < x^\alpha$ for all α) or $1/D_k$ is trivial in $F(J)$.

Since $h_k = A_k \exp \int^x f_k$, it follows similarly using (34) and [3; § 10 (a)] that for each $k \in \{p+1, \ldots, n\}$,

(40) Either h_k is trivial or $1/h_k$ is trivial in $F(J)$.

Finally, if j and k are distinct elements of $\{p+1, \ldots, n-1\}$, then since $D_j/D_k \approx (A_j/A_k) \exp \int^x (f_j-f_k)$, it follows as above that

(41) Either D_j/D_k is trivial or D_k/D_j is trivial in $F(J)$. We now return to (36) and prove,

(42) For each $j \in \{p+1, \ldots, n-1\}$ such that $1/D_j$ is trivial in $F(J)$, we have $\beta_j = 0$.

We prove (42) by contradiction. We assume the contrary and let i_0 be an index such that $1/D_{i_0}$ is trivial but $\beta_{i_0} \neq 0$. Let L be the set of all $j \in \{p+1, \ldots, n-1\}$ for which $\beta_j \neq 0$. For i and j in L with $i \neq j$, we have by (41) that either $D_i < D_j$ or $D_j < D_i$. Since L is a finite set, clearly there exists $k_0 \in L$ such that $D_i < D_{k_0}$ if $i \in L - \{k_0\}$. If $k_0 = i_0$ then $1/D_{k_0}$ is trivial. If $k_0 \neq i_0$ then $D_{i_0} < D_{k_0}$ so again,

(43) $$1/D_{k_0} \text{ is trivial in } F(J).$$

By the property of k_0, we can write $\sum_{j=p+1}^{n-1} \beta_j D_j = \beta_{k_0} D_{k_0}(1+t)$ where $t<1$. Hence by (36), (37) and (38), we obtain in $F(J)$,

(44) $$(\varphi_{j_0}/h_n)(\beta_{j_0}+b)+\beta_{k_0}D_{k_0}(1+t)<x^{\lambda}.$$

Now $D_{k_0}h_n=\varphi_{k_0}$. Thus dividing (44) by D_{k_0} and using (43),

(45) $$(\varphi_{j_0}/\varphi_{k_0})(\beta_{j_0}+b)+\beta_{k_0}(1+t) \text{ is trivial in } F(J).$$

Since β_{k_0} is a non-zero constant, $\beta_{k_0}\approx 1$. If $\beta_{j_0}=0$ (and $b=0$), then (45) is clearly impossible. If $\beta_{j_0}\neq 0$, then since $\beta_{k_0}\approx 1$, we have from (45) that $\beta_{j_0}\varphi_{j_0}/\varphi_{k_0}\sim -\beta_{k_0}$. Thus $\varphi_{j_0}/\varphi_{k_0}\approx 1$ and so $h_{j_0}\approx h_{k_0}$. This is clearly impossible since h_{j_0} is $\sim$ to a function of the form $x^{\alpha}(\log x)^m$ (since $j_0\leq p$), while by (40), either h_{k_0} or $1/h_{k_0}$ is trivial. This contradiction proves (42), which in view of (39) clearly implies,

(46) $$\sum_{k=p+1}^{n-1} \beta_k D_k \text{ is trivial in } F(J).$$

If $\beta_{j_0}=0$ (and $b=0$) in (37), then by (46), the left side of (38) is trivial. Thus by (32), $(h_n-\beta_n V)/h_n$ is trivial and hence is <1 in $F(J)$. Thus $\beta_n\neq 0$ and $h_n\approx V$ proving (30). If $\beta_{j_0}\neq 0$, then $\sum_{j=1}^{p}\beta_j\varphi_j\approx\varphi_{j_0}$. But in view of (46), we have by (38) and (36) that $(1/h_n)\sum_{j=1}^{p}\beta_j\varphi_j<x^{\lambda}$. Hence $\phi_{j_0}/h_n<x^{\lambda}$, so $(1/h_n)<x^{\lambda}/\varphi_{j_0}$. But $\varphi_{j_0}\approx h_{j_0}$ and so (since $j_0\leq p$), φ_{j_0} is $\approx$ to a function of the form $x^{\alpha}(\log x)^m$. Thus $(1/h_n)$ is $<$ some power of x. Hence by (40), $1/h_n$ must be trivial in $F(J)$. Thus $(1/h_n)\sum_{j=1}^{p}\beta_j\varphi_j$ is trivial, so by (46), the left side of (38) is trivial. Hence by (32), $(h_n-\beta_n V)/h_n$ is trivial, whence <1, and so again $\beta_n\neq 0$ and $h_n\approx V$ in $F(J)$ proving (30). Thus in Case II, the conclusion of the lemma holds for Φ, and so the lemma is established by induction.

7. Conclusion of proof of § 5.

Let Ω, q, p, M_j and u_k be as in § 5, where the roots α_j are arranged so that $\operatorname{Re}(\alpha_j) < \operatorname{Re}(\alpha_{j+1})$. By § 4, e.f.d. in $F(a, b)$, the operator $\Phi_0(z) = (1/q!)\Omega((\log x)^q z)$ possesses a factorization $\Phi_0 = E\dot{f}_n \dots \dot{f}_1$ (with f_j as in § 4 (*c*)), and there exists a fundamental set $\{g_1, \dots, g_n\}$ for $\Omega(y)=0$, with $g_j \sim M_j$ for $1 \leq j \leq p$ and $g_k = R_k \exp \int^x f_k$ for $k > p$, such that if $h_j = (\log x)^{-q} g_j$ for each j, then $\Phi = \dot{f}_n \dots \dot{f}_1$ satisfies the hypothesis of § 6 relative to the solution $h_1, \dots, h_n$. Hence by § 6, e.f.d. in $F(a, b)$, there exists a fundamental set $\{\varphi_1, \dots, \varphi_n\}$ for $\Phi(y)=0$ such that $\varphi_j \approx h_j$ and such that the Green's function for Φ is $H_0(x, \zeta) = \sum_{j=1}^n \varphi_j(x) v_j(\zeta)$ where $v_j \approx u_j$ for each j. By § 8 A, the Green's function for Ω is $H(x, \zeta) = (\log x)^q H_0(x, \zeta)/(q!E(\zeta))$. Thus $H(x, \zeta) = \sum_{j=1}^n \psi_j(x) w_j(\zeta)$, where $\psi_j(x) = (\log x)^q \varphi_j(x)$ and $w_j(\zeta) = v_j(\zeta)/(q!E(\zeta))$. Then clearly, $\{\psi_1, \dots, \psi_n\}$ is a fundamental set for $\Omega(y)=0$ and $\psi_j \approx g_j$ (since $\varphi_j \approx h_j$). Hence, $(\psi_1, \dots, \psi_n)$ is an asymptotically canonical fundamental system for Ω in the sense of § 4. Finally, since $E \sim 1$, clearly $w_j \approx u_j$. This concludes the proof of the main theorem.

8. Results needed in the proof of §§ 6.

A. LEMMA. Let f and E be analytic functions having no zeros in a simply-connected region D. Then:

(*a*) If $h(z)$ is analytic function in D such that $\dot{f}(h)=0$ and $h \not\equiv 0$, then the Green's function for $\dot{f}$ is $K(x, \zeta) = h(x) w(\zeta)$, where $w(\zeta) = -f(\zeta)/h(\zeta)$.

(*b*) If $\Omega(y) = \sum_{j=0}^n a_j(x) y^{(j)}$, where the $a_j(x)$ are analytic in D and $a_n(x)$ has no zeros in D, and if $\Lambda(z) = E(x)\Omega(f(x)z)$, then the Green's function $H(x, \zeta)$ for Ω is related to the Green's function $H_1(x, \zeta)$ for Λ by $H_1(x, \zeta) = H(x, \zeta)/(f(x)E(\zeta))$.

PROOF. Since $\{h\}$ is a fundamental set for $\dot{f}(y) = y - (y'/f) = 0$, Part (*a*) follows from the definition of $K(x, \zeta)$.

For (b), set $H_2(x, \zeta)=E(\zeta)f(x)H_1(x, \zeta)$. As in § 3, for each $\zeta \in D$, $H_1(x, \zeta)$ is a solution of $\Lambda(z)=0$ satisfying the following initial conditions at $x=\zeta$: $\partial^k H_1(x, \zeta)/\partial x^k=0$ for $k \leq n-2$, while $\partial^{n-1}H_1(x, \zeta)/\partial x^{n-1}= =1/(E(\zeta)f(\zeta)a_n(\zeta))$ since Efa_n is the leading coefficient of Λ. It is then easily verified that for each ζ, $H_2(x, \zeta)$ is a solution of $\Omega(y)=0$ satisfying the same initial conditions at $x=\zeta$ as the solution $H(x, \zeta)$ (see § 3). Hence $H_2 \equiv H$ by the uniqueness theorem for linear differential equations.

B. Lemma. Let $\Phi_1(y)=\sum_{j=0}^{n} a_j(x)y^{(j)}$ and $\Phi_2(y)=\sum_{j=0}^{m} b_j(x)y^{(j)}$, where the a_j and b_j are analytic in a simply-connected region D, and a_n and b_m have no zeros in D. Let $\Phi_3=\Phi_2\Phi_1$ and for $k=1, 2, 3$ let $H_k(x, \zeta)$ be the Green's function for Φ_k. Then $H_3(x, \zeta)=\int_\zeta^x H_1(x, s)H_2(s, \zeta)ds$, the contour of integration being any rectifiable path in D from ζ to x.

Proof. Set $K(x, \zeta)=\int_\zeta^x H_1(x, s)H_2(s, \zeta)ds$. By the property of the Green's function given in § 1, $K(x, \zeta)$ is for each ζ, a solution of $\Phi_1(y)=H_2(x, \zeta)$, and hence (see § 3), $K(x, \zeta)$ is a solution of $\Phi_3(y)=0$. Furthermore, using the initial conditions at $x=\zeta$ satisfied by H_1 and H_2 (see § 3), a straightforward calculation shows for each $\zeta \in D$, the solution $K(x, \zeta)$ of $\Phi_3(y)=0$ satisfies the same initial conditions at $x=\zeta$ as the solution $H_3(x, \zeta)$ (see § 3). Thus by the uniqueness theorem for linear differential equations $K \equiv H_3$ proving Lemma B.

C. Lemma. Let $\Phi(y)=\sum_{j=0}^{n} a_j(x)y^{(j)}$, where the a_j are analytic in D and a_n is nowhere zero in D. Then if the Green's function for Φ can be written in the form $H(x, \zeta)=\sum_{j=1}^{n} \varphi_j(x)w_j(\zeta)$, where $\varphi_1, \ldots, \varphi_{n-1}$ are linearly independent solutions of $\Phi(y)=0$, then $\{\varphi_1, \ldots, \varphi_n\}$ form a fundamental set of solutions for $\Phi(y)=0$.

Proof. We complete $\{\varphi_1, \ldots, \varphi_{n-1}\}$ to a fundamental set $\{\varphi_1, \ldots, \varphi_{n-1}, g\}$ for $\Phi(y)=0$. Then by definition (§ 3), $H(x, \zeta)=\sum_{j=1}^{n-1} \varphi_j(x)v_j(\zeta)+$

$+g(x)v_n(\zeta)$, and it is proved in [7; p. 38], that $\{v_1, ..., v_n\}$ form a fundamental set for the adjoint equation $\Phi^*(y)=0$. Now for each $\zeta \in D$, $H(x, \zeta)$ solves $\Phi(y)=0$, so clearly,

$$\Phi(\varphi_n(x)w_n(\zeta))\equiv 0 \text{ for each } \zeta \tag{47}$$

If $w_n(\zeta)\equiv 0$, then from the two representations for $H(x, \zeta)$, and the independence of $\{\varphi_1, ..., \varphi_{n-1}, g\}$, we would obtain $v_n(\zeta)\equiv 0$ which would contradict the independence of $\{v_1, ..., v_n\}$. Thus for some $\zeta_0 \in D$, $w_n(\zeta_0)\neq 0$ and so from (47), φ_n is a solution of $\Phi(y)=0$. To show $\{\varphi_1, ..., \varphi_n\}$ is independent, we assume the contrary. Then since $\{\varphi_1, ..., \varphi_{n-1}\}$ is independent, we would have a relation of the form $\varphi_n=\sum_{i=1}^{n-1} c_i\varphi_i$. Thus, $H(x, \zeta)=\sum_{j=1}^{n-1}\varphi_j(x)(w_j(\zeta)+c_jw_n(\zeta))$, which together with the other representation for H and the independence $\{\varphi_1, ..., \varphi_{n-1}, g\}$ again the contradiction $v_n(\zeta)\equiv 0$, thus proving Lemma C.

D. Lemma. Let $R(x)$ be a function such that in some $F(I)$, $R(x)\approx x^\alpha(\log x)^\beta$ for some complex number α and real number β. Then:

(a) If $\alpha\neq -1$, then e.f.d. in $F(I)$, there exists a function $Q(x)\approx \approx xR(x)$ such that $Q'=R$.

(b) If $\alpha=-1$ but $\beta\neq -1$, then e.f.d. in $F(I)$, there exists a function $Q(x)\approx(\log x)^{\beta+1}$ such that $Q'=R$.

Proof. Under the change of variable $y=x^\alpha z$ and division by $x^{\alpha-1}$, the equation $y'=R(x)$ is trasformed into,

$$xz'+\alpha z=T(x), \text{ where } T(x)=x^{1-\alpha}R(x). \tag{48}$$

Thus $T(x)\approx(\log x)^\beta$. If $\alpha\neq -1$, then by [4; § 3], equation (48) possesses, e.f.d. in $F(I)$, a solution $z_0(x)\approx T(x)$. Part (a) then follows by taking $Q(x)=x^\alpha z_0(x)$. If $\alpha=-1$ but $\beta\neq -1$, then by [4; § 3], equation (48) possesses, e.f.d. in $F(I)$, a solution $z_1(x)\approx(\log x)T(x)$. Part ($b$) then follows by taking $Q(x)=x^{-1}z_1(x)$.

Remark. In the case where α is real, Lemma D also follows from [6; Lemma ζ, p. 272].

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