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ON THE ASYMPTOTIC BEHAVIOR OF THE ONE-SIDED GREEN'S FUNCTION FOR A DIFFERENTIAL OPERATOR NEAR A SINGULARITY

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1. Introduction.

In this paper we consider $n_{th}$ order linear differential operators $\Omega$, whose coefficients are complex functions defined and analytic in unbounded sectorial regions, and have asymptotic expansions, as the complex variable $x \to \infty$ in such regions, in terms of real (but not necessarily integral) powers of $x$ and/or functions which are of smaller rate of growth ($<$) than all powers of $x$ as $x \to \infty$. (We are using here the concept of asymptotic equivalence ($\sim$) as $x \to \infty$, and the order relation « $<$ » introduced in [8; § 13]. (A summary of the necessary definitions from [8] appears in § 2 below.) However, it should be noted (see [8; § 128 (g)]) that the class of operators treated here includes, as a special case, those operators whose coefficients are analytic and possess asymptotic expansions (in the customary sense) of the form $\sum c_{i}^{-\lambda_{i}}$ with $\lambda_{i}$ real and $\lambda_{i} \to + \infty$ as $i \to \infty$). More specifically, we are concerned here with the asymptotic behavior of the one-sided Green's function $H(x, \zeta)$ for the operator $\Omega$ (see [7; p. 33] or § 3 below), near the singular point at $\infty$. This function plays a major role in determining the asymptotic behavior near $\infty$ of solutions of the non-homogeneous equation $\Omega(y) = f$ (for functions $f$ analytic in a sectoral region $D$), since

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the function $y(x) = \int_{x_0}^{x} H(x, \zeta) f(\zeta) d\zeta$ is a solution of $\Omega(y) = f$ satisfying zero initial conditions at the point $x_0$ in $D$. (The proof of this fact for the real domain given in [7; p. 34] is easily seen to be valid for the complex simply-connected region $D$, where of course, the contour of integration is any rectifiable path in $D$ from $x_0$ to $x$).

If $\{\psi_1, \ldots, \psi_n\}$ is a fundamental set of solutions for $\Omega(y) = 0$, then the Green's function $H(x, \zeta)$ is a function of the form $\sum_{j=1}^{n} \psi_j(x) w_j(\zeta)$. In this paper, we determine the asymptotic behavior of $H(x, \zeta)$ by determining the asymptotic behavior near $\infty$ of the functions $w_j(\zeta)$, when $\{\psi_1, \ldots, \psi_n\}$ is a particular fundamental set whose existence was proved in [1, 2] and whose asymptotic behavior in subsectorial regions is known. The asymptotic behavior of $\{\psi_1, \ldots, \psi_n\}$ is as follows: Associated with $\Omega$ is a polynomial $P(\alpha)$ of degree $p \leq n$ ([2; § 3 (e)]). If $\alpha_0, \ldots, \alpha_r$ are the distinct roots of $P(\alpha)$ with $\alpha_i$ of multiplicity $m_i$, then $\psi_1, \ldots, \psi_p$ are solutions of $\Omega(y) = 0$ where each $\psi_j$ is $\sim$ to a constant multiple of a distinct function of the form $x^{m} (\log x)^{m-1}$, where $1 \leq m \leq m_i$. For the remaining solutions $\psi_{p+1}, \ldots, \psi_n$, each $\psi_k$ is $\sim$ to a function of the form $\exp \int V_k$ where each $V_k$ is $\sim$ to a function of the form $c_k x^{m} (\log x)^{m-1}$, where $1 \leq m \leq m_i$. (The functions $c_k x^{m} (\log x)^{m-1}$ involved can be determined in advance by an algorithm. For a complete discussion, see § 4 below).

If the above fundamental set $\{\psi_1, \ldots, \psi_n\}$ is used to calculate the Green's function, $H(x, \zeta) = \sum_{j=1}^{n} \psi_j(x) w_j(\zeta)$, directly from the definition of $H(x, \zeta)$ (see § 3 below), the asymptotic behavior of the functions $w_j(\zeta)$ is difficult to determine since each $w_j$ depends on the quotient of the Wronskian of $\{\psi_1, \ldots, \psi_n\} - \{\psi_j\}$ by the Wronskian of $\{\psi_1, \ldots, \psi_n\}$). However, in this paper we do succeed in determining the asymptotic behavior of the functions $w_j(\zeta)$ by taking advantage of a factorization result proved in [1]. It was shown in [1; § 7] that under a simple change of dependent variable and multiplication by a suitable function, the operator $\Omega$ is transformed into an operator $\Phi$ which possesses an exact factorization into first order operators $f_j$ of the form $f_j(y) = y - (y'/f_j)$, where the asymptotic behavior of the functions $f_1, \ldots, f_n$ involved is known.
precisely. Since the Green's functions $K(x, \zeta)$ for a factored operator $\Phi = \Phi_1 \Phi_2$ is related to the Green's functions $K_1$ and $K_2$ for $\Phi_1$ and $\Phi_2$ respectively, by $K(x, \zeta) = \int_{\zeta}^{x} K_2(x, s)K_1(s, \zeta)ds$ (see [7; p. 41] for the proof in the real domain and § 8 B below for the proof in the complex domain), we are in a position to use an inductive proof to determine the behavior of the Green's function for $\Phi$ (see § 6 below), and this easily leads to a result for $\Omega$. In this connection, we make use of results in [3, 4] in determining the asymptotic behavior of the integrals which arise.

Our main result (§ 5) states that if $\Omega$ has been suitably normalized by dividing through by a known function of the form $cx^3$, and if the distinct roots $\alpha_0, \ldots, \alpha_r$ of $P(\alpha)$ also have distinct real parts, then there exists a fundamental set of solutions $\{\psi_1, \ldots, \psi_n\}$ for $\Omega(y) = 0$ having the asymptotic behavior which was previously described such that the asymptotic behavior of each function $\psi_j(\zeta)$, in the Green's function $H(x, \zeta) = \sum_{j=1}^{n} \psi_j(x)w_j(\zeta)$ for $\Omega$, is related to the asymptotic behavior of the corresponding function $\psi_j(x)$ as follows: If $1 \leq j \leq p$, we know $\psi_j(x)$ is $\sim$ to a function of the form $a_jx^{\sigma_i}(\log x)^{m_i-1}$ where $1 \leq m \leq m_i$ and $a_j$ is a constant. We prove that $\psi_j(\zeta)$ is $\sim$ to a constant multiple of $\zeta^{-1-\mu}(\log \zeta)^{m_i-1}$. For $p+1 \leq k \leq n$, we know $\psi_k(x)$ is $\sim$ to a function of the form $\exp \int_{\zeta}^{x} V_k$. We prove that $w_k(\zeta)$ is $\sim$ to a function of the form $\exp \int_{\zeta}^{x} U_k$, where $U_k \sim -V_k$, and in fact, we obtain more detailed information on $U_k$. (The condition concerning distinctness of the real parts of the $\alpha_j$ is needed in the proof since it guarantees that any two of the functions $\psi_1, \ldots, \psi_p$ are comparable with respect to the order relation $< <$ (see § 2 (b))). Since the functions $w_1, \ldots, w_n$ comprise a fundamental set of solutions of the equation $\Omega'(y)=0$ where $\Omega'$ is the adjoint of $\Omega$ (see [7; p. 38]), we have therefore succeeded in also determining the asymptotic behavior of a fundamental set of solutions of the adjoint equation $\Omega^*(y)=0$.

In § 8, we prove certain results which are needed in the proof of the main theorem.
2. Concepts from [5] and [8].

(a) [8; § 94]. Let $-\pi \leq a < b \leq \pi$. For each non-negative real valued function $g$ on $(0, (b-a)/2)$, let $E(g)$ be the union (over $\delta \in (0, (b-a)/2)$) of all sectors, $a+\delta < \text{arg} \,(x-h(\delta)) < b-\delta$ where $h(\delta) = g(\delta) \exp(i(a+b)/2)$. The set of all $E(g)$ (for all choices of $g$) is denoted $F(a, b)$ and is a filter base which converges to $\infty$. Each $E(g)$ is simply-connected by [8; § 93]. If $V(x)$ is analytic in $E(g)$ then the symbol $\int V$ will stand for a primitive of $V(x)$ in $E(g)$. A statement is said to hold except in finitely many directions (briefly e.f.d.) in $F(a, b)$ if there are finitely many points $r_1 < r_2 < \ldots < r_q$ in $(a, b)$ such that the statement holds in each of $F(a, r_1), F(r_1, r_2), \ldots, F(r_q, b)$ separately.

(b) [8; § 13]. If $f$ is analytic in some $E(g)$, then $f \to 0$ in $F(a, b)$ means that for any $\varepsilon > 0$, there is a $g_i$ such that $|f(x)| < \varepsilon$ for all $x \in E(g_i)$. $f < 1$ in $F(a, b)$ means that in addition to $f \to 0$, all functions $\theta_j f \to 0$ where $\theta_j$ is the operator $\theta_j f = (x \log x \ldots \log_{j-1} x)'$. Then $f_1 < f_2$, $f_1 \sim f_2$, $f_1 \approx f_2$, $f_1 \lessapprox f_2$ mean respectively, $f_1/f_2 < 1$, $f_1 - f_2 < f_2$, $f_1 \sim c f_2$ for some constant $c \neq 0$, and finally either $f_1 < f_2$ or $f_1 \approx f_2$. If $\sim c$, we write $f(\infty) = c$, while if $f < 1$, we write $f(\infty) = 0$. The relation $< \sim$ has the property ([8; § 28]) that if $f < 1$ then $\theta_j f < 1$ for all $j$. If $f \sim Kx^{\alpha_0}(\log x)^{\varepsilon_1}$ for complex $\alpha_0$ and $K$ and real $\alpha_1$, then $\delta_0(f)$ will denote $\alpha_0$. It is easily verified that for every $\varepsilon > 0$, $x^{\Re(\alpha_0) - \varepsilon} < x^{\alpha_0} < \lessapprox x^{\Re(\alpha_0) + \varepsilon}$, from which it easily follows that if $\Re(\delta_0(f)) < \Re(\delta_0(h))$ then $f < h$. If $f \sim cx^{-1+d}$ where $c$ is a non-zero constant and $d \geq 0$ then, the indicial function of $f$ is the function on $(-\pi, \pi)$ defined by $I(f)(\varphi) = \cos(d\varphi + \arg c)$. Finally, a function $h$ is called trivial if $h < x^\alpha$ for all real $\alpha$.

(c) [8; § 49] (and [10; § 53]). A logarithmic domain of rank zero (briefly, an LD$_0$) over $F(a, b)$ is a complex vector space $L$ of functions (each analytic in some $E(g)$), which contains the constants, and such that any finite linear combination of elements of $L$, with coefficients which are functions of the form $cx^\alpha$ (for real $\alpha$), is either ~ to a function of this latter form or is trivial.
(d) [5; § 3]. If \( G(z) = \sum_{j=0}^{n} b_j(x)z^j \), where the \( b_j \) belong to an 
\( LD_0 \), then a function \( N(x) \) of the form \( cx^a \) (for real \( a \)) is called a 
critical monomial of \( G \), if there is a function \( h \sim N \) such that \( G(h) \) is 
not \( \sim G(N) \). (An algorithm for finding all critical monomials can be 
found in [5; § 26]). The critical monomial \( N \) of \( G \) is called simple if 
\( N \) is not a critical monomial of \( \partial G / \partial z \).

3. The Green’s function.

If \( \Omega(y) = \sum_{i=0}^{n} a_i(x)y^{(i)} \) where the coefficients \( a_i(x) \) are analytic in a 
simply-connected region \( D \), and \( a_n(x) \) has no zero in \( D \), then the one-sided 
Green’s function for \( \Omega \) is the function \( H(x, \zeta) \) on \( D \times D \) defined as 
follows: If \( B = \{ \psi_1, ..., \psi_n \} \) is a fundamental set of solutions in \( D \) for 
\( \Omega(y) = 0 \), and if \( W \) is the Wronskian of \( B \) while \( W_i \) is the Wronskian 
of \( B - \{ \psi_i \} \), then 
\[
H(x, \zeta) = \sum_{i=1}^{n} \psi_i(x)\nu_i(\zeta)
\]
where
\[
\nu_i(\zeta) = (-1)^{n+i}W_i(\zeta)/(a_n(\zeta)W(\zeta)).
\]

(Remark: It follows from the uniqueness theorem for solutions of linear 
differential equations that the Green’s function is independent of which 
fundamental set is used, since it is easily verified (as in [7; p. 33]) 
that no matter which fundamental set is used, the corresponding \( H(x, \zeta) \) 
is a solution of \( \Omega(y) = 0 \) for each \( \zeta \in D \), satisfying the following initial con-
ditions at \( x = \zeta : \partial^k H(x, \zeta) / \partial^k = 0 \) for \( 0 \leq k \leq n-2 \); \( \partial^{n-1} H(x, \zeta) / \partial x^{n-1} = -(1/a_n(\zeta)). \)

4. Results from [1] and [2].

Let \( \Omega(y) \) be an \( n^{th} \) order linear differential polynomial, coefficients 
in an \( LD_0 \) over \( F(a, b) \). If \( \theta \) is the operator \( \theta y = xy' \), \( \Omega(y) \) may be written 
\( \Omega(y) = \sum_{i=0}^{n} B_i(x)\theta^i y \) where the functions \( B_i \) belong to an \( LD_0 \). We assume 
\( B_n \) is non-trivial. By dividing through by the highest power of \( x \) which
is \sim to a coefficient \(B_j\), we may assume that for each \(j\), \(B_j \leq 1\) and there is an integer \(p \geq 0\) such that \(B_p = 1\) while for \(j > p\), \(B_j < 1\). Let \(q = \min \{j : B_j = 1\}\). By dividing through by \(B_q(\infty)\), we may assume \(B_q \sim 1\). Let \(P(\alpha) = \sum_{j=0}^{n} B_j(\infty)\alpha^j\) and let \(\alpha_1, \ldots, \alpha_r\) be the distinct non-zero roots of \(P(\alpha)\) with \(\alpha_j\) of multiplicity \(m_j\). (Thus \(q + \sum_{j=1}^{r} m_j = p\)). Define \(M_1, \ldots, M_p\) as follows: \(M_j = (\log x)^{j-1}\) if \(1 \leq j \leq q\); \(M_{q+j} = x^a(\log x)^{j-1}\) if \(1 \leq j \leq m_1\), and in general, \(M_{q+m_1 + \ldots + m_{k+1}} = x^{a_{k+1}}(\log x)^{j-1}\) for \(1 \leq k < q\) and \(1 \leq j \leq m_{k+1}\). Define a sequence of integers \(p = t(0) < t(1) < \ldots < t(\sigma) = n\) as follows: \(t(0) = p\) and if \(t(j)\) has been defined and is less than \(n\), let \(t(j+1)\) be the largest \(k\) such that \(t(j) < k \leq n\) and such that \(B_i \leq B_k\) for all \(i\), \(t(j) < i \leq n\). Let \(G(z) = \sum_{j=0}^{\sigma} x^{t(j)}B_{t(j)}z^{t(j)-p}\), and assume that the critical monomials \(N_1, \ldots, N_{n-p}\) of \(G\) are each simple (§ 2 (d)), and are arranged so that \(N_j \leq N_{j+1}\) for each \(j\). Then e.f.d. in \(F(a, b)\), the following conclusions hold:

(a) Each \(N_j\) is of the form \(c_jx^{-1+d_j}\) where \(c_j\) is a non-zero constant and \(d_j > 0\).

(b) The equation \(\Omega(y) = 0\) possesses a linearly independent set of solutions \(\{g_1, \ldots, g_p\}\) where \(g_j \sim M_j\) for \(1 \leq j \leq p\).

(c) If we set \(h_j = (\log x)^{-a}g_j\) for \(1 \leq j \leq p\) and define functions \(f_1, \ldots, f_p, \Psi_0, \ldots, \Psi_{p-1}\) recursively by the formulas, \(\Psi_0 = h_1\) and \(f_{j+1} = \Psi_j/\Psi_i\) where \(\Psi_i = (f_j \ldots f_1)(h_{i+1})\) (recalling that \(f_j(y) = y - (y'/f_j)\)), then there exist functions \(f_{p+1}, \ldots, f_n\) with \(f_k \sim N_{k-p}\) such that,

(i) The equation \(\Omega(y) = 0\) possesses solutions \(g_{p+1}, \ldots, g_n\) such that \(g_k\) is of the form \(g_k = R_k \exp \int f_k\) where \(R_k \sim (\log x)^a \prod_{j=1}^{k-1} (f_j/(f_j - f_k))\) for \(p+1 \leq k \leq n\).

(ii) The solutions \(g_1, \ldots, g_n\) form a fundamental set of solutions for \(\Omega(y) = 0\).

(iii) If \(\Phi_0(z) = (1/q!)\Omega((\log x)^q z)\), then for some function \(E \sim 1\), the operator \(\Phi_0\) possesses the exact factorization \(\Phi_0 = Ef_n \ldots f_1\), where \(f_j(y) = y - (y'/f_j)\).
(iv) If $h_k = (\log x)^{-q}g_k$ for $1 \leq k \leq n$, then $f_k \ldots f_1(h_k) = 0$ for each $k \in \{1, \ldots, n\}$.

(v) The functions $f_1, \ldots, f_p$ have the following asymptotic behavior: $f_j \sim -(q-j+1)x^{-1}(\log x)^{-1}$ if $1 \leq j \leq q$; $f_{q+j} \sim a_k x^{-1}$ if $1 \leq j \leq m_1$, and in general, $f_{q+m_1, \ldots, m_k+j} \sim a_{k+1} x^{-1}$ for $1 \leq k < r$ and $1 \leq j \leq m_{k+1}$.

(REMARK. (a) is proved in [1; § 5]; (b) is proved in [2; §§ 5, 7, 10]; For (c), (i) is proved in [1; § 9] in light of [1; § 8]; (ii) is proved in [1; § 9]; (iii) and (v) are proved in [1; § 7]; (iv) for $1 \leq k \leq p$ follows from the definition of $f_j$, while for $p+1 \leq k \leq n$, it is proved in [1; § 9]).

In view of the above results, and with the above notation, we can make the following definition:

**Definition.** A fundamental system of solutions $(\psi_1, \ldots, \psi_n)$ of $\Omega(y) = 0$ is called asymptotically canonical if $\psi_j \approx M_j$ for $1 \leq j \leq p$ while for $p+1 \leq k \leq n$, $\psi_k$ is $\sim$ to a function of the form $R_k \exp \int f_k$.

5. The Main Theorem.

Let $\Omega(y)$ be an $n$th order linear differential polynomial with coefficient in an $LD_0$ over $F(a, b)$. By dividing through by a convenient function of form $cx^\theta$ (as in § 4), we may assume $\Omega(y) = \sum_{j=0}^{n} B_j(x)^q y^q$, where $\theta$ is the operator $\theta y = xy'$, and where the coefficients $B_j$ belong to an $LD_0$ over $F(a, b)$ and have the following asymptotic properties: $B_j \preceq 1$ for each $j$; For some integers $0 \leq q \leq p$, $B_p \approx 1$, $B_q \sim 1$ and $B_j < 1$ if $j > p$ or $j < q$. Let $B_n$ be non-trivial in $F(a, b)$. Let $P(\alpha) = \sum_{j=0}^{n} B_j(\infty) \alpha^j$ and let $P$ have the property that if $\alpha$ and $\beta$ are roots of $P$ with $\alpha \neq \beta$, then $\alpha$ and $\beta$ have distinct real parts. Let $\alpha_1, \ldots, \alpha_r$ be the distinct non-zero roots of $P$, with $\alpha_j$ of multiplicity $m_j$, and let $M_1, \ldots, M_p$ be as in § 4. Let $G(z)$ be the polynomial constructed as in § 4, and assume, as in § 4, that the critical monomials $N_1, \ldots, N_{n-p}$ of $G(z)$ are each simple
and are arranged so that \( N_j \leq N_{j+1} \) for each \( j \). Define functions \( u(x_1), \ldots, u_n(x) \) e.f.d. in \( F(a, b) \) as follows: 
\[
 u_j(x) = x^{-l} \left( \log x \right)^{q-1} \text{ if } 1 \leq j \leq q;
 u_{q+j}(x) = x^{-l} \left( \log x \right)^{m_j-1} \text{ for } 1 \leq j \leq m_1, \text{ and in general } \]
\[
 u_{q+m_1 + \cdots + m_{k-1} + j}(x) = x^{-l} \left( \log x \right)^{m_{k+1}-1} \text{ for } 1 \leq k < r \text{ and } 1 \leq j \leq m_{k+1}; \]
For \( p+1 \leq k \leq n \), let \( u_k(x) \) be a function of the form \( u_k(x) = E_k(x) \exp \left( - \int f_k \right) \) where 
\[
 E_k = f_k \prod_{j=k+1}^n \left( f_j / (f_j - f_k) \right), \quad \text{the } f_j \text{ being in } \S 4. \]
Then e.f.d. \( F(a, b) \), the following conclusions hold:

1. The equation \( \Omega(y) = 0 \) possesses an asymptotically canonical fundamental system of solutions \( (\psi_1, \ldots, \psi_n) \) in the sense of \( \S 4 \) (i.e. \( \psi_j \approx M_j \) for \( 1 \leq j \leq p \), while \( \psi_k \approx R_k \exp \int f_k \) for \( p+1 \leq k \leq n \)) such that the one-sided Green's function for \( \Omega \) is of the form \( H(x, \zeta) = \sum_{j=1}^n \psi_j(x) w_j(\zeta) \) where \( w_j = u_j \) for each \( j = 1, \ldots, n \).

2. The equation \( \Omega^*(y) = 0 \), where \( \Omega^* \) is the adjoint of \( \Omega \), possesses a fundamental set of solutions \( \{ \psi_1^*, \ldots, \psi_n^* \} \) where \( \psi_j^* \sim u_j \) for each \( j = 1, \ldots, n \).

Remark. It suffices to prove Part (1), since (2) will follow from (1) (see [7; p. 38]). In view of \( \S 4 \) (c) (iii) we first prove a lemma concerning the Green's function for a factored operator, \( \Phi = f_n \ldots f_1 \). The proof will make use of results proved in \( \S 8 \), and the proof of the main theorem will be concluded in \( \S 7 \).

6. Lemma. Let \( 0 \leq q \leq p \leq n \), and let \( m_1, \ldots, m_r \) be positive integers such that \( q + \sum_{j=1}^r m_j = p \). Let \( \alpha_1, \ldots, \alpha_r \) be distinct non-zero complex numbers such that \( \text{Re}(\alpha_j) < \text{Re}(\alpha_{j+1}) \) for each \( j \). If \( q > 0 \), assume also that \( \text{Re}(\alpha_j) \neq 0 \) for each \( j \). Let \( M_1, \ldots, M_p \) be as defined in \( \S 4 \). Let \( I \) be an open subinterval of \( (-\pi, \pi) \) and let \( h_1, \ldots, h_p \) be functions such that \( h_j \sim (\log x)^{-q} M_j \) in \( F(I) \) for \( 1 \leq j \leq p \). Let \( f_1, \ldots, f_p, \Psi_0, \ldots, \Psi_{p-1} \) be
defined as in § 4 (c) and let $f_p$ have the asymptotic behavior described in § 4 (c) (v). Let $N_1, ..., N_{n-p}$ be distinct functions, each of the form $c_jx^{-1+d_j}$ for complex $c_j \neq 0$ and $d_j > 0$, arranged so that $N_j \leq N_{j+1}$ for each $j$. For $p+1 \leq k \leq n$, let $f_k$ be a function $\sim N_{k-p}$ in $F(I)$ and let $h_k$ be a function of the form $h_k = A_k \exp \int f_k$ where $A_k \sim \prod_{j=1}^{k-1} (f_j/f_{j-1} - f_k)$ in $F(I)$. Assume that $h_1, ..., h_n$ are linearly independent and that for each $j \in \{1, ..., n\}$, $f_j$ and $f_j(h_j) = 0$ (where $f_j(y) = y - (y'/f_j)$). Let $\Phi = f_n ... f_1$ and let $u_1, ..., u_n$ be as in § 5. Then, e.f.d. in $F(I)$, there exists a fundamental set of solutions $\{\varphi_1, ..., \varphi_n\}$ of $\Phi(y) = 0$, such that $\varphi_j = h_j$ for $j = 1, ..., n$ and such that the one-sided Green's function for $\Phi$ is of the form $H_0(x, \zeta) = \sum_{j=1}^{n} \varphi_j(x)\varphi_j(\zeta)$ where $\varphi_j = u_j$ for $j = 1, ..., n$.

**PROOF.** The proof will be by induction on $n$. We consider first the case $n = 1$. Here $\Phi = f_1$, and since $f_1(h_1) = 0$, we have by § 8 A that the Green's function for $\Phi$ is,

$$H_0(x, \zeta) = h_1(x)v_1(\zeta)$$

where $v_1(\zeta) = f_1(\zeta)/h_1(\zeta)$.

We distinguish the two cases $p < n$ and $p = n$. If $p < n$ then $p = 0$ (since $n = 1$). Thus by § 5, $u_1(\zeta) = E_1(\zeta) \exp (-\int f_1)$ where $E_1 = f_1$. But since $f_1(h_1) = 0$, clearly $h_1(\zeta) = \exp \int f_1$ and hence by (1), $v_1 = u_1$ so the result holds if $p < n$. If $p = n = 1$, we distinguish the two subcases $q < p$ and $q = p$. If $q < p$, then $q = 0$. Hence $h_1 \sim x^s$ and $f_1 \sim \alpha x^{-1}$. Thus by (1), $v_1(\zeta) \approx \zeta^{-1-a_1}$, so $v_1 = u_1$. If $q = p$, then $h_1 \sim (\log x)^{-1}$ and $f_1 \sim -x^{-1} (\log x)^{-1}$. Hence by (1), $v_1(\zeta) \approx \zeta^{-1}$, so again $v_1 = u_1$. Thus the lemma holds for $n = 1$.

Now let $n > 1$, and assume that the lemma holds for $n - 1$. Let $h_1, ..., h_n$ and $\Phi = f_n ... f_1$ be given as in the statement of the lemma. (We show that the conclusion of the lemma holds for $\Phi$). It follows from the hypothesis, that $h_1, ..., h_{n-1}$ are solutions of $\Phi(y) = 0$ where

$$\Phi_1 = f_{n-1} ... f_1.$$
We distinguish the two cases, $p=n$ and $p<n$.

**Case I.** $p=n$. In this case, we will distinguish three subcases.

**Subcase A.** $q<p$ and $m_r=1$. Then $h_n \sim x^{\alpha_r} (\log x)^{-q}$. It is easily verified that using the given solutions $h_1, \ldots, h_{n-1}$ of $\Phi(y)=0$, the operator $\Phi$ satisfies the induction hypothesis, where the corresponding functions $u_i$ are precisely $u_1, \ldots, u_{n-1}$ as defined in the statement of the lemma (see § 5). Hence by the inductive assumption, there exists e.f.d. in $F(I)$, a fundamental set of solutions $\{\varphi_1, \ldots, \varphi_{n-1}\}$ of $\Phi(y)=0$ such that $\varphi_j \approx h_i$ for each $i$ and such that the Green's function for $\Phi$ is of the form $H_i(x, \zeta) = \sum_{j=1}^{n-1} \varphi_j(x)w_j(\zeta)$ where $w_j \approx u_i$ for each $j$. Now by definition of $f_n$, we have $f_n(\Psi_{n-1})=0$, where $\Psi_{n-1}=f_{n-1} \cdots f_1(h_n)$. In view of the asymptotic relations for the $f_j$ given in § 4 (c) (v), it is easily verified using [1; § 6 (B), (D)] that

\[ \Psi_{n-1} \approx x^{\alpha_r}. \]

Since $f_n(\Psi_{n-1})=0$ and $f_n \sim \alpha_r x^{-1}$, it follows from § 8 A that the Green's function for the operator $f_n$ is $H_2(x, \zeta) = \Psi_{n-1}(x)w(\zeta)$ where (using (3)), $w(\zeta) \approx \zeta^{-1-\alpha_r}$. Since $\Phi=f_n\Phi$ (by (2)), we have by § 8 B that the Green's function for $\Phi$ is $H_0(x, \zeta) = \int_\zeta^x H_1(x, s)H_2(s, \zeta)ds$. Hence,

\[ H_0(x, \zeta) = \sum_{j=1}^{n-1} \varphi_j(x)w_j(\zeta) \int_\zeta^x w_j(s)\Psi_{n-1}(s)ds. \]

Now $w_j= u_i$, so in view of (3), $w_j(\Psi_{n-1}) \approx s^{\alpha_r} u_i(s)$. Hence by the asymptotic relations for the $u_i$ (see § 5), clearly for $1 \leq j \leq n-1$, $(\delta_0(w_j\Psi_{n-1}))$ is either $\alpha_i-1$ or $\alpha_i-1-\alpha_k$ for some $k<r$. Since $\alpha_i \neq 0$ and $\alpha_k \neq \alpha_i$ for $k<r$, we have that $\delta_0(w_j\Psi_{n-1}) \neq -1$ for each $j$. Thus by § 8 D (a), for each $j=1, \ldots, n-1$, there exists e.f.d. in $F(I)$, a function $Q_j(s) \approx s^{\alpha_r+1} u_i(s)$ such that $Q_j' = w_j\Psi_{n-1}$ Hence the right side of (4) is
On the asymptotic behavior of the one-sided Green's, etc.

\[ \sum_{j=1}^{n-1} \phi_j(x)w(\zeta)(Q_j(x) - Q_j(\zeta)), \]

so (4) may be written,

\[ H_0(x, \zeta) = \sum_{j=1}^{n-1} \phi_j(x)v_j(\zeta) + V(x)w(\zeta) \]  

where \( v_j(\zeta) = -w(\zeta)Q_j(\zeta) \) and \( V(x) = \sum_{j=1}^{n-1} \phi_j(x)Q_j(x) \). Since \( w(\zeta) \approx \zeta^{-a-1} \), and \( Q_j(\zeta) \approx \zeta^{\sigma_j + 1}u_j(\zeta) \), clearly \( v_j = u_j \) for \( 1 < j < n - 1 \). Furthermore, since \( u_n(\zeta) \approx \zeta^{-1-n} \), we have \( W = u_n \). Hence in view of (5), the conclusion of the lemma will hold for \( \Phi \), if it can be shown that \( \{ \phi_1, ..., \phi_{n-1}, V \} \) is a fundamental set for \( \Phi(\theta) = 0 \) and that

\[ V(x) = h_n(x). \]

To prove (6), we note first that \( \phi_1, ..., \phi_{n-1} \) are independent solutions of \( \Phi(\theta) = 0 \), since they form a fundamental set for \( \Phi_1(\theta) = 0 \). Hence in view of (5), we have by § 8 (C) that \( \phi_1, ..., \phi_{n-1}, V \) form a fundamental \( \Phi(\theta) = 0 \). Since \( h_1, ..., h_n \) also form a fundamental set, there exist constant \( \beta_k \) and \( \gamma_k \) such that,

\[ V = \sum_{k=1}^{n} \beta_k h_k, \]

and

\[ h_n = \sum_{k=1}^{n-1} \gamma_k \phi_k + \gamma_n V. \]

Now by hypothesis, for \( q + 1 \leq j \leq n - 1 \), we have \( \text{Re} (\delta_0(h_j)) < \text{Re} (\alpha_j) \). Thus \( h_j < h_n \) (see § 2 (b)), and since \( \phi_j = h_j \), we have \( \phi_j < h_n \) also. Hence

\[ U = \sum_{j=q+1}^{n-1} \beta_j h_j < h_n \]

and

\[ W = \sum_{j=q+1}^{n-1} \gamma_j \phi_j < h_n, \]

and so (7) and (8) may be written,

\[ V = \beta_n h_n + \sum_{i=1}^{q} \beta_i h_i + U, \]

where \( U < h_n \), and

\[ h_n = \gamma_n V + \sum_{i=1}^{q} \gamma_i \phi_i + W \]

where \( W < h_n \).
Now if \( q = 0 \), then (6) will follow from (9) if \( \beta_n \neq 0 \). But this is clear, for if \( \beta_n = 0 \), then by (9), \( V < h_n \), and hence from (10) we would obtain \( h_n < h_n \) (since \( q = 0 \)) which is a contradiction. Now consider the case \( q > 0 \). Then by assumption, either \( \text{Re}(\alpha_r) > 0 \) or \( \text{Re}(\alpha_r) < 0 \). If \( \text{Re}(\alpha_r) > 0 \), then for \( 1 \leq i \leq q \), \( h_i < h_n \) (and hence \( \varphi_i < h_n \)) since \( \delta_0(h_i) = 0 < \text{Re}(\alpha_r) \). Thus again, (6) will follow from (9) if \( \beta_n \neq 0 \). But if \( \beta_n = 0 \), then from (9), \( V < h_n \) and so from (10) we would obtain \( h_n < h_n \) which is impossible. If \( \text{Re}(\alpha_r) < 0 \), we consider each term \( \varphi_i Q_i \) in \( V \). Since \( \varphi_i \approx h_i \), we have for \( 1 \leq j \leq n-1 \), \( \varphi_i Q_i = x^{\alpha_r + 1} (\log x)^{-q} M_j u_j \). By the asymptotic relations for \( M_j \) and \( u_j \), clearly \( \delta_0(M_j u_j) = -1 \), and hence,

\[
\delta_0(\varphi_i Q_i) = \alpha_r \quad \text{for} \quad 1 \leq j \leq n-1.
\]

Since \( \text{Re}(\alpha_r) < 0 \) and \( \delta_0(h_k) = 0 \) for \( 1 \leq k \leq q \), we thus obtain \( V < h_k \) and \( h_n < h_k \) for \( 1 \leq k \leq q \). Thus from (9), \( \sum_{i=1}^{q} \beta_i h_i < h_k \) for each \( k \leq q \).

Since \( h_1 < h_2 < \ldots < h_q \), this implies \( \beta_i = 0 \) for \( 1 \leq i \leq q \), for in the contrary case, setting \( j_0 = \max \{ i : 1 \leq i \leq q, \beta_i \neq 0 \} \), we would obtain the contradiction, \( h_{j_0} \approx \sum_{i=1}^{q} \beta_i h_i < h_{j_0} \). Thus from (9), \( V = \beta_n h_n + U \), so (6) will hold if \( \beta_n \neq 0 \). But if \( \beta_n = 0 \), then \( V < h_n \), so since \( \text{Re}(\alpha_r) < 0 \) and \( \varphi_k \approx h_k \), it would follow by (10), that \( \sum_{i=1}^{q} \gamma_i \varphi_i < \varphi_k \) for \( 1 \leq k \leq q \). This would imply, as above that \( \gamma_i = 0 \) for \( 1 \leq i \leq q \), so from (10) (and \( V < h_n \)) we would again obtain the contradiction \( h_n < h_n \). Thus \( \beta_n \neq 0 \) so (6) holds. Thus in this subcase, the conclusion of the lemma holds for \( \Phi \).

**Subcase B.** \( q < p \) and \( m_r > 1 \). Since \( p = n \), we have \( h_n \sim x^{\sigma_r} (\log x)^{-q + m_r - 1} \). For convenience, let \( \sigma(j) = q + m_1 + \ldots + m_r - 1 + j \) for \( 0 \leq j \leq m_r \). As in Subcase A, \( h_1, \ldots, h_{n-1} \) form a fundamental set for \( \Phi_0(y) = 0 \), and we want to calculate the corresponding functions \( u_k \) for \( h_1, \ldots, h_{n-1} \). Now the \( \alpha_i \) and \( m_j \) involved in \( h_1, \ldots, h_{n-1} \) are the same as in the statement for the lemma, and so the corresponding functions \( u_k \) for \( k \leq \sigma(0) \), are precisely \( u_1, \ldots, u_{\sigma(0)} \) as defined in the statement of the lemma. The remaining solutions in \( \{ h_1, \ldots, h_{n-1} \} \) are \( h_{\sigma(i)} \) for \( 1 \leq j \leq m_r - 1 \). Thus the corresponding functions \( u_k \) for these solutions
are obtained by using \( m' = m - 1 \) in place of \( m \) in the definition of \( u_{\sigma(j)} \) given in § 5. Since \( u_{\sigma(j)} = x^{1-a} (\log x)^{m_{r_1} - j} \), using \( m' \) in place of \( m \) clearly results in \( (\log x)^{-1} u_{\sigma(j)} \) as the corresponding \( u \) for \( h_{\sigma(j)} \). Hence, by applying the inductive assumption to \( \Phi_1 \), there exists e.f.d. in \( F(I) \), a fundamental set \( \{ \varphi_1, ..., \varphi_{n-1} \} \) for \( \Phi_1(y) = 0 \) such that \( \varphi_i \approx h_i \) for each \( i \), and such that the Green's function for \( \Phi_1 \) is of the form \( H_1(x, \zeta) = \sum_{k=1}^{n-1} \varphi_k(x)w_k(\zeta) \) where \( w_k \approx u_k \) for \( 1 \leq k \leq \sigma(0) \) while \( w_{\sigma(0)} \approx (\log x)^{-1} u_{\sigma(0)} \) for \( 1 \leq k \leq m - 1 \). Now \( f_n \approx \alpha_n x^{j-1} \), and by using [1; § 6], it is easily verified that \( \Psi_{n-1} \approx x^{\sigma(j)} \). Hence as in Subcase A, the Green's function for the operator \( f_n \) is \( H_n(x, \zeta) = \Psi_{n-1}(x)w(\zeta) \) where \( w(\zeta) \approx \zeta^{1-a} \). Since \( \Phi = f_n \Phi_1 \), we have using § 8 (B) that the Green's function for \( \Phi \) is

\[
H_0(x, \zeta) = \sum_{k=1}^{n-1} \varphi_k(x)w_k(\zeta) \int_\zeta^x w_k(s)\Psi_{n-1}(s)ds.
\]

Now for \( 1 \leq k \leq \sigma(0) \), \( w_k \approx u_k \) and hence \( w_k(s)\Psi_{n-1}(s) \approx s^{\sigma(j)} u_k(s) \). Hence as in Subcase A, \( \delta_0(w_k \Psi_{n-1}) \neq -1 \), and thus by § 8 D (a), for \( 1 \leq k \leq \sigma(0) \), there exists e.f.d. in \( F(I) \), a function \( Q_k(s) \approx s^{\sigma(j)+1} u_k(s) \) such that \( Q' = w_k \Psi_{n-1} \). Now for \( \sigma(1) \leq k \leq n-1 \), say \( k = \sigma(j) \) where \( 1 \leq j \leq m-1 \), we have \( w_k \approx (\log x)^{-1} u_k \). Since \( u_k \approx x^{-1+a} (\log x)^{m_{r_1}-j} \), and also that \( m_{r_1}-j-1 \geq -1 \) (since \( j < m_r \)), and so by § 8 D (b), for \( k = \sigma(j) \) there exists e.f.d. in \( F(I) \), a function \( Q_k(s) \approx (\log s)^{m_{r_1}-j} \) such that \( Q' = w_k \Psi_{n-1} \). Hence the right side of (12) is \( \sum_{k=1}^{n-1} \varphi_k(x)w_k(\zeta)(Q_k(x) - Q_k(\zeta)) \),

\[
H_0(x, \zeta) = \sum_{k=1}^{n-1} \varphi_k(x)v_k(\zeta) + V(x)w(\zeta),
\]

where \( v_k(\zeta) = -w(\zeta)Q_k(\zeta) \) and \( V(x) = \sum_{k=1}^{n-1} \varphi_k(x)Q_k(x) \). Now for \( 1 \leq k \leq \sigma(0) \), \( Q_k(\zeta) \approx \zeta^{\sigma(j)+1} u_k(\zeta) \) and so \( v_k \approx u_k \) since \( w(\zeta) \approx \zeta^{1-a} \). For \( \sigma(1) \leq k \leq n-1 \), say \( k = \sigma(j) \), we have \( Q_k(\zeta) \approx (\log \zeta)^{m_{r_1}-j} \). Thus \( v_k(\zeta) \approx \zeta^{1-a} (\log \zeta)^{m_{r_1}-j} \) and so again \( v_k \approx u_k \). Furthermore \( w \approx u_n \), and so in view of (13), the conclusion of the lemma will hold for \( \Phi \), if it can be shown that \( \{ \varphi_1, ..., \varphi_{n-1}, V \} \) is a fundamental set for \( \Phi(y) = 0 \) and that

\[
V \approx h_n.
\]
The proof of (14) is very similar to the proof of (6) in Subcase A. As in Subcase A, there exist constants $\beta_k$ and $\gamma_k$ such that (7) and (8) hold. By hypothesis for, $q+1 \leq k \leq \sigma(0)$, $\text{Re} (\delta_0(h_k)) < \text{Re} (\alpha_r)$ so $h_k < h_n$. For $\sigma(1) \leq k \leq n-1$, say $k = \sigma(j)$ (where $1 \leq j \leq m_r - 1$), we have $h_k \sim x^a (\log x)^{-q + j - 1} \text{ so } h_k < h_n \text{ since } j < m_r$. Thus setting $U = \sum_{j=q+1}^{n-1} \beta_j h_j$ and $W = \sum_{j=q+1}^{n-1} \gamma_j \phi_j$, we have $U < h_n$ and $W < h_n$, and so we obtain (9) and (10). The proof now proceeds exactly as in Subcase A to establish (14). (We remark that the relation (11) which is needed in the proof is easy to verify, as in Subcase A, by using the definition of $Q_j$.)

**Subcase C.** $q = p$. Thus $q = n$ by this case. As before, $h_1, ..., h_{n-1}$ form a fundamental set for $\Phi(q) = 0$ given by (2). Now $h_j \sim (\log x)^{-q + j - 1}$ for $1 \leq j \leq n-1$, and this does not fit the induction hypothesis for $\Phi$. (i.e. Since $\Phi_1 = \tilde{f}_n ... f_1$, the corresponding $q$ for $\Phi_1$ is $q - 1$, and hence in order to apply the inductive assumption to $\Phi_1$, the $j^{th}$ solution must be $\sim (\log x)^{-(q-1)M_j}$ which is clearly not the case for $h_j$.) To remedy this, we set $\Lambda(z) = \Phi(q^{-1})z$. Then for $1 \leq j \leq n-1$, the functions $h_j^* = (\log x)h_j$ solve $\Lambda(z) = 0$. Clearly, $h_j^* \sim (\log x)^{-q + j}$, so,

$$h_j^* \sim (\log x)^{-(q-1)M_j} \text{ for } 1 \leq j \leq n-1. \tag{15}$$

Define functions $U_1, ..., U_{n-1}, \psi_0, ..., \psi_{n-2}$ recursively by $\psi_0 = h_1^*$ and $U_{j+1} = \psi_j/\psi_j$ where $\psi_j = U_j ... U_1(h_j^*)$. Then clearly,

$$\dot{U}_j ... \dot{U}_1(h_j^*) = 0 \text{ for } 1 \leq j \leq n-1. \tag{16}$$

In view of (15), it follows easily using [1; § 6 (A), (D)] that for $1 \leq j \leq n-1$,

$$U_j \sim -(q-j)x^{-1}(\log x)^{-1} \text{ and } \psi_{j-1} \approx h_j^*. \tag{17}$$

Let $\Lambda_1 = \dot{U}_{n-1} ... \dot{U}_1$. In view of (15), (16), (17), it is clear that $\Lambda_1$, with the solutions $h_1^*, ..., h_{n-1}^*$, satisfies the inductive assumption using $q-1$ for $q$. The corresponding functions $u_j$ are clearly obtained by using $q-1$ for $q$ in the definition of $u_j$ given in § 5. Since
where $w_j=(\log x)^{-1}u_j$ for $1 \leq j \leq n-1$. We now prove,

$$\Lambda(a(x)\Lambda_1, \text{ where } a(x)=\Phi_1((\log x)^{-1}).$$

To prove (19), we apply the division algorithm for linear differential operators ([9; § 2]), and divide $\Lambda$ by $\bar{U}_1$. Since $\bar{U}_1$ is of order one, there exist an operator $\Gamma_1$ and a function $b_1(x)$ such that $\Lambda=\Gamma_1\bar{U}_1+b_1$. Since $\Phi_1$ is of order $n-1$, clearly $\Lambda$ is of order $n-1$ and hence $\Gamma_1$ is of order $n-2$ by [9; § 5(a)]. Since $\Lambda(h_1^*)=0$ and $\bar{U}_1(h_1^*)=0$ (by (16)), we have $b_1h_1^*=0$. Since $h_1^*=0$ by (15), $b_1=0$ so $\Lambda=\Gamma_1\bar{U}_1$. Dividing $\Gamma_1$ by $\bar{U}_2$, there exists an operator $\Gamma_2$ of order $n-3$ and a function $b_2$ such that $\Gamma_1=\Gamma_2\bar{U}_2+b_2$. Since $\Lambda(h_2^*)=0$ and $\bar{U}_2(h_2^*)=0$ (by (16)), we have $b_2U_1(h_2^*)=0$. Since $U_1(h_2^*)=\psi_1$ and $\psi_1\neq 0$ by (17) we obtain $b_2=0$, so $\Lambda=\Gamma_2\bar{U}_2\bar{U}_1$. Continuing this way, we clearly obtain $\Lambda=\Gamma_{n-1}\Lambda_1$ where $\Gamma_{n-1}$ is an operator of order zero. Thus for some function $a(x)$, $\Lambda(z)=a(x)\Lambda_1(z)$. Evaluating at $z=1$ (and noting that $\bar{U}_1(1)=1$), we obtain (19).

From (19) and the definition of $\Lambda$, we have, $\Lambda_1(z)=(1/a(x))\Phi_1((\log x)^{-1}z)$. Thus by § 8 (A), the Green’s function $H_1(x, \zeta)$ for $\Phi_1$ is related to the Green’s function $K(x, \zeta)$ for $\Lambda_1$ by $K(x, \zeta)=a(\zeta)(\log x)H_1(x, \zeta)$. Thus from (18), we obtain,

$$H_1(x, \zeta)=\sum_{j=1}^{n-1} \varphi_j(x)(w_j(\zeta)/a(\zeta)),$$

where $\varphi_j(x)=(\log x)^{-1}\varphi_j^*(x)$. Since $\varphi_j^*=h_j^*$, clearly,

$$\varphi_j\approx h_j \text{ for } 1 \leq j \leq n-1.$$
Now by (19), \( a(x) = f_{n-1} \ldots f_1((\log x)^{-1}) \), and by definition, \( \Psi_{n-1} = f_{n-1} \ldots f_1(h_n) \), where by assumption, \( h_n \sim (\log x)^{-1} \). Since \( q = n \), we have \( f_j \sim -(q-j+1)x^{-1}(\log x)^{-1} \), and so it easily follows using \([1; \S 6 (D)]\), that

\[
(22) \quad a(x) \approx (\log x)^{-1} \quad \text{and} \quad \Psi_{n-1} \approx (\log x)^{-1}.
\]

Since \( f_n(\Psi_{n-1}) = 0 \) and \( f_n \sim -x^{-1}(\log x)^{-1} \), it follows from \( \S 8 \ A \) that the Green’s function for \( f_n \) is \( H_2(x, \zeta) = \Psi_{n-1}(x)w(\zeta) \) where (using (22)), \( w(\zeta) \approx \zeta^{-1} \). Since \( \Phi = f_n\Phi_1 \), we have by \( \S 8 \ B \) and (20) that the Green’s function for \( \Phi \) is,

\[
(23) \quad H_0(x, \zeta) = \sum_{j=1}^{n-1} \phi_j(x)w(\zeta) \int_{\zeta}^{x} (w_j(s)\Psi_{n-1}(s)/a(s))ds.
\]

Now \( w_j(s) \approx (\log s)^{-1}u_j(s) \) and \( \Psi_{n-1}(s)/a(s) \approx 1 \) by (22). Hence since \( u_j(s) \approx s^{-(\log s)^{q-j}} \), we have \( w_j(s)\Psi_{n-1}(s)/a(s) \approx (\log s)^{q-j-1} \) for \( 1 \leq j \leq n-1 \). Since \( q = n \) and \( j < n \), \( q-j-1 \geq -1 \). Thus by \( \S 8 \ D \ (b) \), for each \( j = 1, \ldots, n-1 \), there exists e.f.d. in \( F(I) \), a function \( Q(s) \approx (\log s)^{q-j} \) such that \( Q_j = w_j\Psi_{n-1}/a \). Hence the right side of (23) is \( \sum_{j=1}^{n-1} \phi_j(x)w(\zeta) (Q_j(x) - Q_{j}(\zeta)) \) and so (23) may be written,

\[
(24) \quad H_0(x, \zeta) = \sum_{j=1}^{n-1} \phi_j(x)v_j(\zeta) + V(x)w(\zeta),
\]

where \( v_j(\zeta) = -w(\zeta)Q_j(\zeta) \) and \( V(x) = \sum_{j=1}^{n-1} \phi_j(x)Q_j(x) \). Since \( w(\zeta) \approx \zeta^{-1} \), \( v_j(\zeta) \approx \zeta^{-1}(\log \zeta)^{q-j} \) so \( v_j \approx u_j \) for \( 1 \leq j \leq n-1 \). Furthermore \( w \approx u_n \), so in view of (21) and (24), the conclusion of the lemma will hold for \( \Phi \), if it can be shown that \( \{ \phi_1, \ldots, \phi_{n-1}, V \} \) is a fundamental set of solutions for \( \Phi(y) = 0 \) and that

\[
(25) \quad V \approx h_n.
\]

To prove (25), we note first that since \( \{ \phi_1^*, \ldots, \phi_{n-1}^* \} \) is a fundamental set for \( \Lambda_1(y) = 0 \), clearly \( \{ \phi_1, \ldots, \phi_{n-1} \} \) is a fundamental set for
\( \Phi_1(y) = 0 \). Since \( \Phi = f_1 \Phi_1, \{ \varphi_1, ..., \varphi_{n-1} \} \) is therefore an independent set of solutions of \( \Phi(y) = 0 \), and hence in view of (24), it follows from \( \S \) 8 c that \( \varphi_1, ..., \varphi_{n-1}, V \) form a fundamental set for \( \Phi(y) = 0 \). Since \( h_n \) is a solution \( \Phi(y) = 0 \) by hypothesis, there exist constants \( \gamma_j \) such that,

\[
(26) \quad h_n = \sum_{j=1}^{n-1} \gamma_j \varphi_j + \gamma_n V.
\]

Since \( n = q \), \( h_j \sim (\log x)^{-q+j-1} \) and so \( h_j < h_n \) for \( j < n \). Since \( \varphi_j \approx h_j \) by (21), \( \varphi_j < h_n \) for \( j < n \). Thus \( \gamma_j \neq 0 \), for otherwise by (26), we would obtain the contradiction \( h_n < h_n \). Hence \( \gamma_j \neq 0 \), and so \( h_n \approx V \) by (26). This proves (25), and so the conclusion of the lemma holds for \( \Phi \) in Subcase C, which completes Case I.

**CASE II.** \( p < n \). Then \( h_n = A_n \exp \int f_n \). Now \( h_1, ..., h_{n-1} \) form a fundamental set for \( \Phi_1(y) = 0 \) (see (2)), and we want to calculate the corresponding functions \( u_j \) for \( h_1, ..., h_{n-1} \). Since \( p \leq n - 1 \), the \( \alpha_i \) and \( m_i \) involved in \( h_1, ..., h_p \) are the same as in the statement of the lemma, and so the corresponding functions \( u_j \) are precisely \( u_1, ..., u_p \) as defined in the statement of the lemma. For the remaining solutions \( h_{p+1}, ..., h_{n-1} \), the corresponding functions \( u_k \) are clearly obtained by using \( n - 1 \) in place of \( n \) in the definitions \( u_{p+1}, ..., u_{n-1} \) given in the statement of the lemma (i.e. \( \S \) 5). Since for \( p + 1 \leq k \leq n - 1 \), \( u_k \) is defined as

\[
E_k(x) \exp \left( -\int f_k \right) \text{ where } E_k = f_k \prod_{i=k+1}^{n} \left( f_i / (f_j - f_k) \right), \text{ using } n - 1 \text{ for } n \text{ clearly results in } E_k^* \exp \left( -\int f_k \right), \text{ where } E_k = f_k \prod_{i=k+1}^{n-1} \left( f_i / (f_j - f_k) \right), \text{ as the corresponding } u \text{ for } h_k. \text{ Hence by applying the inductive assumption to } \Phi_1, \text{ there exists e.f.d. in } F(I), \text{ a fundamental set } \{ \varphi_1, ..., \varphi_{n-1} \} \text{ for } \Phi_1(y) = 0 \text{ such that } \varphi_j \approx h_j \text{ for each } j, \text{ and such the Green's function for } \Phi_1 \text{ is of the form } H_1(x, \zeta) = \sum_{j=1}^{n} \varphi_j(x) w_j(\zeta) \text{ where } w_j \approx u_j \text{ for } 1 \leq j \leq p, \text{ while } w_k(\zeta) \approx E_k^*(\zeta) \exp \left( -\int f_k \right) \text{ for } p + 1 \leq k \leq n - 1. \text{ Let } z_0(x) \text{ be a function of the form } \exp \int f_n. \text{ Since } f_n(z_0) = 0, \text{ it follows from } \S \) 8 A that the
Green's function for \( f_n \) is \( H_2(x, \zeta) = z_0(x)w(\zeta) \), where

\[
(27) \\
w(\zeta) = f_n(\zeta) / z_0(\zeta).
\]

Since \( \Phi = \hat{f}_n \Phi_1 \), we have by § 8 (B) that the Green's function for \( \Phi \) is,

\[
(28) \\
H_0(x, \zeta) = \sum_{j=1}^{n-1} \phi_j(x)w(\zeta) \int_{\zeta}^{x} w_j(s)z_0(s)ds.
\]

Now for \( 1 \leq j \leq p \), \( w_j \approx u_j \) so \( w_j \approx \) to a function of the form \( x^d \log x \). Since \( p < n \), \( f_n \approx \) to a function of the form \( cx^{-1+d} \) where \( d > 0 \). Thus clearly (see § 2 (b)), \( IF(f_n) \) has only finitely many zeros on \((-\pi, \pi)\). Since \( z_0(s) = \exp \int f_n \), it follows from [3; § 10 (b)] that for \( 1 \leq j \leq p \), there exists e.f.d. in \( F(I) \), a function of the form \( Q_j(s) = \approx a_j(s)z_0(s) \) where \( a_j \approx w_j/f_n \), such that \( Q'_j = w_jz_0 \). For \( p+1 \leq k \leq n-1 \), \( w_k(s)z_0(s) \approx \) to a function of the form \( E_k(s) \exp \int (f_n-f_k) \). Now for \( p+1 \leq k < j \leq n \), \( f_j - f_k \approx f_i \) (since \( N_{k-p} \leq N_{j-p} \) and \( N_{k-p} \neq N_{j-p} \)), and so it easily follows that \( E_k \) is \( \approx \) to a function of the form \( x^d \log x \). Since \( f_n - f_k \approx f_n \), \( IF(f_n-f_k) \) has only finitely many zeros. Thus it follows from [3; § 10 (b)] that for \( p+1 \leq k \leq n \), there exists e.f.d. in \( F(I) \), a function of the form \( Q_k(s) = T_k(s) \exp \int (f_n-f_k) \), where \( T_k \approx E_k / (f_n - f_k) \) such that \( Q'_k = w_kz_0 \). Hence the right side of (28) is \( \sum_{j=1}^{n-1} \phi_j(x)w(\zeta)(Q_j(x) = Q_j(\zeta)) \), so (28) can be written,

\[
(29) \\
H_0(x, \zeta) = \sum_{j=1}^{n-1} \phi_j(x)v_j(\zeta) + V(x)w(\zeta),
\]

where \( v_j(\zeta) = -w(\zeta)Q_j(\zeta) \) and \( V(x) = \sum_{j=1}^{n-1} \phi_j(x)Q_j(x) \). Now in view of (27), for \( 1 \leq j \leq p \), \( v_j \approx f_n a_j \). Since \( a_j \approx w_j/f_n \), and \( w_j \approx u_j \), we have \( v_j \approx u_j \).
By (27), \( w(\zeta) \approx f_n(\zeta) \exp \left( -\int \zeta f_n \right) \). Thus for \( p+1 \leq k \leq n \), clearly \( v_k(\zeta) \approx (f_n E_k^*/(f_n - f_k)) \exp \left( -\int \zeta f_k \right) \) and hence \( v_k(\zeta) \approx E_k(\zeta) \exp \left( -\int \zeta f_k \right) \). Thus \( v_k \approx u_k \). Furthermore by (27), \( w(\zeta) \approx u_n(\zeta) \), so in view of (29), the conclusion of the lemma will hold for \( \Phi \) if it can be shown that \( \{ \varphi_1, \ldots, \varphi_{n-1}, V \} \) is a fundamental set for \( \Phi(y) = 0 \) and that

\[ \Phi \approx h_n. \quad (30) \]

To prove (30), we note that since \( \{ \varphi_1, \ldots, \varphi_{n-1} \} \) is a fundamental set for \( \Phi_1(y) = 0 \), and since \( \Phi = f_n \Phi_1 \), it follows from (29) and § 8 D that \( \{ \varphi_1, \ldots, \varphi_{n-1}, V \} \) is a fundamental set for \( \Phi(y) = 0 \). Since \( \Phi(h_n) = 0 \), there exist constants \( \beta_i \) such that

\[ h_n = \sum_{j=1}^{n-1} \beta_j \varphi_j + \beta_n V, \quad (31) \]

whence

\[ (h_n - \beta_n V)/h_n = \frac{1}{\sum_{j=1}^{n-1} \beta_j (\varphi_j/h_n)}. \quad (32) \]

We now calculate each term \( \varphi_i Q_i \) in \( V \). For \( 1 \leq j \leq p \), \( \varphi_j Q_j = \varphi_j a_j z_0 \).

For \( p+1 \leq k \leq n-1 \), we have \( \varphi_k \approx h_k \), \( h_k = A_k \exp \int f_k \) and \( Q_k = T_k \exp \int (f_n - f_k) \). Since \( z_0 = \exp \int f_n \), it follows easily that \( \varphi_k Q_k = \Delta_k A_k T_k z_0 \) where \( \Delta_k = 1 \). Thus clearly,

\[ V = U z_0, \quad (33) \]

where \( U = \sum_{j=1}^{p} \varphi_j a_j + \sum_{k=p+1}^{n-1} \Delta_k A_k T_k \). Now \( A_k \approx \Pi_{j=1}^{k-1} (f_j/(f_j - f_k)) \). Since \( f_j - f_k \) is \( \approx f_k \) if \( j < k \) and \( k \geq p+1 \), it follows easily that

\[ A_k \approx \text{a function of the form } x^\lambda (\log x)^\nu. \quad (34) \]
In particular, $A_k$ is some power of $x$. Since $\Delta_k T_k = E_k^p/(f_n - f_k)$, it follows similarly $\Delta_k T_k$ is some power of $x$ for $p+1 \leq k \leq n-1$. Since $\varphi_j = h_j$ and $a_i = u_i/f_n$ for $j \leq p$, it follows easily that $\varphi_j$ and $a_i$ are each some power of $x$. Thus each term in $U$ is some power of $x$, so clearly,

$$U(x) < x^\sigma$$

for some real number $\sigma$.

Since $h_n = A_n \exp \int f_n$, clearly $h_n = cA_n z_0$ for some $c \neq 0$. Hence in view of (33), the left side of (32) is $(cA_n - \beta_n U)/(cA_n)$, which by (34) and (35) is clearly some power of $x$. Thus by (32),

$$\sum_{j=1}^{n-1} \beta_j h_n < x^\lambda$$

for some real number $\lambda$.

Consider $\sum_{j=1}^{p} \beta_j \varphi_j$. Now by hypothesis, $\Re(\alpha_i) < \Re(\alpha_j)$ if $i < j$, and if $q > 0$, $\Re(\alpha_i) \neq 0$. It easily follows (since $\varphi_j = h_j$) that for $1 \leq i < j \leq p$, either $\varphi_i < \varphi_j$ or $\varphi_j < \varphi_i$ (see § 2 (b)). Hence clearly, if not all of $\beta_1, \ldots, \beta_p$ are zero, then there exists an index $j_0 \in \{1, \ldots, p\}$ such that $\beta_{j_0} \neq 0$ and $\varphi_i < \varphi_{j_0}$ if $i \leq p$ and $i \neq j_0$. Thus

$$\sum_{j=1}^{p} \beta_j \varphi_j = \varphi_{j_0}(\beta_{j_0} + b(x)) \text{ where } b < 1.$$  

(If all of $\beta_1, \ldots, \beta_p$ are zero, set $\beta_{j_0}$ and $b$ equal zero so (37) still holds.) For $p+1 \leq k \leq n-1$, set $D_k = \varphi_k / h_n$. Then we may write,

$$\sum_{j=1}^{n-1} \beta_j \varphi_j / h_n = (1/h_n) \sum_{j=1}^{p} \varphi_j \beta_j + \sum_{k=p+1}^{n-1} \beta_k D_k.$$  

Now for $p+1 \leq k \leq n$, clearly $IF(f_k)$ has only finitely many zeros (see § 2 (b)). For $p+1 \leq j < k \leq n$, $f_j - f_k \approx f_k$ so $IF(f_j - f_k)$ also has only finitely many zeros. Thus if we let $\Gamma$ be the union of all zeros in $I$ of all the above functions $IF(f_k)$ and $IF(f_j - f_k)$, then $\Gamma$ is a finite set, say $\varepsilon_1 < \ldots < \varepsilon_m$. If $I = (\varepsilon_0, \varepsilon_{m+1})$, then letting $J$ be any subinterval of any
of the intervals \((\varepsilon_j, \varepsilon_{j+1})\) such that \(\{\varphi_1, ..., \varphi_{n-1}, V\}\) exist on \(F(J)\), we have that (36) is valid on \(F(J)\) and all \(IF(f_k)\) and \(IF(f_i-f_k)\) as above, are nowhere zero on \(J\). Now clearly, since \(\varphi_k = h_k\), we have \(D_k \approx \approx (A_k/A_n) \exp \int f_k \). In view of (34) and the fact that \(IF(f_k-f_n)\) is nowhere zero on \(J\), it follows from [3; § 10 (a)], that for each \(k \in \{p+1, ..., n-1\}\),

(39) Either \(D_k\) is trivial in \(F(J)\) (i.e. \(D_k < x^\alpha\) for all \(\alpha\)) or \(1/D_k\) is trivial in \(F(J)\).

Since \(h_k = A_k \exp \int f_k\), it follows similarly using (34) and [3; § 10 (a)] that for each \(k \in \{p+1, ..., n\}\),

(40) Either \(h_k\) is trivial or \(1/h_k\) is trivial in \(F(J)\).

Finally, if \(j\) and \(k\) are distinct elements of \(\{p+1, ..., n-1\}\), then since \(D_j/D_k \approx (A_j/A_k) \exp \int (f_j-f_k)\), it follows as above that

(41) Either \(D_j/D_k\) is trivial or \(D_k/D_j\) is trivial in \(F(J)\). We now return to (36) and prove,

(42) For each \(j \in \{p+1, ..., n-1\}\) such that \(1/D_j\) is trivial in \(F(J)\), we have \(\beta_j = 0\).

We prove (42) by contradiction. We assume the contrary and let \(i_0\) be an index such that \(1/D_{i_0}\) is trivial but \(\beta_{i_0} \neq 0\). Let \(L\) be the set of all \(j \in \{p+1, ..., n-1\}\) for which \(\beta_j \neq 0\). For \(i\) and \(j\) in \(L\) with \(i \neq j\), we have by (41) that either \(D_i < D_j\) or \(D_j < D_i\). Since \(L\) is a finite set, clearly there exists \(k_0 \in L\) such that \(D_i < D_{k_0}\) if \(i \in L \setminus \{k_0\}\). If \(k_0 = i_0\) then \(1/D_{k_0}\) is trivial. If \(k_0 \neq i_0\) then \(D_{i_0} < D_{k_0}\) so again,
By the property of \( k_0 \), we can write 
\[
\sum_{j=p+1}^{n-1} B_j D_j = B_1 D_1 (1 + t) \text{ where } t < 1.
\]
Hence by (36), (37) and (38), we obtain in \( F(J) \),
\[
(\varphi_{j_0}/h_{j_0})(\beta_{j_0} + b) + \beta_{k_0} D_{k_0} (1 + t) < x^\lambda.
\]
Now \( D_{k_0} h_{j_0} = \varphi_{k_0} \). Thus dividing (44) by \( D_{k_0} \) and using (43),
\[
(\varphi_{j_0}/\varphi_{k_0})(\beta_{j_0} + b) + \beta_{k_0} (1 + t) \text{ is trivial in } F(J).
\]
Since \( \beta_{k_0} \) is a non-zero constant, \( \beta_{k_0} \approx 1 \). If \( \beta_{j_0} = 0 \) (and \( b = 0 \)), then (45) is clearly impossible. If \( \beta_{j_0} \neq 0 \), then since \( \beta_{k_0} \approx 1 \), we have from (45) that \( \beta_{j_0} \varphi_{j_0}/\varphi_{k_0} \approx -\beta_{k_0} \). Thus \( \varphi_{j_0}/\varphi_{k_0} \approx 1 \) and so \( h_{j_0} \approx h_{k_0} \). This is clearly impossible since \( h_{j_0} \) is \( \approx \) to a function of the form \( x^a (\log x)^m \) (since \( j_0 \leq p \)), while by (40), either \( h_{k_0} \) or \( 1/h_{k_0} \) is trivial. This contradiction proves (42), which in view of (39) clearly implies,
\[
\sum_{k=p+1}^{n-1} \beta_k D_k \text{ is trivial in } F(J).
\]
If \( \beta_{j_0} = 0 \) (and \( b = 0 \)) in (37), then by (46), the left side of (38) is trivial. Thus by (32), \( (h_n - \beta_n V)/h_n \) is trivial and hence is \( < 1 \) in \( F(J) \).
Thus \( \beta_n \neq 0 \) and \( h_n \approx V \) proving (30). If \( \beta_{j_0} \neq 0 \), then \( \sum_{j=1}^{p} \beta_j \varphi_j \approx \varphi_{j_0} \). But in view of (46), we have by (38) and (36) that \( (1/h_n) \sum_{j=1}^{p} \beta_j \varphi_j < x^\lambda \). Hence \( \phi_{j_0}/h_n < x^\lambda \), so \( (1/h_n) < x^\lambda/\varphi_{j_0} \). But \( \varphi_{j_0} \approx h_{j_0} \) and so (since \( j_0 \leq p \), \( \varphi_{j_0} \) is \( \approx \) to a function of the form \( x^a (\log x)^m \). Thus \( 1/h_n \) is \( < \) some power of \( x \). Hence by (40), \( 1/h_n \) must be trivial in \( F(J) \). Thus \( (1/h_n) \sum_{j=1}^{p} \beta_j \varphi_j \)
is trivial, so by (46), the left side of (38) is trivial. Hence by (32), \( (h_n - \beta_n V)/h_n \) is trivial, whence \( < 1 \), and so again \( \beta_n \neq 0 \) and \( h_n \approx V \) in \( F(J) \) proving (30). Thus in Case II, the conclusion of the lemma holds for \( \Phi \), and so the lemma is established by induction.
7. Conclusion of proof of § 5.

Let \( \Omega, q, p, M_i \) and \( u_k \) be as in § 5, where the roots \( \alpha_i \) are arranged so that \( \Re (\alpha_j) \leq \Re (\alpha_{j+1}) \). By § 4, e.f.d. in \( F(a, b) \), the operator \( \Phi_0(z) = (1/q!) \Lambda ((\log x)z) \) possesses a factorization \( \Phi_0 = \Phi f \) (with \( f \) as in § 4 (c)), and there exists a fundamental set \( \{ g_1, ..., g_n \} \) for \( \Omega(y) = 0 \), with \( g_j \sim M_j \) for \( 1 \leq j \leq p \) and \( g_k = R_k \exp \int f_k \) for \( k > p \), such that if
\[
 h_j = (\log x)^{-q} g_j \text{ for each } j, \text{ then } \Phi = f_n \cdots f_1 \text{ satisfies the hypothesis of § 6 relative to the solution } h_1, ..., h_n. \text{ Hence by § 6, e.f.d. in } F(a, b), \text{ there exists a fundamental set } \{ \varphi_1, ..., \varphi_n \} \text{ for } \Phi(y) = 0 \text{ such that } \varphi_j \approx h_j \text{ and such that the Green's function for } \Phi \text{ is } H_0(x, \zeta) = \sum_{j=1}^n \varphi_j(x) v_j(\zeta) \text{ where } v_j = u_j \text{ for each } j. \text{ By § 8 A, the Green's function for } \Omega \text{ is } H(x, \zeta) = (\log x)^q H_0(x, \zeta)/(q! E(\zeta)). \text{ Thus } H(x, \zeta) = \sum_{j=1}^n \psi_j(x) w_j(\zeta), \text{ where } \psi_j(x) = (\log x)^q \varphi_j(x) \text{ and } w_j(\zeta) = v_j(\zeta)/(q! E(\zeta)). \text{ Then clearly, } \{ \psi_1, ..., \psi_n \} \text{ is a fundamental set for } \Omega(y) = 0 \text{ and } \psi_j = u_j \text{ (since } \varphi_j \approx h_j). \text{ Hence, } \{ \psi_1, ..., \psi_n \} \text{ is an asymptotically canonical fundamental system for } \Omega \text{ in the sense of § 4. Finally, since } E \sim 1, \text{ clearly } w_j = u_j. \text{ This concludes the proof of the main theorem.}

8. Results needed in the proof of §§ 6.

A. LEMMA. Let \( f \) and \( E \) be analytic functions having no zeros in a simply-connected region \( D \). Then:

(a) If \( h(z) \) is analytic function in \( D \) such that \( \dot{h}(h) = 0 \) and \( h \not= 0 \), then the Green's function for \( \dot{h} \) is \( K(x, \zeta) = h(x)w(\zeta), \text{ where } w(\zeta) = -f(\zeta)/h(\zeta). \)

(b) If \( \Lambda(y) = \sum_{j=0}^n a_j(y) y^{(j)}, \text{ where the } a_j(x) \text{ are analytic in } D \text{ and } a_n(x) \text{ has no zeros in } D, \text{ and if } \Lambda(z) = E(x) \Omega(f(z)), \text{ then the Green's function } H(x, \zeta) \text{ for } \Omega \text{ is related to the Green's function } H_1(x, \zeta) \text{ for } \Lambda \text{ by } H_1(x, \zeta) = H(x, \zeta)/(f(x) E(\zeta)). \)

PROOF. Since \( \{ h \} \) is a fundamental set for \( \dot{h}(y) = y - (y'/f) = 0 \), Part (a) follows from the definition of \( K(x, \zeta). \)
For (b), set \( H_2(x, \zeta) = E(\zeta)f(x)H_1(x, \zeta) \). As in § 3, for each \( \zeta \in D \), \( H_1(x, \zeta) \) is a solution of \( \Lambda(z) = 0 \) satisfying the following initial conditions at \( x = \zeta \): \( \partial^k H_1(x, \zeta) / \partial x^k = 0 \) for \( k \leq n-2 \), while \( \partial^{n-1} H_1(x, \zeta) / \partial x^{n-1} = -1/(E(\zeta)f(\zeta)a_n(\zeta)) \) since \( Ef a_n \) is the leading coefficient of \( \Lambda \). It is then easily verified that for each \( \zeta \), \( H_2(x, \zeta) \) is a solution of \( \Omega(y) = 0 \) satisfying the same initial conditions at \( x = \zeta \) as the solution \( H(x, \zeta) \) (see § 3). Hence \( H_2 = H \) by the uniqueness theorem for linear differential equations.

**B. LEMMA.** Let \( \Phi_1(y) = \sum_{j=0}^{n} a_j(x)y^{(j)} \) and \( \Phi_2(y) = \sum_{j=0}^{m} b_j(x)y^{(j)} \), where the \( a_j \) and \( b_j \) are analytic in a simply-connected region \( D \), and \( a_n \) and \( b_m \) have no zeros in \( D \). Let \( \Phi_3 = \Phi_2 \Phi_1 \) and for \( k = 1, 2, 3 \) let \( H_k(x, \zeta) \) be the Green's function for \( \Phi_k \). Then \( H_3(x, \zeta) = \int_{\zeta}^{x} H_1(x, s)H_2(s, \zeta)ds \), the contour of integration being any rectifiable path in \( D \) from \( \zeta \) to \( x \).

**PROOF.** Set \( K(x, \zeta) = \int_{\zeta}^{x} H_1(x, s)H_2(s, \zeta)ds \). By the property of the Green's function given in § 1, \( K(x, \zeta) \) is for each \( \zeta \), a solution of \( \Phi_1(y) = H_2(x, \zeta) \), and hence (see § 3), \( K(x, \zeta) \) is a solution of \( \Phi_3(y) = 0 \). Furthermore, using the initial conditions at \( x = \zeta \) satisfied by \( H_1 \) and \( H_2 \) (see § 3), a straightforward calculation shows for each \( \zeta \in D \), the solution \( K(x, \zeta) \) of \( \Phi_3(y) = 0 \) satisfies the same initial conditions at \( x = \zeta \) as the solution \( H_3(x, \zeta) \) (see § 3). Thus by the uniqueness theorem for linear differential equations \( K = H_3 \) proving Lemma B.

**C. LEMMA.** Let \( \Phi(y) = \sum_{j=0}^{n} a_j(x)y^{(j)} \), where the \( a_j \) are analytic in \( D \) and \( a_n \) is nowhere zero in \( D \). Then if the Green's function for \( \Phi \) can be written in the form \( H(x, \zeta) = \sum_{j=1}^{n} \varphi_j(x)w_j(\zeta) \), where \( \varphi_1, ..., \varphi_{n-1} \) are linearly independent solutions of \( \Phi(y) = 0 \), then \( \{ \varphi_1, ..., \varphi_{n} \} \) form a fundamental set of solutions for \( \Phi(y) = 0 \).

**PROOF.** We complete \( \{ \varphi_1, ..., \varphi_{n-1} \} \) to a fundamental set \( \{ \varphi_1, ..., \varphi_{n-1}, g \} \) for \( \Phi(y) = 0 \). Then by definition (§ 3), \( H(x, \zeta) = \sum_{j=1}^{n-1} \varphi_j(x)v_j(\zeta) + \)
\( g(x)v_n(\zeta) \), and it is proved in [7; p. 38], that \( \{v_1, \ldots, v_n\} \) form a fundamental set for the adjoint equation \( \Phi'(y) = 0 \). Now for each \( \zeta \in D \), \( H(x, \zeta) \) solves \( \Phi(y) = 0 \), so clearly,

\[
(47) \quad \Phi(\varphi_n(x)w_n(\zeta)) = 0 \text{ for each } \zeta
\]

If \( w_n(\zeta) = 0 \), then from the two representations for \( H(x, \zeta) \), and the independence of \( \{\varphi_1, \ldots, \varphi_{n-1}, g\} \), we would obtain \( v_n(\zeta) = 0 \) which would contradict the independence of \( \{v_1, \ldots, v_n\} \). Thus for some \( \zeta \in D \), \( w_n(\zeta) \neq 0 \) and so from (47), \( \varphi_n \) is a solution of \( \Phi(y) = 0 \). To show \( \{\varphi_1, \ldots, \varphi_n\} \) is independent, we assume the contrary. Then since \( \{\varphi_1, \ldots, \varphi_{n-1}\} \) is independent, we would have a relation of the form

\[
\varphi_n = \sum_{i=1}^{n-1} c_i \varphi_i.
\]

Thus, \( H(x, \zeta) = \sum_{i=1}^{n-1} \varphi_i(x)(w_i(\zeta) + c_i w_n(\zeta)) \), which together with the other representation for \( H \) and the independence \( \{\varphi_1, \ldots, \varphi_{n-1}, g\} \) again the contradiction \( v_n(\zeta) = 0 \), thus proving Lemma C.

**D. LEMMA.** Let \( R(x) \) be a function such that in some \( F(I) \), \( R(x) \approx x^\alpha (\log x)^\beta \) for some complex number \( \alpha \) and real number \( \beta \). Then:

(a) If \( \alpha \neq -1 \), then e.f.d. in \( F(I) \), there exists a function \( Q(x) \approx x^\alpha R(x) \) such that \( Q' = R \).

(b) If \( \alpha = -1 \) but \( \beta \neq -1 \), then e.f.d. in \( F(I) \), there exists a function \( Q(x) \approx (\log x)^{\beta+1} \) such that \( Q' = R \).

**PROOF.** Under the change of variable \( y = x^\alpha z \) and division by \( x^{\alpha-1} \), the equation \( y' = R(x) \) is trasformed into,

\[
(48) \quad xz' + \alpha z = T(x), \text{ where } T(x) = x^{1-\alpha} R(x).
\]

Thus \( T(x) \approx (\log x)^\beta \). If \( \alpha \neq -1 \), then by [4; § 3], equation (48) possesses, e.f.d. in \( F(I) \), a solution \( z_0(x) \approx T(x) \). Part (a) then follows by taking \( Q(x) = x^\alpha z_0(x) \). If \( \alpha = -1 \) but \( \beta \neq -1 \), then by [4; § 3], equation (48) possesses, e.f.d. in \( F(I) \), a solution \( z_1(x) \approx (\log x)T(x) \). Part (b) then follows by taking \( Q(x) = x^{-1} z_1(x) \).

**REMARK.** In the case where \( \alpha \) is real, Lemma D also follows from [6; Lemma \( \zeta \), p. 272].
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