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Rings of continuous functions with values in an archimedean ordered field


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Introduction.

The purpose of this paper is to study the ring $C(X, F)$ of all continuous functions on a topological space $X$ with values in a proper subfield $F$ of the real number field $R$. The ring $C(X) \equiv C(X, R)$ of all real-valued continuous functions on $X$ has been extensively studied; a standard reference to this work is the book $[GJ]$. Furthermore, Pierce, in his paper $[P]$, laid the foundations of a theory of rings of integer-valued functions $C(X, Z)$.

It seems natural to study the rings intermediate to $C(X)$ and $C(X, Z)$; if the study of the ring $C(X, Z)$ gives some information about the special properties of $C(X)$ which depend on $R$ being a field, a study of the rings $C(X, F)$ should bring to light the properties of $C(X)$ which depend on $R$ being an order-complete field.

Our results can be summarized as follows: With regard to the relationship between $C(X, F)$ and the underlying topological space $X$, $C(X, F)$ behaves much like $C(X, Z)$. On the other hand, the residue class fields of $C(X, F)$ are similar to those of $C(X)$.

Many of the results in this paper are not surprising, however, answering as they do a natural question, we think that they are worthy of some study.
1. Preliminaries - Structure spaces.

1.1. Let $X$ be a topological space and let $C(X, F)$ be the set of all continuous functions on $X$ with values in an archimedean ordered field $F$. It is well-known that $F$ is (canonically isomorphic to) a subfield of $R$. We assume that $F \cong R$. The set $C(X, F)$ has the natural structure of a lattice ordered ring under pointwise lattice and algebraic operations. In this paper we shall be concerned with some subrings of $C(X, F)$, namely: The subring $C^*(X, F)$ of all bounded continuous functions on $X$ with values in $F$; the subring $C^{**}(X, F)=\{f \in C(X, F) : \text{cl} \text{supp}[X] \text{ is compact}\}$, and the subring $C^0(X, F)$ of all functions in $C(X, F)$ such that $f[X]$ is finite. For simplicity we write $C$, $C^*$, $C^{**}$ and $C^0$ when no specification of the space $X$ is required.

If $V$ is clopen (open and closed) in $X$ we write $\chi_V$ for the characteristic function of $V$. Clearly $\chi_V \in C^0$ and is an idempotent. Also if $e \in C(X, F)$ is an idempotent the $V = e[1]$ is clopen in $X$ and $e = \chi_V$. A subset $E \subseteq C(X, F)$ of idempotents such that $\sum_{e \in E} e = 1$ is called a partition of unity into idempotents. Throughout this paper the term partition unity will mean such a set.

1.2. Lemma. For each prime ideal $P$ of $C^0$, $C^0/P = F$. Hence every prime ideal of $C^0$ is maximal.

Proof. Let $f \in C^0(X, F)$, then $f[X] = \{q_1, \ldots, q_n\}$ ($q_i \in F$, $q_i \neq q_j$ unless $i = j$). Let $V_i = f^{-1}[q_i]$, $e_i = \chi_{V_i}$. Then $\{e_1, \ldots, e_n\}$ is a finite partition of unity into idempotents and $f = \sum_{i=1}^{n} q_i e_i$. For exactly one $k$ ($1 \leq k \leq n$), we have $e_k \notin P$. Hence, $P(f) = P(e_k q_k) = q_k P(e_k) = q_k$, since $P(e_k) = 1$, being a non-zero idempotent of the integral domain $C^0/P$.

If $P$ is a prime ideal of one of the rings $C$, $C^*$, then proofs similar to those in [GJ], chapters 2, 5 and 14 show that $P$ is absolutely convex, that the residue class ring is totally ordered under the quotient ordering and that the prime ideals containing $P$ form a chain.

1.3. Denote by $\mathfrak{F}$, $\mathfrak{M}$, $\mathfrak{F}^*$, $\mathfrak{M}^*$, $\mathfrak{F}^0 = \mathfrak{M}^0$, the prime and maximal ideal spaces of the rings $C$, $C^*$, $C^0$ respectively. Proofs similar to those in [DMO] show that $\mathfrak{M}$, $\mathfrak{M}^*$, $\mathfrak{M}^0$ are compact Hausdorff spaces.
and that the mapping $\lambda : \mathfrak{M} \rightarrow \mathfrak{M}^*$ which sends every maximal ideal $M$ of $C$ into the unique maximal ideal of $C^*$ containing $M \cap C^*$ is a homeomorphism. Furthermore, a maximal ideal $M^*$ of $C^*$ is of the form $M \cap C^*$ (where $M$ is a maximal ideal of $C$) if and only if $M^*$ does not contain a unit of $C$. (Again see [DMO]).

**Lemma (a).** Let $M_1$, $M_2$ be distinct maximal ideals of $C$, then there exists an idempotent $e \in C$ such that $e \in M_1 \setminus M_2$.

**Proof.** Choose $f \in M_1$ such that $1 - f \in M_2$. Let $\alpha \in R \setminus F$, $0 \leq \alpha \leq 1$ and let $V = f^{-1}[(\alpha, \infty)]$, $e = \chi_V$. Then $e$ is an idempotent of $C$ and it is easily shown that $e$ is a multiple of $f$ in $C$, $1 - e$ is a multiple of $1 - f$ in $C$.

**Lemma (b).** The map $\lambda_0 : \mathfrak{M} \rightarrow \mathfrak{M}^0$ given by $\lambda_0 M = M \cap C^0$ is a homeomorphism.

**Proof.** Clearly $\lambda_0$ is continuous and by lemma 1.3 (a), $\lambda_0$ is one-to-one. Let $M^0 \in \mathfrak{M}^0$; it is easily seen that $M^0$ generates a proper ideal $I$ in $C$. If $M$ is any maximal ideal of $C$ containing $I$, $\lambda_0 M = M \cap C \supset M^0$. Hence, $\lambda_0 M = M^0$. Since $\mathfrak{M}$ and $\mathfrak{M}^0$ are compact Hausdorff spaces, $\lambda_0$ is a homeomorphism.

**Remark.** In a similar way it can be shown that $\mathfrak{M}^*$ and $\mathfrak{M}^0$ are homeomorphic.

1.4. **Lemma.** The space $\mathfrak{M}$ has a base of clopen sets.

**Proof.** A base for the closed sets of $\mathfrak{M}$ is given by the family of sets $V(f) = \{M \in \mathfrak{M} : f \in M\}$. If $e$ is an idempotent of $C$, then $V(e)$ is clopen since $\mathfrak{M} \setminus V(e) = V(1 - e)$. By lemma 1.3 (a), the sets of the form $V(e)$ (where $e$ is an idempotent of $C$) separate the points of $\mathfrak{M}$. The result follows from the compactness of $\mathfrak{M}$.

For any $p \in X$, put $M_p = \{f \in C(X, F) : f(p) = 0\}$. Clearly $M_p$ is a maximal ideal of $C(X, F)$ and $M^*_p = M_p \cap C^*(X, F)$, $M^0_p = M_p \cap C^0(X, F)$ are maximal ideals of $C^*$ and $C^0$ respectively. These ideals are called fixed maximal ideals.

**Theorem.** The map $\theta : X \rightarrow \mathfrak{M}$ which sends every point $p \in X$ into the fixed maximal ideal $M_p$ is continuous and maps clopen subsets
of $X$ into clopen subsets of $\emptyset[X]$. The mapping $\theta' : C(\emptyset[X], F) \to C(X, F)$ given by $\theta'(g) = g \circ \theta$ where $g \in C(\emptyset[X], F)$ is an isomorphism of $C(\emptyset[X], F)$ onto $C(X, F)$. Furthermore, $\emptyset[X]$ is dense in $\mathfrak{M}$, $\theta$ is one-to-one if and only if $C(X, F)$ separates the points of $X$ and is a homeomorphism onto $\emptyset[X]$ if and only if $X$ is a $T_\sigma$-space with a base of clopen sets.

**Proof.** All of these statements are either obvious or are already known in slightly different contexts (e.g. [P] or [GJ]).

1.5. By the preceding theorem, if $X$ is a $T_\sigma$-space with a base of clopen sets, the subspaces of $\mathfrak{M}$, $\mathfrak{M}^*$ and $\mathfrak{M}^0$ of fixed maximal ideals are homeomorphic to $X$. We shall identify $X$ with these subspaces; hence $X$ is dense in $\mathfrak{M}$, $\mathfrak{M}^*$ and $\mathfrak{M}^0$ and the mappings $\lambda : \mathfrak{M} \to \mathfrak{M}^*$ $\lambda_0 : \mathfrak{M} \to \mathfrak{M}^0$ and $\lambda_0^* : \mathfrak{M}^* \to \mathfrak{M}^0$ already defined, are homeomorphisms which preserve $X$.

**Lemma.** Let $X$ be a $T_\sigma$-space with a base of clopen sets and let $\tau : X \to Y$ be a continuous map on $X$ into a compact totally disconnected space $Y$. Then there is a continuous mapping $\bar{\tau} : \mathfrak{M}^0 \to Y$ such that $\bar{\tau} | X = \tau$.

**Proof.** Let $\varphi : C^0(Y, F) \to C^0(X, F)$ be defined by $\varphi(g) = g \circ \tau$ (where $g \in C^0(Y, F)$). Then $\varphi$ is a ring homomorphism which induces a continuous map $\psi$ on the prime ideal space of $C^0(X, F)$ into the prime ideal space of $C^0(Y, F)$ ($\psi\!(P) = \varphi[P]$). Hence, $\psi$ is a map on $\mathfrak{M}^0(= = \mathfrak{P})$ into $\mathfrak{M}^0(Y)$ which can be identified with $Y$ since $Y$ is compact and totally disconnected. It is clear that $\psi$ is the desired extension of $\tau$.

If $X$ is a $T_\sigma$-space with a base of clopen sets, $\mathfrak{M}$ is a totally disconnected compactification of $X$. The preceding lemma shows that $\mathfrak{M}$ is the largest totally disconnected compactification of $X$. It is easily seen that $\mathfrak{M}$ coincides with the space $\delta X$ of [P] (theorem 1.5.2), hence, $\mathfrak{M}$ is homeomorphic to the maximal ideal space of the Boolean algebra $\mathfrak{B}(X)$ of all clopen subsets of $X$ ([P], theorem 1.6.1). We shall hereafter write $\delta X$ in place of $\mathfrak{M}$.

1.6. **Theorem.** Let $X$ be a $T_\sigma$-space with a base of clopen sets. Then $C(\delta X, F) = C^{**}(X, F)$. Hence, $\delta X$ is homeomorphic to $\mathfrak{M}^{**}$ (under a homeomorphism which preserves $X$).
PROOF. Map $C(\delta X, F)$ into $C(X, F)$ via the restriction $f \mapsto f|X$, where $f \in C(\delta X, F)$. It is easily seen that this homomorphism is one-to-one and by lemma 1.5, its range is all of $C^{**}(X, F)$. (If $g \in C^{**}(X, F)$, then $cl_{tg}[X]$ is a compact totally disconnected space).

REMARK. It is not difficult to show that if $X$ is a $T_0$-space with a base of clopen sets, then $\delta X$ is the smallest compactification of $X$ in which $X$ is $C^{**}$-embedded.

1.7. We now establish the relationship between $\beta X$ and $\delta X$. First we need a lemma.

LEMMA. Let $X$ be a topological space. A subset $Z \subset X$ is of the form $Z(f)$ for some $f \in C(X, F)$ if and only if $Z$ is a countable intersection of clopen sets.

PROOF. If $Z = Z(f)$ for some $f \in C(X, F)$, then $Z = \bigcap_{n \in \mathbb{N}} \{ x \in X : |f(x)| < \alpha/n \}$, where $\alpha$ is some positive element of $R \setminus F$.

Conversely, if $Z = \bigcap_{n \in \mathbb{N}} V_n$, where $V_n$ is clopen for each $n \in \mathbb{N}$, then assuming that the $V_n$ are nested and putting $W_n = V_n \setminus V_{n+1}$, we can define $u$ to be $1/n$ on $W_n$, $1$ on $X \setminus V_1$ and $0$ on $Z$. Clearly $u \in C(X, F)$.

THEOREM. Let $X$ be a $T_0$-space with a base of clopen sets. The following are equivalent:

1) $\beta X = \delta X$.

2) $\beta X$ is totally disconnected.

3) Any two disjoint zero sets in $X$ are contained in disjoint clopen sets.

4) Any zero set in $X$ is a countable intersection of clopen sets.

5) The mapping $M \mapsto M \cap C(X, F)$ maps $\mathfrak{N}_R$ homeomorphically onto $\mathfrak{N}$ (where $\mathfrak{N}_R$ is the maximal ideal space of $C(X)$).

PROOF. 1) implies 2). Obvious.

2) implies 3). See [GJ], theorem 16.17.

3) implies 4). Let $Z$ be a zero set of $X$, $Z = Z(f)$ say. Define
Each $Z_n$ is a zero set disjoint from $Z$ and hence there exists a clopen set $V_n$ such that $Z \subset V_n$ and $Z_n \cap V_n = \emptyset$. Then $Z \subset \bigcap_{n} V_n \subset \bigcup_{n} (X \setminus Z_n) = Z$.

4) implies 5). This is clear since from lemma 1.7, every zero set is an an $F$-zero set. (That is to say, a zero set of a function in $C(X, F)$).

5) implies 1). Obvious.

2. Residue class fields.

2.1. In this paragraph we investigate some properties of the residue class fields of the rings $C(X, F)$ and $C^*(X, F)$.

Firstly, observe that if $M^*$ is a maximal ideal of $C^*(=C^*(X, F))$ then $C^*/M^*$ is an archimedean ordered field hence canonically embeddable in $R$.

**Lemma.** Let $M^*$ be a maximal ideal of $C^*$, and suppose that $M^*(f) = \alpha \in R$. Then $\alpha \in cl_{R}[X]$, and if $\alpha \notin f[X]$, then $M^*$ contains a countable partition of unity.

**Proof.** Suppose that $\alpha \notin f[X]$. If $q, s \in F$, $q < \alpha < s$, then $f - (f \vee q) \land s \in M^*$, since $M^*(f \vee q) \land s = (M^*(f) \vee q) \land s = (\alpha \vee q) \land s = \alpha$. Consider two sequences $(\alpha_i), (\beta_i)$, $\alpha_i, \beta_i \in R \setminus F$, for all $i \in N$, the first strictly increasing the second strictly decreasing, both converging to $\alpha$, and such that $\alpha_i < f(x) < \beta_i$ for all $x \in X$. Put $V_i = f^*[\langle (\alpha_i, \alpha_{i+1}) \cup (\beta_{i+1}, \beta_i)\rangle]$, and $e_i = \chi_{V_i}$. Thus $\{e_i : i \in N\}$ is a countable partition of unity.

Let $q_i, s_i \in F$ be such that $\alpha_{i+1} < q_i < \alpha < s_i < \beta_{i+1}$, and put $g_i = f - (f \vee q_i) \land s_i$. Then $g_i \in M^*$ and if $x \in V_i$, then $|g_i(x)| \geq \max \{q_i - \alpha_{i+1}, \beta_{i+1} - s_i\}$. Hence if we define $h_i(x) = 1/g_i(x)$ for each $x \in V_i$, $h_i(x) = 0$ for $x \notin V_i$, $h_i \in C^*$ and $e_i = h_i g_i \in M^*$. It is clear that $\alpha \in cl_{R}[X]$.

**Theorem.** Let $M^*$ be a maximal ideal of $C^*$. The following are equivalent:

1) $C^*/M^*$ is a proper extension of $F$.

2) $M^*$ contains a countable partition of unity.

3) $M^*$ contains a unit of $C$.

4) $C^*/M^* = R$. 
PROOF. 1) implies 2). Lemma 2.1.

2) implies 3). Let \( \{e_i : i \in \mathbb{N}\} \) be a partition contained in \( M^* \). Define \( u = \sum_{i=1}^{n} e_i \). Then \( u \) is a unit of \( C \) and \( 0 \leq M^*(u) = M^*(\sum_{i \geq 1/n} e_i) \leq 1/n \). Hence \( M^*(u) = 0 \), that is to say \( u \in M^* \).

3) implies 2). If \( u \in M^* \) and \( u \) is a unit of \( C \), \( M^*(u) = 0 \) but \( 0 \notin \mathbb{U}[X] \). The result follows from lemma 2.1.

2) implies 4). Let \( \alpha \) be any real number, \( (q_i) \) a sequence of elements of \( \mathbb{F} \) converging to \( \alpha \), \( \{e_i : i \in \mathbb{N}\} \) a countable partition of unity contained in \( M^* \). Put \( f = \sum q_i e_i \). Then \( f \in C^* \) and for each \( n \in \mathbb{N} \), \( M^*(f) = M^*(\sum_{i \geq N} q_i e_i) \); hence \( M^*(f) = \alpha \), since \( \{\alpha\} = \bigcap_{n} \text{cl}(q_i : i \geq n) \).

4) implies 1). Obvious.

2.2 LEMMA. Let \( P \) (\( P^* \)) be a prime, non-maximal, ideal of \( C \) (\( C^* \)). Then \( C/P(C^*/P^*) \) contains infinitely small elements.

PROOF. Let \( M \) be the maximal ideal of \( C \) containing \( P \), and let \( u \in M \setminus P, \ u \geq 0 \). For each \( n \in \mathbb{N} \), \( (u - u \wedge 1/n)(u \vee 1/n - u) = 0 \), and \( u \vee 1/n - u \notin M \), since \( u \vee 1/n \) is a unit, being bounded away from 0. Hence \( u - u \wedge 1/n \notin P \), so that \( 0 < P(u) = P(u \wedge 1/n) \leq 1/n \), for each \( n \in \mathbb{N} \).

THEOREM. Let \( M \) be a maximal ideal of \( C \). The following are equivalent:

1) \( C/M = \mathbb{F} \).

2) \( P^* = M \cap C^* \) is maximal in \( C^* \).

3) \( M \) contains no countable partition of unity.

4) \( M \) contains no partition of unity of non-measurable cardinal.

PROOF. 1) is equivalent to 2). \( C/M \) contains a canonical copy of the ordered ring \( C^*/P^* \) and is the field of fractions of this copy. It follows from lemma 2.2 that if \( P^* \) is not maximal, then \( C^*/P^* \) contains infinitely small elements.

2) implies 3). \( P^* \) contains no countable partition of unity since it contains no unit of \( C \).
3) implies 2). If $P^*$ is not maximal, the maximal ideal $M^*$ of $C^*$ containing $P^*$ contains a unit of $C$ (see 1.3), hence it also contains a countable partition of unity $E$. It is clear that $E \subset P^*$, hence $E \subset M$.

3) implies 4). Suppose that $E$ is a partition of unity of non-measurable cardinal, contained in $M$. Consider $E$ as a topological space with the discrete topology. Define $\mathcal{F} = \{ E \mid E \subset E, \sum_{e \in E} e \in M \}$. It is easy to show that $\mathcal{F}$ is a free ultrafilter on $E$, and since $E$ is realcompact, $\mathcal{F}$ is hyper-real. Choose a countable subfamily of $\mathcal{F}$ with empty intersection, $\{ Z_i : i \in \mathbb{N} \}$. Then $E \setminus Z_i \notin \mathcal{F}$ and $\bigcup_i (E \setminus Z_i) = E$. Thus $V_i = (E \setminus Z_i) \setminus \bigcup_{i < i} (E \setminus Z_i)$ is a countable partition of $E$ and $V_i \notin \mathcal{F}$ for each $i \in \mathbb{N}$. Clearly $\{ \sum_{e \in V_i} e : i \in \mathbb{N} \}$ is a countable partition of unity contained in $M$.

4) implies 3). Obvious.

2.3. Theorem. Let $M$ be a maximal ideal of $C$ such that $C/M = K(\neq F)$. Then $K$ is an $\eta_1$-field in which $F$ is algebraically closed. (We identify $F$ with the image of the constant functions).

Proof. Suppose that $u \in K$ is algebraic over $F$, and let $p(t)$ be its minimum polynomial over $F$. If $f \in C$ is such that $u = M(f)$ then $0 = p(u) = M(p(f))$, that is to say, $p(f) \in M$. But this implies that $Z(p(f)) \neq \emptyset$, i.e. $p(t)$ must have a root in $F$. Since $p(t)$ is irreducible, it follows that $p(t) = t - q$ (for some $q \in F$). Hence $u = M(f) = q \in F$.

The fact that $K$ is an $\eta_1$-field can be shown by using the same argument as in [GJ], 13.7. and 13.8. However, the presence of partitions of unity allows a considerably simpler argument. The theorem will be a consequence of [GJ], 13.8 and the following lemma.

Lemma. Let $P$ a prime ideal contained in a maximal ideal $M$ such that $C/M \neq F$. Then if $A$, $B$ are countable subsets of $C/P$, with $A \subset B$, there exists $u \in C/P$ such that $A \leq u \leq B$.

Proof. Suppose that $A$ and $B$ are non-empty. By [GJ], 13.5, we can find an increasing sequence $f_1 \leq f_2 \leq \ldots$ and a decreasing sequence $g_1 \geq g_2 \geq \ldots$ of elements of $C$ such that $f_n \leq g_n$ for each $n \in \mathbb{N}$, and $\{P(f_j) : i \in \mathbb{N} \}$ is a cofinal subset of $A$, and $\{P(g_i) : i \in \mathbb{N} \}$ is a cofinal subset of $B$. Let $\{ e_i : i \in \mathbb{N} \}$ be a countable partition of unity contained in $P$ and let $f = \sum_{i}^{\mathcal{Z}} (f_i + g_i) e_i$. It is easy to show that $u = P(f)$ satisfies
the required condition. If either $A$ or $B$ is empty, a simple modification of the preceding argument shows that either $B$ is not cointial or $A$ is not cofinal.

**Remark.** If $M$ is a maximal ideal of $C$ such that $C/M \neq F$, then $|C/M| \geq c$. Furthermore in the special case $F = Q$, $C/M$ contains no copy of $R$.

3. *F-realcompactness.*

3.1. Let $X$ be a $T_{0}$-space with a base of clopen sets. Let $\mathfrak{M} (= \delta X)$ be the maximal ideal space of $C (= C(X, F))$. Denote by $\nu X$ the subspace of $\mathfrak{M}$ consisting of all those ideals $M$ for which $C/M = F$; $\nu X$ is obviously a $T_{0}$-space with a base of clopen sets in which $X$ is dense and $C(X, F)$-embedded. Hence $(C(X, F) = C(\nu X, F)$ and every ideal $M$ of $C(\nu X, F)$ for which $C/M = F$ is fixed. We call a space $F$-realcompact if every ideal $M$ of $C(X, F)$ for which $C/M = F$ is fixed.

**Theorem.** Let $X$ be a $T_{0}$-space with a base of clopen sets. If $X$ is $F$-realcompact, then it is realcompact.

**Proof.** Suppose that $M'$ is a real maximal ideal of $C(X)$; $M' \cap C(X, F)$ is a maximal ideal of $C(X, F)$ with the property that $C(X, F)/(M' \cap C(X, F)) = F$. (Observe that this field is embedded in $C(X)/M'$ as an ordered subring). Hence $M' \cap C(X, F) = M_p \cap C(X, F)$. It follows that $M' = M_p$.

**Remark.** Dr. Peter Nyikos has communicated to one of the authors that the space considered in $[R]$ is not $F$-realcompact. Hence a space can be realcompact and have a base of clopen sets without being $F$-realcompact. However, if $X$ is zero-dimensional and realcompact, then it is $F$-realcompact (see 1.7).

3.2. **Example.** The space $\Delta_1$ of $[GJ]$, $16M$ is a $T_{0}$-space with a base of clopen sets whose dimension is 1. Hence $\beta \Delta_1$ is not totally disconnected. That is to say, $\beta \Delta_1 \neq \delta \Delta_1$. It is known from $[D]$ and $[GJ]$ that $\Delta_1$ is dense and $C$-embedded in a space $\Delta$ such that $\Delta \setminus \Delta_1$ is a copy of $[0, 1]$. Also, the quotient space $\Delta_0$ of $\Delta$ obtained by identifying the points of $\Delta \setminus \Delta_1$ is zero dimensional. It is easy to see that
and In fact $A_0$ is the space obtained from $A$ by the method of theorem 1.4. ($A_0 = \emptyset[A]$). We show that $A$ is realcompact.

Let $\pi$ be the restriction to $A$ of the canonical projection of $W^*[0,1]$ onto $W^*$; for each $\tau < \omega_1$ let $A_\tau = \pi^{-1}[W(\tau+1)]$; $A_\tau$ is a clopen subspace of $A$ (hence $C$-embedded in $A$) homeomorphic to a subspace of $R^2$. Hence $A_\tau$ is realcompact. Consider a real $\omega$-ultrafilter $\mathcal{F}$ on $A$; if for each $Z \in \mathcal{F}$, $\pi[Z]$ is cofinal in $W^*$, then (from [GJ], 16M.4), $\omega_1 \in \pi[Z]$, for each $Z \in \mathcal{F}$. Hence $\{Z : \pi[Z] \text{ is cofinal in } W^* \}$ is a filter on $\{\omega_1\} \times [0,1]$. Hence $\mathcal{F}$ is fixed. If for some $Z \in \mathcal{F}$, $\pi[Z]$ has an upper bound $\tau$ in $W$, then $\{Z : \pi[Z] \text{ is fixed} \}$ is a real $\omega$-ultrafilter on $A_\tau$ and so is fixed.

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