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RINGS OF CONTINUOUS FUNCTIONS WITH VALUES IN AN ARCHIMEDEAN ORDERED FIELD

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Introduction.

The purpose of this paper is to study the ring $C(X, F)$ of all continuous functions on a topological space X with values in a proper subfield F of the real number field R. The ring $C(X)[=C(X, R)]$ of all real-valued continuous functions on X has been extensively studied; a standard reference to this work is the book [GJ]. Furthermore, Pierce, in his paper [P], laid the foundations of a theory of rings of integer-valued functions $C(X, Z)$.

It seems natural to study the rings intermediate to $C(X)$ and $C(X, Z)$; if the study of the ring $C(X, Z)$ gives some information about the special properties of C(X) which depend on R being a field, a study of the rings $C(X, F)$ should bring to light the properties of $C(X)$ which depend on R being an order-complete field.

Our results can be summarized as follows: With regard to the relationship between $C(X, F)$ and the underlying topological space X, $C(X, F)$ behaves much like $C(X, Z)$. On the other hand, the residue class fields of $C(X, F)$ are similar to those of $C(X)$.

Many of the results in this paper are not surprising, however, answering as they do a natural question, we think that they are worthy of some study.

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1. Preliminaries - Structure spaces.

1.1. Let X be a topological space and let $C(X, F)$ be the set of all continuous functions on X with values in an archimedean ordered field F. It is well-known that F is (canonically isomorphic to) a subfield of R. We assume that $F \neq R$. The set $C(X, F)$ has the natural structure of a lattice ordered ring under pointwise lattice and algebraic operations. In this paper we shall be concerned with some subrings of $C(X, F)$, namely: The subring $C^*(X, F)$ of all bounded continuous functions on X with values in F; the subring $C^{**}(X, F) = \{f \in C(X, F) : cl_F f[X]$ is compact$\}$, and the subring $C^0(X, F)$ of all functions in $C(X, F)$ such that $f[X]$ is finite. For simplicity we write C, C^*, C^{**} and C^0 when no specification of the space X is required.

If V is clopen (open and closed) in X we write χ_V for the characteristic function of V. Clearly $\chi_V \in C^0$ and is an idempotent. Also if $e \in C(X, F)$ is an idempotent the $V = e^{\leftarrow}[1]$ is clopen in X and $e = \chi_V$. A subset $E \subset C(X, F)$ of idempotents such that $\sum_{e \in E} e = 1$ is called a partition of unity into idempotents. Throughout this paper the term partition unity will mean such a set.

1.2. Lemma. *For each prime ideal P of C^0, $C^0/P = F$. Hence every prime ideal of C^0 is maximal.*

Proof. Let $f \in C^0(X, F)$, then $f[X] = \{q_1, \ldots q_n\}$ ($q_i \in F$, $q_i \neq q_j$ unless $i = j$). Let $V_i = f^{\leftarrow}[q_i]$, $e_i = \chi_{V_i}$. Then $\{e_1, \ldots e_n\}$ is a finite partition of unity into idempotents and $f = \sum_{i=1}^{n} q_i e_i$. For exactly one k $(1 \leqslant k \leqslant n)$, we have $e_k \notin P$. Hence, $P(f) = P(e_k q_k) = q_k P(e_k) = q_k$, since $P(e_k) = 1$, being a non-zero idempotent of the integral domain C^0/P.

If P is a prime ideal of one of the rings C, C^*, then proofs similar to those in [GJ], chapters 2, 5 and 14 show that P is absolutely convex, that the residue class ring is totally ordered under the quotient ordering and that the prime ideals containing P form a chain.

1.3. Denote by $\mathfrak{S}$, $\mathfrak{M}$, $\mathfrak{S}^*$, $\mathfrak{M}^*$, $\mathfrak{S}^0 = \mathfrak{M}^0$, the prime and maximal ideal spaces of the rings C, C^*, C^0 respectively. Proofs similar to those in [DMO] show that $\mathfrak{M}$, $\mathfrak{M}^*$, $\mathfrak{M}^0$ are compact Hausdorff spaces and that the mapping $\lambda : \mathfrak{M} \to \mathfrak{M}^*$ which sends every maximal ideal M of C into the unique maximal ideal of C^* containing $M \cap C^*$ is a homeomorphism. Furthermore, a maximal ideal M^* of C^* is of the form $M \cap C^*$ (where M is a maximal ideal of C) if and only if M^* does not contain a unit of C. (Again see [*DMO*]).

LEMMA (a). *Let M_1, M_2 be distinct maximal ideals of C, then there exists an idempotent $e \in C$ such that $e \in M_1 \backslash M_2$.*

PROOF. Choose $f \in M_1$ such that $1-f \in M_2$. Let $\alpha \in R \backslash F$, $0 \leqslant \alpha \leqslant 1$ and let $V = f^{\leftarrow}[(\alpha, \infty)]$, $e = \chi_V$. Then e is an idempotent of C and it is easily shown that e is a multiple of f in C, $1-e$ is a multiple of $1-f$ in C.

LEMMA (b). *The map $\lambda_0 : \mathfrak{M} \to \mathfrak{M}^0$ given by $\lambda_0 M = M \cap C^0$ is a homeomorphism.*

PROOF. Clearly λ_0 is continuous and by lemma 1.3 (a), λ_0 is one-to-one. Let $M^0 \in \mathfrak{M}^0$; it is easily seen that M^0 generates a proper ideal I in C. If M is any maximal ideal of C containing I, $\lambda_0 M = M \cap C^0 \supset M^0$. Hence, $\lambda_0 M = M^0$. Since $\mathfrak{M}$ and $\mathfrak{M}^0$ are compact Hausdorff spaces, λ_0 is a homeomorphism.

REMARK. In a similar way it can be shown that $\mathfrak{M}^*$ and $\mathfrak{M}^0$ are homeomorphic.

1.4. LEMMA. *The space $\mathfrak{M}$ has a base of clopen sets.*

PROOF. A base for the closed sets of $\mathfrak{M}$ is given by the family of sets $V(f) = \{M \in \mathfrak{M} : f \in M\}$. If e is an idempotent of C, then $V(e)$ is clopen since $\mathfrak{M} \backslash V(e) = V(1-e)$. By lemma 1.3 (a), the sets of the form $V(e)$ (where e is an idempotent of C) separate the points of $\mathfrak{M}$. The result follows from the compactness of $\mathfrak{M}$.

For any $p \in X$, put $M_p = \{f \in C(X, F) : f(p) = 0\}$. Clearly M_p is a maximal ideal of $C(X, F)$ and $M^*_p = M_p \cap C^*(X, F)$, $M_p^0 = M_p \cap C^0(X, F)$ are maximal ideals of C^* and C^0 respectively. These ideals are called fixed maximal ideals.

THEOREM. *The map $\theta : X \to \mathfrak{M}$ which sends every point $p \in X$ into the fixed maximal ideal M_p is continuous and maps clopen subsets* *of X into clopen subsets of $\theta[X]$. The mapping $\theta' : C(\theta[X], F) \to C(X, F)$ given by $\theta'(g) = g \circ \theta$ where g $C(\theta[X], F))$ is an isomorphism of $C(\theta[X], F)$ onto $C(X, F)$. Furthermore, $\theta[X]$ is dense in $\mathfrak{M}$, θ is one-to-one if and only if $C(X, F)$ separates the points of X and is a homeomorphism onto $\theta[X]$ if and only if X is a T_0-space with a base of clopen sets.*

PROOF. All of these statements are either obvious or are already known in slightly different contexts (e.g. [P] or [GJ]).

1.5. By the preceding theorem, if X is a T_0-space with a base of clopen sets, the subspaces of $\mathfrak{M}$, $\mathfrak{M}^*$ and $\mathfrak{M}^0$ of fixed maximal ideals are homeomorphic to X. We shall identify X with these subspaces; hence X is dense in $\mathfrak{M}$, $\mathfrak{M}^*$ and $\mathfrak{M}^0$ and the mappings $\lambda : \mathfrak{M} \to \mathfrak{M}^*$ $\lambda_0 : \mathfrak{M} \to \mathfrak{M}^0$ and $\lambda_0^* : \mathfrak{M}^* \to \mathfrak{M}^0$ already defined, are homeomorphisms which preserve X.

LEMMA. *Let X be a T_0-space with a base of clopen sets and let $\tau : X \to Y$ be a continuous map on X into a compact totally disconnected space Y. Then there is a continuous mapping $\bar{\tau} : \mathfrak{M}^0 \to Y$ such that $\bar{\tau} \mid X = \bar{\tau}$.*

PROOF. Let $\varphi : C^0(Y, F) \to C^0(X, F)$ be defined by $\varphi(g) = g \circ \tau$ (where $g \in C^0(Y, F)$). Then φ is a ring homomorphism which induces a continuous map ψ on the prime ideal space of $C^0(X, F)$ into the prime ideal space of $C^0(Y, F)$ $(\psi(P) = \varphi^{\leftarrow}[P])$. Hence, ψ is a map on $\mathfrak{M}^0(= \mathfrak{F}^0)$ into $\mathfrak{M}^0(Y)$ which can be identified with Y since Y is compact and totally disconnected. It is clear that ψ is the desired extension of τ.

If X is a T_0-space with a base of clopen sets, $\mathfrak{M}$ is a totally disconnected compactification of X. The preceding lemma shows that $\mathfrak{M}$ is the largest totally disconnected compactification of X. It is easily seen that $\mathfrak{M}$ coincides with the space δX of [P] (theorem 1.5.2), hence, $\mathfrak{M}$ is homeomorphic to the maximal ideal space of the Boolean algebra $\mathfrak{B}(X)$ of all clopen subsets of X ([P], theorem 1.6.1). We shall hereafter write δX in place of $\mathfrak{M}$.

1.6. THEOREM. *Let X be a T_0-space with a base of clopen sets. Then $C(\delta X, F) = C^{**}(X, F)$. Hence, δX is homeomorphic to $\mathfrak{M}^{**}$ (under a homeomorphism which preserves X).*

PROOF. Map $C(\delta X, F)$ into $C(X, F)$ via the restriction $f \rightarrow f \mid X$, where $f \in C(\delta X, F)$. It is easily seen that this homomorphism is one-to-one and by lemma 1.5, its range is all of $C^{**}(X, F)$. (If $g \in C^{**}(X, F)$, then $cl_F g[X]$ is a compact totally disconnected space).

REMARK. It is not difficult to show that if X is a T_0-space with a base of clopen sets, then δX is the smallest compactification of X in which X is C^{**}-embedded.

1.7. We now establish the relationship between βX and δX. First we need a lemma.

LEMMA. *Let X be a topological space. A subset $Z \subset X$ is of the form $Z(f)$ for some $f \in C(X, F)$ if and only if Z is a countable intersection of clopen sets.*

PROOF. If $Z = Z(f)$ for some $f \in C(X, F)$, then $Z = \bigcap_{n \in N} \{x \in X : |f(x)| < \alpha/n\}$, where α is some positive element of $R \backslash F$.

Conversely, if $Z = \bigcap_{n \in N} V_n$, where V_n is clopen for each $n \in N$, then assuming that the V_n are nested and putting $W_n = V_n \backslash V_{n+1}$, we can define u to be $1/n$ on W_n, 1 on $X \backslash V_1$ and 0 on Z. Clearly $u \in C(X, F)$.

THEOREM. *Let X be a T_0-space with a base of clopen sets. The following are equivalent:*

1) $\beta X = \delta X$.

2) *βX is totally disconnected.*

3) *Any two disjoint zero sets in X are contained in disjoint clopen sets.*

4) *Any zero set in X is a countable intersection of clopen sets.*

5) *The mapping $M \rightarrow M \cap C(X, F)$ maps $\mathfrak{M}_R$ homeomorphically onto $\mathfrak{M}$ (where $\mathfrak{M}_R$ is the maximal ideal space of $C(X)$).*

PROOF. 1) implies 2). Obvious.

2) implies 3). See [GJ], theorem 16.17.

3) implies 4). Let Z be a zero set of X, $Z = Z(f)$ say. Define

$Z_n=\{x\in X : |f(x)|\geq 1/n\}$. Each Z_n is a zero set disjoint from Z and hence there exists a clopen set V_n such that $Z\subset V_n$ and $Z_n\cap V_n=\emptyset$. Then $Z\subset \bigcap_n V_n\subset \bigcup_n (X\backslash Z_n)=Z$.

4) implies 5). This is clear since from lemma 1.7, every zero set is an an F-zero set. (That is to say, a zero set of a function in $C(X, F)$).

5) implies 1). Obvious.

2. Residue class fields.

2.1. In this paragraph we investigate some properties of the residue class fields of the rings $C(X, F)$ and $C^*(X, F)$.

Firstly, observe that if M^* is a maximal ideal of $C^*(=C^*(X, F))$ then C^*/M^* is an archimedean ordered field hence canonically embeddable in R.

LEMMA. *Let M^* be a maximal ideal of C^*, and suppose that $M^*(f)=\alpha(\in R)$. Then $\alpha\in cl_R f[X]$, and if $\alpha\notin f[X]$, then M^* contains a countable partition of unity.*

PROOF. Suppose that $\alpha\notin f[X]$. If q, $s\in F$, $q<\alpha<s$, then $f-(f\vee q)\wedge s\in M^*$, since $M^*((f\vee q)\wedge s)=(M^*(f)\vee q)\wedge s=(\alpha\vee q)\wedge s=\alpha$. Consider two sequences (α_i), (β_i) (α_i, $\beta_i\in R\backslash$F, for all $i\in N$), the first strictly increasing the second strictly decreasing, both converging to α, and such that $\alpha_1<f(x)<\beta_1$ for all $x\in X$. Put $V_i=f^{\leftarrow}[(\alpha_i, \alpha_{i+1})\cup(\beta_{i+1}, \beta_i)]$, and $e_i=\chi_V$. Thus $\{e_i : i\in N\}$ is a countable partition of unity.

Let q_i, $s_i\in F$ be such that $\alpha_{i+1}<q_i<\alpha<s_i<\beta_{i+1}$, and put $g_i=f-(f\vee q_i)\wedge s_i$. Then $g_i\in M^*$ and if $x\in V_i$, then $|g_i(x)|\geq$ max $\{q_i-\alpha_{i+1}, \beta_{i+1}-s_i\}$. Hence if we define $h_i(X)=1/g_i(x)$ for each $x\in V_i$, $h_i(x)=0$ for $x\notin V_i$, $h_i\in C^*$ and $e_i=h_i g_i\in M^*$. It is clear that $\alpha\in cl_R f[X]$.

THEOREM. *Let M^* be a maximal ideal of C^*. The following are equivalent:*

1) *C^*/M^* is a proper extension of F.*

2) *M^* contains a countable partition of unity.*

3) *M^* contains a unit of C.*

4) *$C^*/M^*=R$.*

PROOF. 1) implies 2). Lemma 2.1.

2) implies 3). Let $\{e_i : i \in N\}$ be a partition contained in M^*. Define $u = \sum_i i^{-1} e_i$. Then u is a unit of C and $0 \leq M^*(u) = M^*(\sum_{i \geq 1/n} i^{-1} e_i) \leq \leq 1/n$. Hence $M^*(u) = 0$, that is to say $u \in M^*$.

3) implies 2). If $u \in M^*$ and u is a unit of C, $M^*(u) = 0$ but $0 \notin u[X]$. The result follows from lemma 2.1.

2) implies 4). Let α be any real number, (q_i) a sequence of elements of F converging to α, $\{e_i : i \in N\}$ a countable partition of unity contained in M^*. Put $f = \sum_i q_i e_i$. Then $f \in C^*$ and for each $n \in N$, $M^*(f) = = M^*(\sum_{i \geq N} q_i e_i)$; hence $M^*(f) = \alpha$, since $\{\alpha\} = \bigcap_n cl_R\{q_i : i \geq n\}$.

4) implies 1). Obvious.

2.2 LEMMA. *Let P (P^*) be a prime, non-maximal, ideal of C (C^*). Then $C/P(C^*/P^*)$ contains infinitely small elements.*

PROOF. Let M be the maximal ideal of C containing P, and let $u \in M \backslash P$, $u \geq 0$. For each $n \in N$, $(u - u \wedge 1/n)(u \vee 1/n - u) = 0$, and $u \vee 1/n - u \notin M$, since $u \vee 1/n$ is a unit, being bounded away from 0. Hence $u - u \wedge 1/n \in P$, so that $0 < P(u) = P(u \wedge 1/n) \leq 1/n$, for each $n \in N$.

THEOREM. *Let M be a maximal ideal of C. The following are equivalent:*

1) *$C/M = F$.*

2) *$P^* = M \cap C^*$ is maximal in C^*.*

3) *M contains no countable partition of unity.*

4) *M contains no partition of unity of non-measurable cardinal.*

PROOF. 1) is equivalent to 2). C/M contains a canonical copy of the ordered ring C^*/P^* and is the field of fractions of this copy. It follows from lemma 2.2 that if P^* is not maximal, then C^*/P^* contains infinitely small elements.

2) implies 3). P^* contains no countable partition of unity since it contains no unit of C.

3) implies 2) . If P^* is not maximal, the maximal ideal M^* of C^* containing P^* contains a unit of C (see 1.3), hence it also contains a countable partition of unity E. It is clear that $E \subset P^*$, hence $E \subset M$.

3) implies 4). Suppose that E is a partition of unity of non-measurable cardinal, contained in M. Consider E as a topological space with the discrete topology. Define $\mathfrak{F} = \{Z : Z \subset E, \sum_{e \in Z} e \notin M\}$. It is easy to show that $\mathfrak{F}$ is a free ultrafilter on E, and since E is realcompact, $\mathfrak{F}$ is hyperreal. Choose a countable subfamily of $\mathfrak{F}$ with empty intersection, $\{Z_i : i \in N\}$. Then $E \setminus Z_i \notin \mathfrak{F}$ and $\bigcup_i (E \setminus Z_i) = E$. Thus $V_i = (E \setminus Z_i) \setminus \bigcup_{j<i} (E \setminus Z_j)$ is a countable partition of E and $V_i \notin \mathfrak{F}$ for each $i \in N$. Clearly $\{\sum_{e \in V_i} e : i \in N\}$ is a countable partition of unity contained in M.

4) implies 3). Obvious.

2.3. THEOREM. *Let M be a maximal ideal of C such that $C/M = K (\neq F)$. Then K is an η_1-field in which F is algebraically closed. (We identify F with the image of the constant functions).*

PROOF. Suppose that $u \in K$ is algebraic over F, and let $p(t)$ be its minimum polynomial over F. If $f \in C$ is such that $u = M(f)$ then $0 = p(u) = M(p(f))$, that is to say, $p(f) \in M$. But this implies that $Z(p(f)) \neq \emptyset$, *i.e.* $p(t)$ must have a root in F. Since $p(t)$ is irreducible, it follows that $p(t) = t - q$ (for some $q \in F$). Hence $u = M(f) = q \in F$.

The fact that K is an η_1-field can be shown by using the same argument as in [GJ], 13.7. and 13.8. However, the presence of partitions of unity allows a considerably simpler argument. The theorem will be a consequence of [GJ], 13.8 and the following lemma.

LEMMA. *Let P a prime ideal contained in a maximal ideal M such that $C/M \neq F$. Then if A, B are countable subsets of C/P, with $A < B$, there exists $u \in C/P$ such that $A \leq u \leq B$.*

PROOF. Suppose that A and B are non-empty. By [GJ], 13.5, we can find an increasing sequence $f_1 \leq f_2 \leq \ldots$ and a decreasing sequence $g_1 \geq g_2 \geq \ldots$ of elements of C such that $f_n \leq g_n$ for each $n \in N$, and $\{P(f_i) : i \in N\}$ is a cofinal subset of A, and $\{P(g_i) : i \in N\}$ is a coinitial subset of B. Let $\{e_i : i \in N\}$ be a countable partition of unity contained in P and let $f = \sum_i 2^{-1}(f_i + g_i)e_i$. It is easy to show that $u = P(f)$ satisfies the required condition. If either A or B is empty, a simple modification of the preceding argument shows that either B is not coinitial or A is not cofinal.

REMARK. If M is a maximal ideal of C such that $C/M \neq F$, then $|C/M| \geq c$. Furthermore in the special case $F=Q$, C/M contains no copy of R.

3. F-realcompactness.

3.1. Let X be a T_0-space with a base of clopen sets. Let $\mathfrak{M}$ $(=\delta X)$ be the maximal ideal space of C $(=C(X, F))$. Denote by νX the subspace of $\mathfrak{M}$ consisting of all those ideals M for which $C/M=F$; νX is obviously a T_0-space with a base of clopen sets in which X is dense and $C(X, F)$-embedded. Hence $(C(X, F)=C(\nu X, F)$ and every ideal M of $C(\nu X, F)$ for which $C/M=F$ is fixed. We call a space F-realcompact if every ideal M of $C(X, F)$ for which $C/M=F$ is fixed.

THEOREM. *Let X be a T_0-space with a base of clopen sets. If X is F-realcompact, then it is realcompact.*

PROOF. Suppose that M' is a real maximal ideal of $C(X)$; $M' \cap C(X, F)$ is a maximal ideal of $C(X, F)$ with the property that $C(X, F)/(M' \cap C(X, F))=F$. (Observe that this field is embedded in $C(X)/M'$ as an ordered subring). Hence $M' \cap C(X, F)=M_p \cap C(X, F)$. It follows that $M'=M_p$.

REMARK. Dr. Peter Nyikos has communicated to one of the authors that the space considered in [R] is not F-realcompact. Hence a space can be realcompact and have a base of clopen sets without being F-realcompact. However, if X is zero-dimensional and realcompact, then it is F-realcompact (see 1.7).

3.2. EXAMPLE. The space Δ_1 of [GJ], 16M is a T_0-space with a base of clopen sets whose dimension is 1. Hence $\beta\Delta_1$ is not totally disconnected. That is to say, $\beta\Delta_1 \neq \delta\Delta_1$. It is known from [$D$] and [$GJ$] that Δ_1 is dense and C-embedded in a space Δ such that $\Delta \setminus \Delta_1$ is a copy of $[0, 1]$. Also, the quotient space Δ_0 of Δ obtained by identifying the points of $\Delta \setminus \Delta_1$ is zero dimensional. It is easy to see that

$\upsilon\Delta_1=\Delta$, $\nu\Delta_1=\Delta_0$ and $\delta\Delta_1=\beta\Delta_0$. In fact Δ_0 is the space obtained from Δ by the method of theorem 1.4. ($\Delta_0=\theta[\Delta]$). We show that Δ is realcompact.

Let π be the restriction to Δ of the canonical projection of W^* $[0, 1]$ onto W^*; for each $\tau<\omega_1$ let $\Delta_\tau=\pi^{\leftarrow}[W(\tau+1)]$; Δ_τ is a clopen subspace of Δ (hence C-embedded in Δ) homeomorphic to a subspace of R^2. Hence Δ_τ is realcompact. Consider a real z-ultrafilter $\mathfrak{F}$ on Δ; if for each $Z\in\mathfrak{F}$, $\pi[Z]$ is cofinal in W^*, then (from [*GJ*], 16M.4), $\omega_1\in\pi[Z]$, for each $Z\in\mathfrak{F}$. Hence $\{Z\cap(\{\omega_1\}\times[0, 1]) : Z\in\mathfrak{F}\}$ is a filter on $\{\omega_1\}\times[0, 1]$. Hence $\mathfrak{F}$ is fixed. If for some $Z\in\mathfrak{F}$, $\pi[Z]$ has an upper bound τ in W, then $\{Z\cap\Delta_\tau : Z\in\mathfrak{F}\}$ is a real z-ultrafilter on Δ_τ and so is fixed.

REFERENCES

[*D*] Dowker, C. H.: *Local dimension of normal spaces*, Quart. J. Math., Oxford Ser. (2) 6, 1955, 101-120.

[*DMO*] De Marco, G. and Orsatti, A.: *On spectra of commutative rings etc.*, Paper submitted to Proc. A.M.S.

[*GJ*] Gillman, L. and Jerison, M.: *Rings of continuous functions*, Van Nostrand, New York, 1960.

[*P*] Pierce, R. S.: *Rings of integer-valued continuous functions*, Trans. A.M.S., 100, 1961, 371-394.

[*R*] Roy, P.: *Nonequality of dimension for metric spaces*, Trans. A.M.S., 134. 1968, 117-132.

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