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ON RADICAL RINGS

CHRISTIAN W. KROENER *)

Introduction.

If $J(R)$ denotes the Jacobson radical of a ring R, R is said to be a radical ring (in the sense of Jacobson) if $R = J(R)$. Radical rings occupy a position intermediate between nil rings and semiradical rings. In general these three classes of rings are distinct. However it is well known that in the presence of the minimum condition for left ideals (d. c. c.) these three classes coincide with the class of nilpotent rings.

Radical rings differ from nil and semiradical rings among other things by the fact that subrings need not be of the same type, but subrings are always semiradical rings. The converse question of embedding a given semiradical in a radical ring was raised by Andrunakievitch [1] who gave a criterion similar to the Ore condition. In this paper we give several other conditions which lead to the same result. They give further evidence that radical rings play the same role with respect to quasi multiplication ($x \circ y = x + y - xy$) as simple rings with d. c. c. for left ideals with respect to ordinary multiplication. We also show that if a ring R can be embedded in a radical ring $Q(R)$, its matrix ring R_n can also be embedded and $Q(R_n) = (Q(R))_n$.

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I. Basic Definitions and Notations.

Let R denote an associative not necessarily commutative ring R.

DEFINITION 1. Let $a, b \in R$. The binary operation $o : R \times R \rightarrow R$ defined by $(a, b) \rightarrow a \circ b = a + b - ab$ is called quasi multiplication. $a \circ b$ is called the quasi product of a, b.

DEFINITION 2. $a \in R$ is called left (right) semiradical if $a \circ x = a \circ \bar{x}$ $(x \circ a = \bar{x} \circ a)$ implies $x = \bar{x}$.

It is easy to see that this definition is equivalent to the following: $ax = x$ $(xa = x)$ implies $x = 0$.

$a \in R$ is called semiradical if it is left and right semiradical. R is called semiradical if every $a \in R$ is semiradical.

DEFINITION 3. An element $a \in R$ is called left (right) radical if there exists an element $a'_l\,(a'_r) \in R$ such that $a'_l \circ a = 0$ $(a \circ a'_r = 0)$. a'_l (a'_r) is called left (right) quasi inverse of a.

We remark: if a is both left and right radical then $a'_l = a'_r = a'$ is unique.

An element $a \in R$ is called radical if it is left and right radical. Its unique quasi inverse is denoted by a'. R is called radical if every $a \in R$ is radical.

Remark 1. A semiradical ring forms a monoid, a radical ring a group with respect to quasi multiplcation.

Remark 2. The definitions of semiradical and radical element are as in Andrunakievitch [1]. In the Englisch literature a radical element is usually called quasi regular. (Perlis [1], Jacobson [4]). However the term « quasi-regular » seems to be inappropriate for the following reasons:

We recall that $a \in R$ is called regular (Jacobson [5], Herstein [3]) if it is not a left or right zero divisor with respect to ordinary multiplication. It is the semiradical element rather than the radical element which should be called quasi regular i. e. regular with respect to quasi multiplication because it is not a left or right zero divisor with respect to quasi multiplication. A radical element should then be called a quasi unit because it has a quasi left and right inverse. This term is further justified by the following result of Andrunakievitch: In a principal quasi ideal ring every element x can be represented as a quasi product of prime factors and this representation is unique up *to radical elements.*

Although « quasi regular » and « quasi unit » in the sense just mentioned are more closely related to the properties of quasi multiplication, we will use — to avoid confusion — the terms semi-radical element, radical element and quasi inverse as defined above.

Let Z denote the ring of rational integers.

DEFINITION 4. A subset $I \subseteq R$ is called a left quasi ideal of R if

(i) $a \in I$ implies $r \circ a \in I$ for all $r \in R$

(ii) $a_1, a_2, \dots, a_n \in I$ implies

$$\sum_{i=1}^{n} q_i a_i \in I \text{ where } q_i \in Z \ (i = 1, \dots, n) \text{ such that}$$

$$\sum_{i=1}^{n} q_i = 1.$$

In a similar way one defines right and two sided quasi ideals. Two sided quasi ideals will simply be called quasi ideals.

Remark 3. Every ring contains the quasi ideals R and Φ (the empty set), which satisfy the above conditions trivially. The formulation of condition (ii) is motivated by the desire for distributivity with respect to quasi multiplication.

Remark 4. The significance of quasi ideals lies in the fact that they arise as « quasi kernels » of homomorphism in the same way as ordinary ideals arise as kernels of homomorphism. (Theorem 8 [1]). To include the empty set as quasi ideal we added the condition on the identity of R:

If a ring R without identity is mapped by a homomorphism onto a ring R^* the set of all elements of R mapped onto the identity of R^* will be a quasi ideal I. Conversely, for every quasi ideal I of R there exists a homomorphism onto a ring R^* such that the elements of I and only those are mapped onto the idenlity of R^*.

PROOF. If R^* does not have an identity, the quasi ideal $I = \Phi$ and we may take $R^* = R$ in the converse. If $1 \in R^*$ one verifies that the set of elements of R mapped on to 1 satisfies the conditions of Definition 4. If $I \neq \Phi$ is a quasi ideal of R one shows that $I^* = \{(a - b)\}a, b \in I\}$ is an ideal of R in the ordinary sense, that in $\bar{R} = R/I^*$ the element I is of the form $a + I^*$ for any $a \in I$ and constitutes the identity of $\bar{R}$.

It can be seen easily that if R contains an identity there is a $1 - 1$ correspondence between (left) ideals of R and nonempty (left) quasi ideals of R. For if A is a left ideal of R, then $1 - A = \{1 - a \mid a \in A\}$ is a nonempty left quasi ideal of R and if $A \neq \Phi$ is a left quasi ideal of R then $1 - A$ is a left ideal of R. This also shows that conditions imposed on quasi ideals imply conditions on ideals and vice versa in rings with identity.

DEFINITION 5. The left quasi ideal generated by a subset $M \subset R$ is the intersection of all left quasi ideals of R containing M.

It is easy to show that it consists of all finite sums of the form $\Sigma q_i (r_i \circ m_i)$, where $r_i \in M$, $m_i \in M$, $q_i \in Z$ such that $\Sigma q_i = 1$.

In particular, the left quasi ideal generated by $m \in R$ will consist of all elements of the form $r \circ m, r \in R$. It will be called left principal ideal and will be denoted by $R \circ m$.

Remark 5. The ring $R = R \circ O$. In a radical ring every nonempty left quasi ideal is a left principal quasi ideal, since every non-empty left quasi ideal coincides with R. In [1] it is shown that in the ring of even integers every non-empty left quasi ideal is a left principal quasi ideal.

DEFINITION 6. The sum $A + B$ of two left quasi ideals A, B is the left quasi ideal generated by the union of the sets A, B.

Apparently $A + B =$

$$\left\{ \sum_{i=1}^{n} p_i a_i + \sum_{j=1}^{m} q_j b_j \mid a_i \in A, b_j \in B, p_i, q_j \in Z \text{ s. t.} \sum_{i=1}^{n} p_i + \sum_{j=1}^{m} q_j = 1 \right\}.$$

This may be extended to infinite sums.

DEFINITION 7. A sum $A = \sum_{i \in I} A_i$ of left quasi ideals is said to be direct if for each $(i, j) \in I \times I \; A_i \cap A_j = \Phi, i \neq j$.

DEFINITION 8. The product $A \circ B$ of two left quasi ideals A, B is the left quasi ideal generated by all product $a \circ b$, $a \in A, b \in B$.
Apparently $A \circ B =$

$$\left\{ \sum_{i=1}^{n} p_i (a_i \circ b_i) \mid a_i \in A, b_i \in B, p_i \in Z, \sum p_i = 1 \right\}.$$

Remark 6. As for ordinary ideals one can introduce the maximum and minimum condition for left quasi ideals and show their equivalence to the ascending chain condition (a. c. c.) and descending chain condition (d. c. c.) for left quasi ideals, respectively.

DEFINITION 9. R is called quasi simple if there exist no two sided quasi ideals in R other than R and Φ.

II. Left Quasi Quotient Rings.

It will be seen that the embedding of a given semiradical ring in a radical ring resembles the well known formation of quotient rings. We shall therefore start more generally be defining left quasi quotient rings.

DEFINITION. A ring $Q(R) \supset R$ is called a left quasi quotient ring of R if the following conditions are satisfied:

(i) If $a \in R$ is semiradical, then a is radical in $Q(R)$.

(ii) Every element $x \in Q(R)$ is of the form $a' \circ b$, where $a, b \in R$, a semiradical, a' the quasi inverse of a in Q.

Lemma 1. A necessary and sufficient condition that a ring R have a left quasi quotient ring $Q(R)$ is: for every semiradical $b \in R$ and every $a \in R$ there exists a semiradical $b_1 \in R$ and $a_1 \in R$ such that $b_1 \circ a = a_1 \circ b$. The ring $Q(R)$ is uniquely determined up to isomorphism by R.

Proof. If $Q(R)$ exists then for $b \in R$ semiradical, $a \circ b' \in Q$, hence $a \circ b' = b_1' \circ a_1$, with b_1 semiradical in R, $a_1 \in R$. Thus $b_1 \circ a = a_1 \circ b$ and the condition is necessary. This condition is also sufficient: Let S denote the set of semiradical elements of R. Clearly S is closed under quasi multiplication. One verifies that if $b_1 \circ d = d_1 \circ b$, where $b_1, b, d \in S$ then also $d_1 \in S$. Consider $R \times S = \{(a, b) \mid a \in R, b \in S\}$. Let $(a, b), (c, d) \in R \times S$. By assumption there exist $b_1, d_1 \in R$, $b_1 \in S$ such that $d_1 \circ b = b_1 \circ d$. Define $(a, b) \backsim (c, d)$ if and only if $(d_1 \circ a, d_1 \circ b) = (b_1 \circ c, b_1 \circ d)$. One checks that $\backsim$ is an equivalence relation independent of the choice of $b_1, d_1 \in R$. In $Q(R) = (R \times S)/\backsim$ define sum and quasi product in the following way:

$$(a, b) + (c, d) = (d_1 \circ a + b_1 \circ c - d_1 \circ b, d_1 \circ b)$$

where $d_1 \circ b = b_1 \circ d$

$$(a, b) \circ (c, d) = (a_1 \circ c, d_1 \circ b)$$

where $a_1 \circ d = d_1 \circ a$, $d_1 \in S$.

As usual one verifies that addition and quasi multiplication are well defined. If we denote the class represented by (a, b) by $\frac{a}{b}$ and define sum and quasi product of the new symbols by

$$\frac{a}{b} + \frac{c}{d} = \frac{d_1 \circ a + b_1 \circ e - d_1 \circ b}{d_1 \circ b}$$

$d_1 \circ b = b_1 \circ d$

$$\frac{a}{b} \circ \frac{c}{d} = \frac{a_1 \circ c}{d_1 \circ b}$$

$$a_1 \circ d = d_1 \circ a, \, d_1 \in S$$

then one can show that $(Q(R), +, \circ)$ satisfies the following laws:

(i) $(Q(R), +)$ is an abelian group

(ii) $(x \circ y) \circ z = x \circ (y \circ z)$

(iii) $(x + y) \circ z = x \circ z + y \circ z - z$

$$z \circ (x + y) = z \circ x + z \circ y - z$$

for all $x, y, z \in Q(R)$. It is a trivial exercise to show that this is equivalent to the fact that $(Q(R), +, \cdot)$ is a ring.

Finally, the subring $\widetilde{R}$ of all elements of the form $\frac{a}{0}$ is a ring isomorphic to R. For

$$\frac{a}{0} = \frac{b}{0} \text{ implies } a = b.$$

Furthermore

$$\frac{a}{0} + \frac{b}{0} = \frac{a + b - 0}{0} = \frac{a + b}{0}$$

$$\frac{a}{0} \circ \frac{b}{0} = \frac{a_1 \circ b}{0_1 \circ 0}$$

where $a_1 \circ 0 = 0_1 \circ a$ i. e. $a_1 = 0_1 \circ a$.

Thus

$$\frac{a}{0} \circ \frac{b}{0} = \frac{0_1 \circ a \circ b}{0_1 \circ 0} = \frac{a \circ b}{0}.$$

If we identify $\widetilde{R}$ with R we may write a instead of $\frac{a}{0}$, hence $R \subset Q(R)$. Since

(i) for every semiradical $a \in R$ there exists an element $a' = \frac{0}{a} \in Q(R)$ such that $a' \circ a = 0 = a \circ a'$ and

(ii) every element $\frac{a}{b} \in Q(R)$ can be written in the form $\frac{a}{b} = \frac{0}{b} \circ$
$\circ \frac{a}{0} = b' \circ a.$

the ring $Q(R)$ is a left quasi quotient ring of R.

If R_1 and R_2 have left quasi quotient rings $Q(R_1)$ and $Q(R_2)$ it is easy to see that any isomorphism S of R_1 onto R_2 determines an isomorphism $Q(R_1)$ onto $Q(R_2)$ having the form $b' \circ a \rightarrow S(b)' \circ S(a)$, $a, b \in R$, b semiradical.

In the same way we may also define a right quasi quotient ring. It seems likely that the existence of a right quasi quotient ring is independent of the existence of a left quasi quotient ring. At the moment, however, we can only prove the following.

PROPOSITION 1. If a ring possesses a left and a right quasi quotient ring then they coincide.

PROOF. Let $Q(R)$ be the left quasi quotient ring of R, $x = b' \circ a \in Q(R)$, $a, b \in R$, b semiradical. Since the right quasi quotient ring $\overline{Q(R)}$ exists there are elements $\bar{a}, \bar{b} \in R$, $\bar{b}$ semiradical such that $a \circ \bar{b} = b \circ \bar{a}$. Then $x \circ \bar{b} = b' \circ a \circ \bar{b} = b' \circ b \circ \bar{a} = \bar{a} \in R$ and $x = \bar{a} \circ \bar{b}'$ thus $x \in \overline{Q(R)}$. Similarly one shows $\overline{Q(R)} \subseteq Q(R)$.

PROPOSITION 2. If R possesses a left quasi quotient ring $Q(R)$ then every left quasi ideal $I \neq \Phi$ of R which contains a semiradical element possesses a left quasi quotient ring $Q(I)$ and $Q(I) = Q(R)$.

PROOF. Let $x \in Q(R)$. Then $x = b' \circ a$, $a, b \in R, b$ semiradical. Let $c \in I$ be semiradical. By the lemma there exist $b_1, c_1 \in R$, b_1 semiradical such that $c_1 \circ b = b_1 \circ c$. Since b, b_1, c semiradical, c_1 is too. Moreover we can find $r, s \in R$, r semiradical such that $r \circ (c_1 \circ a) = s \circ c$. Thus

$$\begin{aligned} x = b' \circ a &= b' \circ c_1' \circ r' \circ r \circ c_1 \circ a \\ &= (r \circ c_1 \circ b)' \circ (r \circ c_1 \circ a) \\ &= (r \circ b_1 \circ c)' \circ (s \circ c) \\ &= y' \circ z \end{aligned}$$

where $y = r \circ b_1 \circ c \in I$ and $z = s \circ c \in I$. Thus $Q(R) \subseteq Q(I)$. Clearly $Q(I) \subseteq Q(R)$.

For later use we include here a technical

LEMMA 2. Suppose R possesses a left quasi quotient ring $Q(R)$. Given $y_1, \ldots, y_n \in Q(R)$ there exists a semiradical $b \in R$ and $a_1, \ldots$ $\ldots, a_n \in R$ such that $b \circ y_i = a_i (i = 1, \ldots, n)$.

PROOF. If $y_1, \ldots, y_n \in Q(R)$, there exist semiradical elements $b_1, \ldots, b_n \in R$ and $x_1, \ldots, x_n \in R$ such that $b_i \circ y_i = x_i$ $(i = 1, \ldots, n)$. We use induction on n; $n = 1$ is true by assumption. Suppose there exist semiradical $s \in R$, and $z_1, \ldots, z_{n-1}$ such that $s \circ y_i = z_i (i = 1, \ldots$ $\ldots, n-1)$. By Lemma 1 we can find $c, d \in R$, c semiradical, such that $b = c \circ s = d \circ b_n$. Then $b \circ y_i = c \circ s \circ y_i = c \circ z_i = a_i (i = 1, \ldots$ $\ldots, n-1)$ $b \circ y_n = d \circ b_n \circ y_n = d \circ x_n = a_n$. Clearly b is semiradical and $b, a_i \in R$.

III. Radical Rings as Left Quasi Quotient Rings.

THEOREM 1. *The following conditions on R are equivalent.*

(i) *R is semiradical and for $a, b \in R$ there exist $a_1, b_1 \in R$ such that $b_1 \circ a = a_1 \circ b$*

(ii) *R is semiradical and does not contain any infinite direct sums of left quasi ideals*

(iii) *R satisfies (1) every non-empty left quasi ideal of R contains a semiradical element and (2) R does not contain any infinite direct sums of left quasi ideals*

(iv) *R possesses a left quasi quotient ring $Q(R)$ which is radical ring.*

PROOF.

(i) $\Longrightarrow$ (ii) the condition on the element of R implies that every two non-empty left quasi ideals have a non-empty intersection

(ii) $\Longrightarrow$ (iii) trivial

(iii) $\Longrightarrow$ (iv) there exists some left quasi ideal $I \neq \Phi$ in R such that any two non-empty left quasi ideals of R lying in I have non-empty intersection. If not, there are two non-empty left quasi ideals I_1, I_2 of R such that $I_1 \cap I_2 = \Phi$. If also in I_1 there exists two non-empty left quasi ideals of R, say I_{11}, I_{12}

with empty intersection then $I_{11} + I_{12} + I_2$ is a direct sum of left quasi ideals in R. Continuing this way with I_{11} we can produce longer and longer direct sums, contradicting assumption (2) on R. Thus $I \neq \Phi$ exists. Suppose A, B are non-empty left quasi ideals of R. Let $x \in I$ be semiradical and consider $A \circ x$, $B \circ x$, non-empty left quasi ideals of R lying in I. By the above $A \circ x \cap B \circ x \neq \Phi$, hence there exist $a \in A, b \in B$ such that $a \circ x = b \circ x$, which implies $a = b \in A \cap B$, hence $A \cap B \neq \Phi$. Now let $a, b \in R$, b semiradical. Then $R \circ a \cap R \circ b \neq 0$ implies that the left quasi ideal $M = \{x \mid x \circ a \in R \circ b\}$ is non-empty. There exists a semiradical $b_1 \in M$ such that $b_1 \circ a = a_1 \circ b$ and R possesses a left quasi quotient ring $Q(R)$ by lemma 1. Finally, we show $Q(R)$ is radical ring. Let $L \neq \Phi$ be any left quasi ideal of $Q(R)$.

It is easy to see that $L = Q \circ (R \cap L)$ and that $L \cap R \neq \Phi$ is a left quasi ideal of R. Let $y \in L \cap R$ be semiradical, then $0 = y' \circ y \in Q \circ (L \cap R) = L$ gives $L = Q(R)$. It follows that every $x \in Q(R)$ is left radical, hence it is radical. Thus $Q(R)$ is a radical ring.

(iv) $\Longrightarrow$ (i) $R \subseteq Q(R)$ is semiradical, since $Q(R)$ radical. The rest follows from lemma 1.

The equivalence (i) $\Longleftrightarrow$ (iv) was shown first by Andrunakievitch. By a result of Kurotchkin [6] R is a radical ring if and only if it is a quasi simple ring with d. c. c. on left quasi ideals. The equivalence (iii) $\Longleftrightarrow$ (iv) establishes therefore a very close analogue to the first Goldie theorem: A prime ring R possesses a left quotient ring Q which is a simple ring with d. c. c. on left ideals if and only if R satisfies (1) the a. c. c. on left annihilators and (2) does not contain any infinite direct sums of left ideals.

THEOREM 2. *If R is a semiradical ring in which every non-empty left quasi ideal is a left principal quasi ideal, then R possesses a left quasi quotient ring which is a radical ring.*

PROOF. Let $a, b \in R$ and consider $R \circ a$, $R \circ b$.
$R \circ a + R \circ b = R \circ d$ by assumption. Thus we have the following relations.

(1) $$a = a_1 \circ d$$

(2) $$b = b_1 \circ d$$

(3) $$d = \Sigma p_i (r_i \circ a) + \Sigma q_j (s_j \circ b) \text{ where } \Sigma p_i + \Sigma q_j = 1.$$

Using (1) and (2) in (3) we obtain

$$d = \Sigma p_i (r_i \circ a_1 \circ d) + \Sigma q_j (s_j \circ b_1 \circ d)$$

$$= [\Sigma p_i (r_i \circ a_1) + \Sigma q_j (s_j \circ b_1)] \circ d.$$

Since R is semiradical this implies

(4) $$0 = \Sigma p_i (r_i \circ a_1) + \Sigma q_j (s_j \circ b_1).$$

Multiplying (4) by b_1 and a_1 respectively, we get:

(5) $$b_1 = \Sigma p_i (b_1 \circ r_i \circ a_1) + \Sigma q_j (b_1 \circ s_j \circ b_1)$$

(6) $$a_1 = \Sigma p_i (a_1 \circ r_i \circ a_1) + \Sigma q_j (a_1 \circ s_j \circ b_1).$$

Subtraction of (6) from (5) gives:

$$b_1 - \Sigma q_j (b_1 \circ s_j \circ b_1) + \Sigma q_j (a_1 \circ s_j \circ b_1) =$$

$$a_1 - \Sigma p_i (a_1 \circ r_i \circ a_1) + \Sigma p_i (b_1 \circ r_i \circ a_1)$$

Since $1 - \Sigma p_i + \Sigma p_i = 1$ and $1 - \Sigma q_j + \Sigma q_j = 1$

we may write

$$[0 - \Sigma q_j (b_1 \circ s_j) + \Sigma q_j (a_1 \circ s_j)] \circ b_1$$

$$= [0 - \Sigma p_i (a_1 \circ r_i) + \Sigma p_i (b_1 \circ r_i)] \circ a_1$$

which upon right quasi multiplication by d establishes $x \circ b = y \circ a$ where

$$x = \Sigma\, q_j (a_1 \circ s_j) - \Sigma\, q_j (b_1 \circ s_j)$$

$$y = \Sigma\, p_i (b_1 \circ r_i) - \Sigma\, p_i (a_1 \circ r_i)$$

and $a_1, b_1, r_i, s_j \in R, p_i, q_j \in Z$ as determined by (1), (2) and (3). By Theorem 1 (i) R possesses a left quasi quotient ring which is a radical ring.

As example for Theorem 2 we may take the ring of even integers E which can be embedded in $Q(E) = \left\{ \frac{2n}{2m+1} \mid n, m \in Z \right\}$, a radical ring.

Finally we turn to the $n \times n$ matrix ring R_n over R. It is well known that if R is a radical ring so is R_n, because $J(R_n) = (J(R))_n$ [5]. The following theorem is a generalization of this result.

THEOREM 3. *If R possesses a left quasi quotient ring which is a radical ring, then R_n is semiradical, possesses a left quasi quotient ring $Q(R_n)$ which is also radical and $Q(R_n) = (Q(R))_n$.*

PROOF. The monomophism $R \longrightarrow Q(R)$ induces a monomophism $R_n \longrightarrow (Q(R))_n$. But $(Q(R))_n$ is radical so R_n is semiradical. Clearly $Q(R_n) \subseteq (Q(R))_n$. Now let $Y = (y_{ij}) \in (Q(R))_n$. We must determine $B \in R_n$ such that $B \circ Y = A \in R_n$.

Since $y_{ij} \in Q(R)$ there exist $b_{ij}, x_{ij} \in R$ such that

$$b_{ij} \circ y_{ij} = x_{ij} \qquad (i, j = 1, \dots, n).$$

By Lemma 2 we can find $b \in R, z_{ij} \in R$ such that

$$b + y_{ij} - by_{ij}$$

$$b \circ y_{ij} = z_{ij} \qquad (i, j = 1, \dots, n).$$

Claim: $B = (b_{ij})$ where

$$b_{ii} = b \qquad (i = 1, \dots, n)$$

$$b_{ij} = c(1-b)$$

$$= c - cb \qquad (i, j = 1, \dots, n, i \neq j)$$

$$0 \neq c \in R.$$

Clearly $B \in R_n$. We show $B \circ Y = A \in R_n$

$$a_{rr} = b_{rr} + y_{rr} - \sum_{k=1}^{n} b_{rk} y_{kr}$$

$$= b + y_{rr} \quad - c(1-b) y_{1r}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$- c(1-b) y_{r-1, r}$$

$$- \qquad b \; y_{rr}$$

$$- c(1-b) y_{r+1, r}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$- c(1-b) y_{nr}$$

$$= b + y_{rr} - by_{rr} - c(y_{1r} - by_{1r})$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$- c(y_{r-1, r} - by_{r-1, r})$$

$$- c(y_{r+1, r} - by_{r+1, r})$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$- c(y_{nr} - by_{nr})$$

$$\begin{aligned}
&= z_{rr} - c\,(z_{1r} - b) \\
&\qquad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\qquad - c\,(z_{r-1,\,r} - b) \\
&\qquad - c\,(z_{r+1,\,r} - b) \\
&\qquad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\qquad - c\,(z_{nr} - b) \in R.
\end{aligned}$$

Let $r \neq s$,

$$\begin{aligned}
a_{rs} &= b_{rs} + y_{rs} - \sum_{k=1}^{n} b_{rk}\, y_{ks} \\
&= c\,(1 - b) + y_{rs} - c\,(1 - b)\, y_{1s} \\
&\qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\qquad - c\,(1 - b)\, y_{r-1,\,s} \\
&\qquad - b\ y_{rs} \\
&\qquad - c\,(1 - b)\, y_{r+1,\,s} \\
&\qquad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\qquad - c\,(1 - b)\, y_{ns} \\
&= c\,(1 - b) + y_{rs} - b y_{rs} \\
&\qquad - c\,(y_{1s} - b y_{1s}) \\
&\qquad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\qquad - c\,(y_{r-1,\,s} - b y_{r-1,\,s}) \\
&\qquad - c\,(y_{r+1,\,s} - b y_{r+1,\,s}) \\
&\qquad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\qquad - c\,(y_{ns} - b y_{ns})
\end{aligned}$$

$$
\begin{aligned}
= c(1 - b) + z_{rs} - b \\
- c(z_{1s} - b) \\
\cdots \\
- c(z_{r-1,s} - b) \\
- c(z_{r+1,s} - b) \\
\cdots \\
- c(z_{ns} - b) \in R.
\end{aligned}
$$

Thus $B \circ Y \in R_n$, hence $(Q(R))_n \subseteq Q(R_n)$.

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