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CERTAIN SUMS OF DOUBLE HYPERGEOMETRIC SERIES

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1. Let Kampé de Fériét function [1] in a modified notation be defined as

$$F_{\varrho,\nu}^{\lambda,\mu}\begin{pmatrix} a_\lambda : b_\mu , b'_\mu ; \\ c_\varrho : d_\nu , d'_\nu \end{pmatrix} = \sum_{m,n=0} \frac{(a_\lambda)_{m+n} (b_\mu)_m (b'_\mu)_n}{(c_\varrho)_{m+n} (d_\nu)_m (d'_\nu)_n \, m\,!\, n\,!}$$

where $\lambda + \mu \leq 1 + \varrho + \nu$ and a_λ stands for the sequence $a_1, a_2, \ldots, a_\lambda$; and $(a)_m = a(a+1)\ldots(a+m-1), (a)_0 = 1$.

Carlitz [3-6] has given four sums of $F_{1,1}^{1,2}$ and a sum of $F_{0,2}^{1,2}$ under different sets of condition. Jain [7] has given a sum of $F_{1,1}^{0,3}$ viz.

$$(1.1) \quad F_{1,1}^{0,3}\begin{pmatrix} - : c-a, a ; \; c-b, b ; \; -m, -n ; \\ c : 1-a-b+c-m, 1+a+b-c-n ; \end{pmatrix}$$

$$= \frac{(a)_m (c-a)_n (b)_m (c-b)_n}{(c)_{m+n} (a+b-c)_m (c-a-b)_n}$$

where $a+b-c$ is not an integer.

Recently Srivastava [1]) [9] studied the sum of $F_{1,1}^{1,2}$ in a more systematic way and gave a number of results.

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[1]) I am thankful to prof. G. P. Srivastava who has so kindly allowed me to go through his unpublished manuscript [9].

The object of this paper is to study the sum of $F_{1,1}^{0,3}$, in detail.

2. $$F = F_{1,1}^{0,3}\left(\begin{matrix} - : a, a' ; b, b' ; -m, -n ; \\ d : c, c' \end{matrix}\right)$$

$$= \sum_{p,q=0} \frac{(a)_p (a')_q (b)_p (q')_q (-m)_p (-n)_q}{(d)_{p+q} (c)_p (c')_q \, p!\, q!}$$

$$= \sum \frac{(a)_p (b)_p (-m)_p}{(d)_p (c)_p \, p!} \, {}_3F_2 \left[\begin{matrix} a', b', -n \\ d+p, c' \end{matrix}\right].$$

Now using the transformation [2]

(2.1) $${}_3F_2 \left[\begin{matrix} a, b, -m \\ d, c \end{matrix}\right] = \frac{(c-b)_m}{(c)_m} \, {}_3F_2 \left[\begin{matrix} d-a, b, -m ; \\ d, 1+b-c-m \end{matrix}\right]$$

we get

(2.2) $$F = \frac{(c'-b')_n}{(c')_n} \sum_{p=0}^{m} \frac{(a)_p (b)_p (-m)_p}{(d)_p (c)_p \, p!} \, {}_3F_2 \left[\begin{matrix} d-a'+p, b', -n ; \\ d+p, 1+b'-c'-n \end{matrix}\right]$$

$$= \frac{(c'-b')n}{(c')_n} \sum_{q=0}^{n} \frac{(d-a')_q (b')_q (-n)_q}{(d)_q (1+b'-c'-n)_q \, q!} \, {}_4F_3 \left[\begin{matrix} d-a'+q, a, b, -m ; \\ d+q, c, d-a' \end{matrix}\right].$$

Choosing

(2.3) $$d = a + a'$$

and

(2.4) $$d + c = 1 + a + b - m$$

${}_4F_3$ on the right reduces to ${}_3F_2$ Saalschützian.

Using Saalschützian theorem [8], we get

$$F = \frac{(d-a)_m (d-b)_m (c'-b')_n}{(d)_m (d-a-b)_m (c')_n} \, {}_4F_3 \left[\begin{matrix} a, b', d-b+m, -n ; \\ d+m, 1+b'-c'-n, d-b \end{matrix}\right].$$

This ${}_4F_3$ reduces to ${}_2F_1$ under six different sets of conditions and the resulting ${}_2F_1$ can be summed by Gauss theorem.

Thus assuming the conditions (2.3) & (2.4) are satisfied, we get

$$(2.5) \qquad F = \frac{(d-a)_m\,(d-b)_m\,(a)_n\,(b)_n}{(d)_{m+n}\,(d-a-b)_m\,(a+b-d)_n}$$

if $d = b + b'$ and $a = 1 + b' - c' - n$.

which is the same as (1.1)

$$(2.6) \qquad = \frac{(d-a)_{m+n}\;(d-b)_{m+n}}{(d)_{m+n}\,(d-a-b)_m\;(m)_n}$$

if $b' = d - b$ and $c' = 1 - n - m$.

$$(2.7) \qquad = \frac{(d-b)_m\,(c')_{m+n}\;m\,!}{(d)_m\,(1+b)_m\,(c')_m\,(c')_n}$$

if $d = a - m$ and $b' = d - b$.

$$(2.8) \qquad = \frac{(d-a)_m\,(c'-b')_m\,(c')_{m+n}}{(d)_m\,(c)_m\,(c')_m\,(c')_n}$$

if $d = a - m$ and $a - b + m = 1 + b' - c' - n$.

$$(2.9) \qquad = \frac{(d-a)_m\,(d-b)_m\,(a)_n\;m\,!}{(d)_m\,(d-a-b)_m\,(c')_n\,(d-b)_n\,(m-n)\,!}$$

if $m \geq n$ ⎫ if $a = 1 + b' - c' - n$ and $d = b' - m$

$= 0.$ if $m < n$ ⎭

$$(2.10) \qquad = \frac{(d-a)_m\,(d-b)_m\,(b'-b)_n\,(d-a-b)_n}{(d)_m\,(d-a-b)_m\,(-b)_n\,(d-b)_n}$$

if $d = b' - m$ and $c' = 1 + b - n$.

provided b is not a positive integer $< n$.

Now using (2.1) in the right hand side of (2.2), assuming $d = b + b'$ and changing the order of summation, we get

$$(2.11) \qquad F = \frac{(b')_n (b + c' - a')_n}{(c')_n (d)_n} \sum_{q=0}^{n} \frac{(c' - b')_q (b)_q (-n)_q}{(1 - b' - n)_q (b + c' - a')_q \, q!}$$

$$\cdot {}_4F_3 \left[\begin{matrix} b + c' - a' + n, b + q, a, -m \\ c, d + n, b + c' - a' + q \end{matrix} \right].$$

This ${}_4F_3$ is Saalschützian if (2.4) is satisfied.

If $c = b + c' - a' + n$, ${}_4F_3$ reduces to ${}_3F_2$ which can be summed by Saalschützian theorem and we get

$$F = \frac{(b + c' - a' - a)_m (b')_n (b + c' - a')_n (c' - a')_m}{(b + c' - a')_m (c')_n (d)_n (c' - a' - a)_m}$$

$$\cdot {}_3F_2 \left[\begin{matrix} c' - b', b, -n \\ c - n + m, c - a - n \end{matrix} \right].$$

If further $c = b + n - m$ it reduces to ${}_2F_1$ and is summed by Gauss theorem. Hence

$$(2.12) \qquad F = \frac{(1 + a - b)_{m-n} (d - b)_n (d - a - a')_m \, m!}{(1 - b)_{m-n} (d)_n (a' - m)_n (1 + a)_m}.$$

If $b' = d - b$, $d + c = 1 + a + b - m$, $c' = a' - m$, $c = b + n - m$.

Similarly if we take $d = a - n$ in (2.11) and assume condition (2.4) is satisfied, then summing ${}_3F_2$ by Saalschütz's theorem we get

$$F = \frac{(d - b)_n (b + c' - a')_n (c' - a')_m (-n)_m}{(c')_n (d)_n (-b - n)_m (b + c' - a')_m}$$

$$\cdot {}_3F_2 \left[\begin{matrix} c' - b', b, -n + m, \\ 1 - b' - n, b + c' - a' + m \end{matrix} \right].$$

This ${}_3F_2$ reduces to ${}_2F_1$ if either $a' = d + m$ or $a' = c' + m$, which can be summed by Gauss theorem. Thus

$$(2.13)\qquad F = \frac{(b')_m\,(c'-b')_{n-m}\,(c'-a')_m\,(1+b)_n\,n\,!}{(c')_n\,(d)_m\,(1+b)_{n-m}\,(n-m)\,!} \qquad \text{if}\, n \geq m$$

$$= 0 \qquad \text{if}\, n < m$$

provided $d = b + b', d + c = 1 + a + b - m, a = d + n, a' = d + m$ and

$$(2.14)\qquad F = \frac{(b')_m\,(b)_{n+1}\,m\,!\,n\,!}{(1-a')_m\,(d)_n\,(n-m)\,!\,(b+n-m)} \qquad \text{if}\, n \geq m$$

$$= 0 \qquad \text{if}\, n < m$$

provided $d = b + b', d + c = 1 + a + b - m, a = d + n, a' = c' + m.$

3. In this section, we consider that case of $F_{1,1}^{0,3}$ in which one parameter in the numerator cancell one parameter in the denominator.

$$f = F_{1,1}^{0,3}\left(\begin{matrix} - : a, a' ; b, b' ; c, c' ; \\ d : e, a' ; \end{matrix}\right) = \sum_{p=0}^{\infty} \frac{(a)_p\,(b)_p\,(c)_p}{(d)_p\,(e)_p\,p\,!}\ {}_2F_1\left[\begin{matrix} b', c', \\ d+p \end{matrix}\right]$$

which by Gauss theorem is

$$(3.1)\qquad = \frac{\Gamma(d-b'-c')\,\Gamma(d)}{\Gamma(d-b'),\Gamma(d-c')}\ {}_4F_3\left[\begin{matrix} a, b, c, d-b'-c' ; \\ e, d-b', d-c' \end{matrix}\right]$$

choosing $d = b + b'$, ${}_4F_3$ in (3.1) reduces to ${}_3F_2$.

Thus

$$f = \frac{\Gamma(b-c')\,\Gamma(d)}{\Gamma(b)\,\Gamma(d-c')}\ {}_3F_2\left[\begin{matrix} a, c, b-c' ; \\ e, d-c' \end{matrix}\right].$$

It can be summed by Gauss theorem if $d = c + c'$ or $e = b - c'$.

Thus

$$(3.2)\qquad f = \frac{\Gamma(b-c')\,\Gamma(d)\,\Gamma(e)\,\Gamma(e-a-b-c')}{\Gamma(b)\,\Gamma(c)\,\Gamma(e-a)\,\Gamma(e-b+c')}$$

if $d = b + b' = c + c'.$

$$(3.3) \qquad = \frac{\Gamma(e)\,\Gamma(d)\,\Gamma(d-a-c-c')}{\Gamma(b)\,\Gamma(d-a-c')\,\Gamma(d-c-c')} \quad \text{if } b = d - b = e + c'.$$

Using Saalschütz's theorem, we get

$$(3.4) \qquad = \frac{\Gamma(b-c')\,\Gamma(d)\,\Gamma(b'-c)\,\Gamma(b'-a)\,\Gamma(d-a-c-c')}{\Gamma(b)\,\Gamma(b'-a-c)\,\Gamma(d-a-c')\,\Gamma(d-c-c')\,\Gamma(b')}$$

provided $d = b + b'$, $d + e = 1 + a + b + c$ and a or c is a negative integer.

Using Dixon theorem, we get

$$(3.5) \quad = \frac{\Gamma(b-c')\,\Gamma(d)\,\Gamma\left(1+\frac{1}{2}a\right)\Gamma(e)\,\Gamma\left(1+\frac{1}{2}a-b-c+c'\right)}{\Gamma(b)\,\Gamma(1+a)\,\Gamma\left(e-\frac{1}{2}a\right)\Gamma\left(d-c'-\frac{1}{2}a\right)\Gamma(1+a-b-c+c')}$$

provided when $d = b + b'$ and

either $e = 1 + a - c$, $d = 1 + a - b + 2c'$

or $e = 1 + a - b + c'$, $d = 1 + a - c + c'$.

The above ${}_3F_2$ can be summed by Watson theorem in four different sets of conditions. Sum under one set of condition is

$$(3.6) \qquad f =$$

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(b-c')\,\Gamma(d)\,\Gamma\left(c+\frac{1}{2}\right)\Gamma(1-d+c+c')}{\Gamma(b)\,\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}b-\frac{1}{2}c'\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}a+c\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}b+\frac{1}{2}c'+c\right)}$$

if $d = b + b'$, $e = 2c$, $2d = 1 + a + b + c'$.

The ${}_3F_2$ can also be summed by Whipples theorem in two different sets of conditions. Sum in one case is

$$(3.7) \quad = \frac{\pi\, 2^{1-2c}\,\Gamma(b-c')\,\Gamma(d)\,\Gamma(e)}{\Gamma(b)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}d-\frac{1}{2}c'\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}c\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}d-c'\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}c-\frac{1}{2}c'\right)}$$

if $d=b+b', a+b-c'=1$, and $d+e=1+2c+c'$.

Taking $c=d-b'-c'$, ${}_4F_3$ in (3.1) reduces to ${}_3F_2$ which can be summed by Saalschützs, Dixon, Whipples theorem, hence giving at least four different sums of $F_{1,1}^{0,3}$.

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